SOLUTIONS

10

Consumer Surplus and Dead Weight Loss

Solutions for *Microeconomics: An Intuitive Approach with Calculus*

Apart from end-of-chapter exercises provided in the student *Study Guide*, these solutions are provided for use by instructors. (End-of-Chapter exercises with solutions in the student *Study Guide* are so marked in the textbook.)

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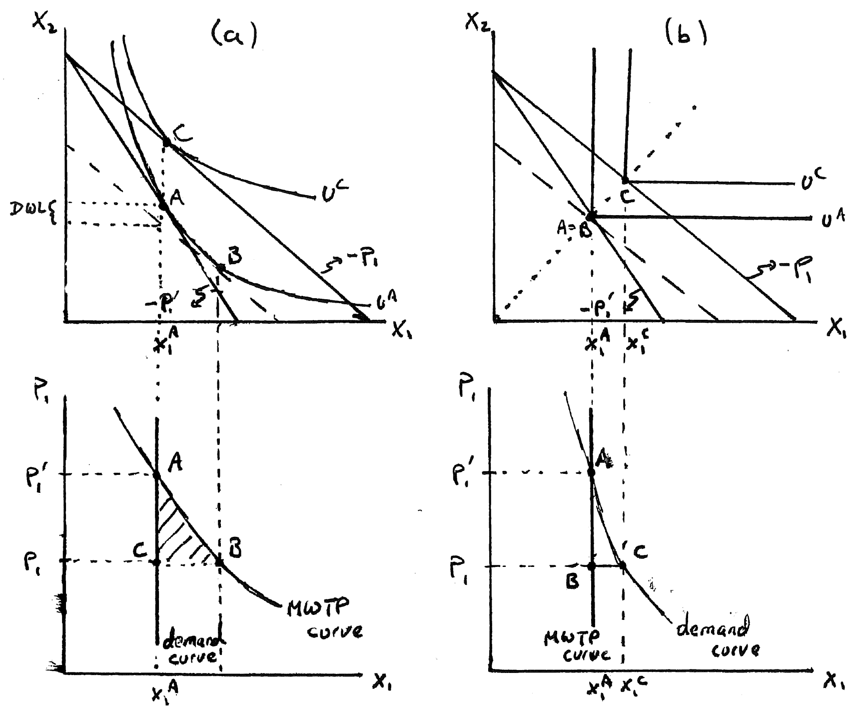
- Each end-of-chapter exercise begins on a new page. This is to facilitate maximum flexibility for instructors who may wish to share answers to some but not all exercises with their students.
- If you are assigning only the A-parts of exercises in *Microeconomics: An Intuitive Approach with Calculus*, you may wish to instead use the solution set created for the companion book *Microeconomics: An Intuitive Approach*.
- *Solutions to Within-Chapter Exercises are provided in the student* Study Guide.

10.1 *Consider a good x_1 in a model where a consumer chooses between x_1 and a composite good x_2.*

A: *Explain why the following either cannot happen or, if you think it can happen, how:*

(a) *Own price demand for a good is perfectly vertical but taxing the good produces a dead weight loss.*

Answer: Panel (a) of Graph 10.1 illustrates two different prices — p_1 and p_1' — for x_1, with $p_1' > p_1$. (Thus, p_1' would be the tax-inclusive price.) The optimal consumption of x_1 is x_1^A when price is p_1' as well as when price is p_1 — thus, price has no impact on the demand for x_1. In the lower graph of panel (a), this translates into a perfectly vertical demand curve. However, there is still a substitution effect that gives rise to the deadweight loss. This deadweight loss can be either seen as the vertical distance labeled DWL in the upper graph — or as the shaded triangle in the lower graph. Thus, we have an example where the demand is vertical but there is still a deadweight loss from taxation.



Graph 10.1: Demand and Deadweight Loss from Taxation

(b) *Own price demand is downward sloping (not vertical) and there is no deadweight loss from taxing the good.*

Answer: In panel (b) of Graph 10.1, we again illustrate two different prices for x_1 but this time model the tastes as those for perfect complements. The demand for x_1 is higher at the lower price — where x_1^C is chosen — than at the higher price — where x_1^A is chosen. This translates into a downward sloping demand curve in the lower graph. But now there is no substitution effect — which leads to a vertical $MWTP$ curve and the disappearance of the deadweight loss triangle. Put differently, the tax revenues raised through this tax are exactly the same that we could raise through a lump sum tax that makes the consumer equally well off. We therefore have an example of a downward sloping demand curve with no deadweight loss from taxation.

B: *Now suppose that the consumer's tastes can be summarized by the CES utility function* $u(x_1,x_2) = (0.5x_1^{-\rho}+0.5x_2^{-\rho})^{-1/\rho}$.

(a) *Are there values for* ρ *that would result in the scenario described in A(a)?*

Answer: To solve for the demand for x_1, we have to solve

$$\max_{x_1,x_2} (0.5x_1^{-\rho}+0.5x_2^{-\rho})^{-1/\rho} \text{ subject to } I = p_1x_1+x_2. \tag{10.1}$$

This gives us

$$x_1 = \frac{I}{p_1+p_1^{1/(\rho+1)}} \text{ and } x_2 = \frac{p_1^{1/(\rho+1)I}}{p_1+p_1^{1/(\rho+1)}} = \frac{I}{p_1^{\rho/(\rho+1)}+1}. \tag{10.2}$$

Regardless of ρ, x_1 is an inverse function of p_1, implying a downward sloping demand curve. Since there is no way for CES demands to be perfectly vertical, there are no values for ρ that would result in the scenario in A(a).

(b) *Are there values for* ρ *that would result in the scenario described in A(b)?*

Answer: In order for there to be no deadweight loss, there cannot be a substitution effect. The only way there is no substitution effect is if the tastes are for perfect complements — i.e. the elasticity of substitution is 0. And CES utility functions have elasticity of substitution of zero only when $\rho=\infty$. Thus, in order for the scenario to work, $\rho=\infty$. (In that case, $x_1 = I/(p_1+1)$ — which implies the demand for x_1 falls as price increases; i.e. demand for x_1 is downward sloping.)

(c) *Would either of the scenarios work with tastes that are quasilinear in* x_1*?*

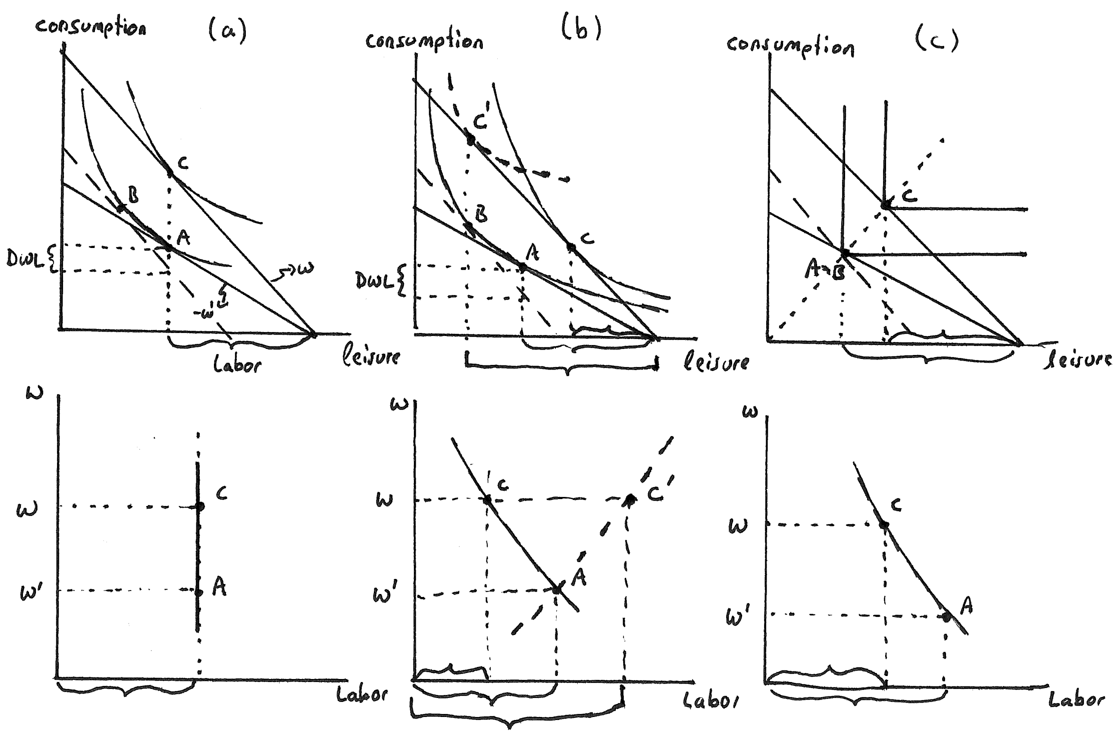
Answer: No. The scenario in A(a) would not work because quasilinearity in x_1 implies no income effect — which would put C in panel (a) of Graph 10.1 directly above B (rather than above A). The substitution effect then implies that A lies to the left of C — which implies a downward sloping (and not a vertical) demand curve. The scenario in A(b) won't work because quasilinear tastes have substitution effects — and thus give rise to deadweight losses from taxation.

10.2 *Suppose that both consumption and leisure are always normal goods. Keep in mind the underlying cause for deadweight losses from wage-distoring taxation as you answer the questions below.*

A: *Explain why the following either cannot happen or, if you think it can happen, how:*

(a) *Labor supply is perfectly vertical but there is a significant dead weight loss from taxing wages.*

Answer: In panel (a) of Graph 10.2, we illustrate a high wage w and a low (after-tax) wage w'. Tastes are drawn such that the optimal amount of leisure is the same under both wages — which translates in the lower graph to a vertical labor supply curve. However, there is a substitution effect that gives rise to a dead weight loss that is labeled DWL on the vertical axis in the upper graph. (In Chapter 19, we will show how to illustrate this in terms of the lower graph.)



Graph 10.2: Labor Supply and Deadweight Loss from Taxation

(b) *Labor supply is perfectly vertical and there is no dead weight loss from taxing wages.*

Answer: The only way this could happen is if the indifference curves in panel (a) are "tangent" at the same points A and C but the indifference curve that is tangent at A has a sufficient kink to cause $B = A$ — i.e. to eliminate a substitution effect. In principle, this is possible if the kink is not at a right angle (as in the case of perfect complements) — it has to be less sharp than that, but sufficiently sharp to eliminate the substitution effect.

(c) *Labor supply is downward sloping and there is a deadweight loss from taxation of wages.*

Answer: This is illustrated in panel (b) of Graph 10.2. Here, the consumer demands more leisure at the higher wage at bundle C than at the lower wage at bundle A — which results in a downward slope of the labor supply curve. Since there is a substitution effect similar to the one in panel (a), we still have the deadweight loss from the wage tax.

(d) *Labor supply is upward sloping and there is a deadweight loss from taxing wages.*

Answer: This can also happen — all we would have to change in panel (b) of the graph is to change the optimal indifference curve at the high wage to the dashed curve that is tangent at C'. This would result in less leisure demand at the higher wage than at the lower wage — which results in an upward sloping labor supply curve. Since we have not changed the optimal indifference curve at the lower wage, the substitution effect remains, as does the deadweight loss.

(e) *Labor supply is downward sloping and there is no deadweight loss from taxing wages.*

Answer: This is illustrated in panel (c) of Graph 10.2. The substitution effect that causes the deadweight loss is eliminated by the assumption that tastes over consumption and leisure are now perfect complements. At the higher wage, the worker takes more leisure and thus works less — giving rise to a downward sloping labor supply curve.

(f) *Labor supply is upward sloping and there is no deadweight loss from taxing wages.*

Answer: In principle, this is possible — but only if leisure is an inferior good (which the question presumes it is not). We would still need a kink at the optimal after-tax point A — a kink that is sufficiently large to eliminate the substitution effect. Making this kink less sharp than it is in panel (c) would give us some room on the pre-tax budget to locate C to the left of $A = B$. That would give us an upward slope of the labor supply curve while preserving the absence of a substitution effect that causes the deadweight loss. But it would imply leisure is an inferior good whose consumption declines as the budget line shifts out.

B: *Now suppose that tastes can be summarized by the CES utility $u(c,\ell) = (0.5c^{-\rho} + 0.5\ell^{-\rho})^{-1/\rho}$, where c is consumption and ℓ is leisure.*

(a) *Are there values for ρ that would result in the scenario in A(a)*

Answer: For the labor supply to be perfectly vertical, it would have to be the case that the optimal amount of leisure does not depend on the wage rate. To solve for the optimal amount of leisure, we solve the problem

$$\max_{c,\ell} = (0.5c^{-\rho} + 0.5\ell^{-\rho})^{-1/\rho} \text{ subject to } w(L-\ell) = c \tag{10.3}$$

where L is the leisure endowment. This gives us

$$\ell = \frac{L}{1 + w^{-\rho/(\rho+1)}}. \tag{10.4}$$

When ρ is set to zero, this reduces to $\ell = L/2$ — which implies leisure demand becomes independent of the wage rate and tastes are Cobb-Douglas. This further implies that labor supply becomes independent of the wage rate — i.e. the labor supply curve is vertical. The reason for this is that the substitution effect is exactly offset by the wealth effect, but the substitution effect still gives rise to a deadweight loss from wage taxation. Thus, the scenario in A(a) arises when $\rho = 0$.

(b) *Are there values for ρ that would result in the scenario in A(b)?*

The scenario in A(b), on the other hand, cannot arise — because the only way we get a perfectly vertical labor supply curve under these CES tastes is if $\rho = 0$ — but this implies the elasticity of substitution is 1 which introduces the substitution effect that causes the deadweight loss.

(c) *Are there values for ρ that would result in the scenario in A(c)?*

Answer: In order for labor supply to slope down, it has to be the case that leisure increases as the wage increases. We calculated the optimal leisure amount in equation (10.4). To see how this responds to an increase in the wage, we can take the derivative with respect to wage to get

$$\frac{\partial \ell}{\partial w} = \frac{\rho}{\rho+1}\left(\frac{L}{\left(1 + w^{-\rho/(\rho+1)}\right)^2 w^{(2\rho+1)/(\rho+1)}}\right). \tag{10.5}$$

In order for leisure to increase as the wage increases, this derivative has to be positive. The term in parentheses is positive (since $L > 0$ and $w > 0$) — which implies that the whole derivative is positive if and only if $\rho > 0$. Thus, as long as $\rho > 0$, the labor supply curve is downward sloping. And, as long as $\rho \neq \infty$ (which would put the elasticity of substitution at zero and eliminate substitution effects), there will be a deadweight loss. The scenario in A(c) therefore arises for $0 < \rho < \infty$.

(d) *Are there values for ρ that would result in the scenario in A(d)?*

Answer: The only way that labor supply is upward sloping is if leisure falls with an increase in the wage. We can therefore again look at the derivative of optimal leisure demand, which we calculated in equation (10.5). In order for leisure demand to fall as wage increases, this derivative has to be negative. And, since the term in parenthesis is positive, this happens if and only if $-1 < \rho < 0$. We furthermore know that the elasticity of substitution is positive so long as $\rho \neq \infty$ — which means it is positive when ρ falls between 0 and -1. Thus, the scenario in A(d) arises for $-1 < \rho < 0$.

(e) *Are there values for ρ that would result in the scenario in A(e) and A(f)?*

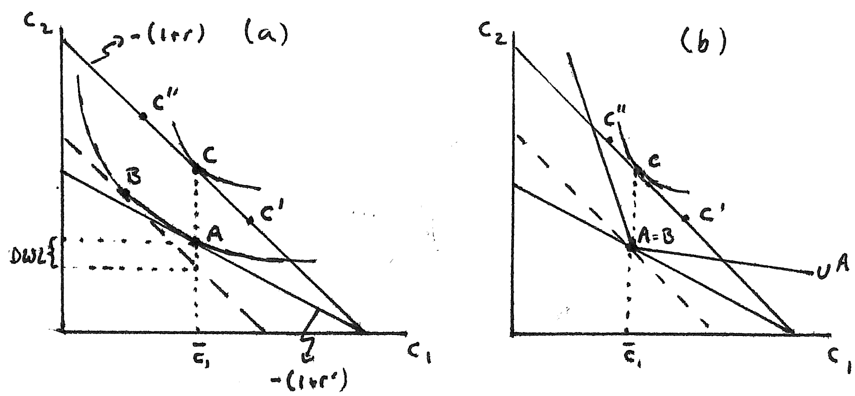
Answer: In order for there to be no deadweight loss, it has to be that there are no substitution effects. The only way to eliminate the substitution effect under the CES utility in this problem is for $\rho = \infty$. And, as ρ approaches ∞, the leisure demand from equation (10.4) converges to $\ell = L/(1 + w^{-1})$ which has a positive derivative with respect to w. Thus, when $\rho = \infty$, leisure demand increases with increases in the wage — which implies labor supply decreases with an increase in the wage; i.e. labor supply slopes down. Thus the scenario in A(e) arises when $\rho = \infty$ but the scenario in A(f) cannot arise.

10.3 *Suppose that consumption takes place this period and next period, and consumption is always a normal good. Suppose further that income now is positive and income next period is zero.*

A: *Explain why the following either cannot happen or, if you think it can happen, how:*

(a) *Savings behavior is immune to changes in the interest rate, but taxing interest income causes a dead weight loss.*

Answer: In panel (a) of Graph 10.3, we graph two budgets, one with a high interest rate r and one with a low (after-tax) interest rate r'. If tastes are such that the optimal level of consumption in period 1 (i.e. c_1) is unaffected by the interest rate, the optimal bundles A and C lie at the same level $\overline{c}_1$ (as drawn in panel (a)). Thus, a tax on interest income (which lowers the effective interest rate) causes no change in savings. At the same time, there is still a substitution effect that gives rise to a dead weight loss which is indicated as DWL on the vertical axis.



Graph 10.3: Savings Behavior and Deadweight Loss from Interest Taxation

(b) *Savings behavior is immune to changes in the interest rate, and taxing interest income causes no dead weight loss.*

Answer: This is also in principle possible — but the indifference curve that is "tangent" at A must now have a kink sufficiently large to eliminate the substitution effect with $B = A$. This is illustrated in panel (b) of the graph. The kink cannot be as sharp as a kink for perfect complements but would need to be sufficiently sharp to eliminate the substitution effect.

(c) *Savings decreases with increases in the interest rate and there is a deadweight loss from taxation of interest.*

Answer: Savings decreases with an increase in the interest rate if A lies to the left of C — i.e. if consumption now (c_1) increases as the interest rate increases. This can be accomplished in panel (a) of Graph 10.3 by simply re-drawing the indifference curve on the higher interest rate r budget to be tangent at C'. It does not require us to change anything about the indifference curve that is tangent at A — which implies we will continue to have the same substitution effect that gives rise to the deadweight loss.

(d) *Savings increases with increases in the interest rate and there is a deadweight loss from taxation of interest.*

Answer: Savings increases with an increase in the interest rate if A lies to the right of C — i.e. if consumption now (c_1) decreases as the interest rate increases. This can be accomplished in panel (a) of Graph 10.3 by simply re-drawing the indifference curve on the higher interest rate r budget to be tangent at C''. It does not require us to change anything about the indifference curve that is tangent at A — which implies we will continue to have the same substitution effect that gives rise to the deadweight loss.

(e) *Savings decreases with an increase in the interest rate and there is no deadweight loss.*

Answer: We now have to change the optimal bundle on the high interest rate r budget in panel (b) of Graph 10.3. If savings decreases with an increase in the interest rate, A has to lie to the left of C — so that consumption now (c_1) increases as the interest rate goes up. Thus, if we change the higher indifference curve to be tangent at C', we have the desired savings response. Since we did not change the lower (kinked) indifference curve, there is still no substitution effect — and thus no deadweight loss from the tax on interest. This would also work for tastes that treat c_1 and c_2 as perfect complements.

(f) *Savings increases with an increase in the interest rate and there is no deadweight loss.*

Answer: We again have to change the optimal bundle on the high interest rate r budget in panel (b) of Graph 10.3. If savings increases with an increase in the interest rate, A has to lie to the right of C — so that consumption now (c_1) decreases as the interest rate goes up. Thus, if we change the higher indifference curve to be tangent at C'', we have the desired savings response. Since we did not change the lower (kinked) indifference curve, there is still no substitution effect — and thus no deadweight loss from the tax on interest. (This would not work for perfect complements, however — because the sharp kink required for perfect complements would not permit A to lie to the right of C.) So the statement is in fact possible — except for the fact that the problem assumed at the outset that consumption now and in the future is normal. This is not the case at C'' — which implies the statement can be true only if consumption now is an inferior good.

B: *Now suppose that tastes can be summarized by the CES utility function $u(c_1,c_2) = (0.5c_1^{-\rho} + 0.5c_2^{-\rho})^{-1/\rho}$, where c_1 is consumption in the first period and c_2 is consumption in the second period.*

(a) *Are there values for ρ that would result in the scenario in A(a) and A(b)?*

Answer: In order for savings behavior to be immune to changes in the interest rate, it must be that consumption now (c_1) is immune to changes in the interest rate. We can calculate the optimal level of c_1 by solving the problem

$$\max_{c_1,c_2}(0.5c_1^{-\rho} + 0.5c_2^{-\rho})^{-1/\rho} \text{ subject to } (1+r)(I-c_1) = c_2 \tag{10.6}$$

where I is current income. This gives us

$$c_1 = \frac{I}{1+(1+r)^{-\rho/(\rho+1)}}. \tag{10.7}$$

When ρ is set to zero — i.e. when tastes become Cobb-Douglas — we then get that $c_1 = I/2$ and current consumption is therefore immune to the interest rate. This then implies that savings is immune to the interest rate if and only if $\rho = 0$. Since the elasticity of substitution is positive when $\rho = 0$, this implies there are substitution effects that result in a dead weight loss from a tax on interest. The scenario in A(a) is therefore possible, but the scenario in A(b) is not possible with these CES tastes.

(b) *Are there values for ρ that would result in the scenario in A(c)?*

Answer: In order for savings to decrease with an increase in the interest rate, it must be the case that consumption now (c_1) increases with an increase in the interest rate. Equation (10.7) describes how c_1 changes with the interest rate — and by taking the derivative with respect to r, we can tell whether c_1 increases or decreases as r increases. This derivative is

$$\frac{\partial c_1}{\partial r} = \frac{\rho}{(\rho+1)}\left(\frac{I}{\left(1+(1+r)^{-\rho/(\rho+1)}\right)^2 (1+r)^{(2\rho+1)/(\rho+1)}}\right). \tag{10.8}$$

Consumption now will increase (and savings will decrease) with the interest rate if this derivative is positive. And, since the term in parenthesis is positive (given that I and r are positive), the derivative is positive if and only if $\rho > 0$. Furthermore, so long as $\rho \neq \infty$, the elasticity of substitution is positive — which implies the existence of substitution effects that create dead weight losses from taxing interest income. Therefore, the scenario in A(c) arises whenever $0 < \rho < \infty$.

(c) *Are there values for ρ that would result in the scenario in A(d)?*

Answer: Savings increases with an increase in the interest rate if and only if consumption now (c_1) falls with an increase in the interest rate. This occurs only if the derivative from equation (10.8) is negative, which in turn occurs only so long as $-1 < \rho < 0$. Since the elasticity of substitution is positive for all such values of ρ, this further implies the emergence of dead weight losses from taxation of interest income. Thus, the scenario in A(d) emerges whenever $-1 < \rho < 0$.

(d) *Are there values for ρ that would result in the scenario in A(e) or A(f)?*

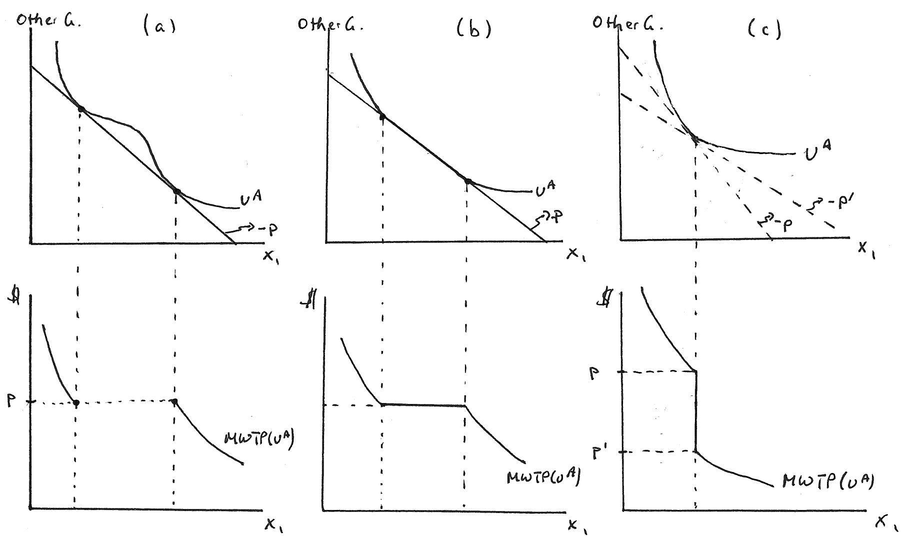
Answer: The only way that there are no substitution effects that give rise to dead weight losses is if the elasticity of substitution is zero — which only happens if $\rho = \infty$. As ρ approaches infinity, the optimal level of current consumption approaches to

$$c_1 = \frac{I}{1+(1+r)^{-1}}. \tag{10.9}$$

The derivative of c_1 with respect to r is positive. Thus, when $\rho = \infty$ (which is necessary for there to be no deadweight loss from taxation of interest), c_1 will increase (and savings will therefore decrease) with an increase in the interest rate. Scenario A(e) therefore arises when $\rho = \infty$ and scenario A(f) is not possible under this CES specification of tastes.

10.4 *Suppose that your tastes do not satisfy the convexity assumption. In particular, suppose the indifference curve corresponding to utility level u^A has a shape like the indifference curves depicted in Graph 6.8, with good x_1 on the horizontal axis and "other consumption" on the vertical. Illustrate what the $MWTP$ (or compensated demand) curve corresponding to utility level u^A would look like. How would your answer change if the indifference curve instead satisfied convexity but contained a "flat" portion along which the MRS is constant? What if the indifference curve were strictly convex but contained a kink?*

Answer: The answers are depicted in panels (a), (b) and (c) of Graph 10.4. In the case of the non-convexity, the $MWTP$ curve would contain a discontinuity at the price level at which the budget is tangent in two places. In the case of the convex indifference curve with a flat portion, the $MWTP$ curve would contain a horizontal element at the price level at which the budget is tangent to the flat portion of the indifference curve. Finally, in the case of the kinked indifference curve, we would get a strictly vertical element to the $MWTP$ curve over the range of prices at which the kink point in the indifference curve is the optimum.



Graph 10.4: Odd-Shaped Indifference Curves and their $MWTP$ Curves

10.5 Everyday Application: *Teacher Pay and Pro-Basketball Salaries: Do we have our priorities in order? We trust our school aged children to be taught by dedicated teachers in our schools, but we pay those teachers only about $50,000 per year. At the same time, we watch pro-basketball games as entertainment — and we pay some of the players 400 times as much!*

A: *When confronted with these facts, many people throw their hands up in the air and conclude we are just hopelessly messed up as a society — that we place more value on our entertainment than on the future of our children.*

(a) *Suppose we treat our society as a single individual. What is our marginal willingness to pay for a teacher? What is our marginal willingness to pay for a star basketball player?*

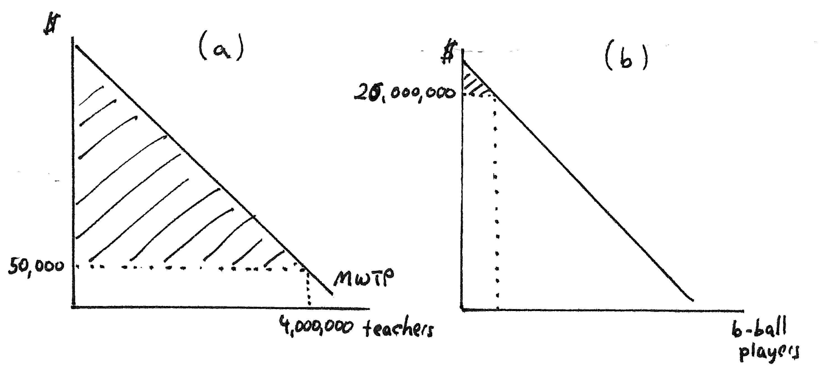
Answer: Our marginal willingness to pay for a teacher is $50,000 per year, and our marginal willingness to pay for a star basketball player is $20,000,000 per year.

(b) *There are about 4 million teachers that work in primary and secondary schools in the United States. What is the smallest dollar figure that could represent our total willingness to pay for teachers?*

Answer: By paying $50,000 per year to 4 million teachers, we are paying a total of $200 billion (i.e. $200,000,000,000). Given we are choosing to pay this amount, that is the least that our total willingness to pay for teachers could be.

(c) *Do you think our* actual *total willingness to pay for teachers is likely to be much greater than that minimum figure? Why or why not?*

Answer: $200 billion would be our total willingness to pay if our marginal willingness to pay curve were perfectly horizontal at $50,000; i.e. if our marginal willingness to pay for the first teacher were the same as our marginal willingness to pay for each additional teacher. But that is almost certainly not the case — rather, our marginal willingness to pay for the first teacher is likely to be very high — with decreasing marginal willingness to pay for each additional teacher. Only with the 4 millionth teacher does our marginal willingness to pay reach $50,000. Since total willingness to pay is the entire area under the marginal willingness to pay curve (and not just the amount we actually pay), our total willingness to pay for our teachers is likely much higher than $200 billion per year. In panel (a) of Graph 10.5, the *additional* amount we are willing to pay for teachers (above the $200 billion we actually pay) is depicted as the large shaded area.



Graph 10.5: Marginal and Total Willingness to Pay

(d) *For purposes of this problem, assume there are 10 star basketball players at any given time. What is the least our total willingness to pay for star basketball players could be?*

Answer: Since we are paying $20 million to each, our total willingness to pay for the 10 star basketball players is at least $200 million.

(e) *Is our actual total willingness to pay for basketball players likely to be much higher than this minimum?*

Answer: Our actual total willingness to pay is somewhat higher than that since our $MWTP$ curve for basketball players is downward sloping (just as it is for teachers). But since there are only 10 such players, the $MWTP$ for the 10th player is probably not nearly as much lower than the $MWTP$ of the first player as the $MWTP$ of the 4 millionth teacher is lower than the $MWTP$ for the first teacher. Thus, $200 million is a closer approximation to our total willingness to pay for basketball players than $200 billion is to our total willingness to pay for teachers. This can be seen graphically by comparing the shaded areas in panels (a) and (b) of Graph 10.5 — where the shaded area in each graph depicts the amount we are willing to pay *in addition* to what we had to pay.

(f) *Do the facts cited at the beginning of this question really warrant the conclusion that we place more value on our entertainment than on the future of our children?*

Answer: We concluded that our total willingness to pay for star basketball players is approximately $200 million — and our total willingness to pay for teachers is substantially larger than $200 billion. While it is true that we are willing to pay more for a star basketball player *on the margin*, the total value we place on the services from the basketball players is therefore substantially less than what we are willing to pay for the services of teachers. Put differently, not only do we actually pay more for all the teachers than we do for basketball players, but the consumer surplus we get from teachers is many orders of magnitude larger than what we get from basketball players.

(g) *Adam Smith puzzled over an analogous dilemma: He observed that people were willing to pay exorbitant amounts for diamonds but virtually nothing for water. With water essential for sustaining life and diamonds just an item that appeals to our vanity, how could we value diamonds so much more than water? This became known as the* diamond-water paradox. *Can you explain the paradox to Smith?*

Answer: Exactly the same reasoning holds as does for teachers and star basketball players. The two panels of Graph 10.5 could simply be re-labeled, with the first representing water and the second diamonds. The only difference is that the example is even more extreme — our $MWTP$ for the last gallon of water is close to zero but we consume a lot of water — causing our total willingness to pay to be represented by a very large triangle under the $MWTP$ curve. But most of us consume very few diamonds. On the margin, we value a diamond more than a gallon of water, but we place much more value on our water consumption than on our diamond consumption.

B: *Suppose our marginal willingness to pay for teachers (x_1) is given by $MWTP = A - \alpha x_1$ and our marginal willingness to pay for star basketball players (x_2) is given by $MWTP = B - \beta x_2$.*

(a) *Given the facts cited above, what is the lowest that A and B could be?*

Answer: If $MWTP$ curves were perfectly horizontal, then $A = 50,000$ and $B = 20,000,000$.

(b) *If A and B were as you just concluded, what would α and β be?*

Answer: Since $MWTP$ curves can't slope up, it would have to be that they are perfectly flat — with slope of zero. Thus, if $A = 50,000$ and $B = 20,000,000$, then $\alpha = \beta = 0$.

(c) *What would be our marginal and total willingness to pay for teachers and star basketball players?*

Answer: Under these parameters, our total willingness to pay for teachers would be $200 billion and our total willingness to pay for star basketball players would be $200 million.

(d) *Suppose $A = B = \$100$ million. Can you tell what α and β must be?*

Answer: In order for the marginal willingness to pay of the 4 millionth teacher to be $50,000, the slope term α would have to be $99,950,000/4,000,000 = 24.9875$. In order for the marginal willingness to pay for the 10th basketball player to be $20,000,000, the slope term β must be $80,000,000/10 = 8,000,000$.

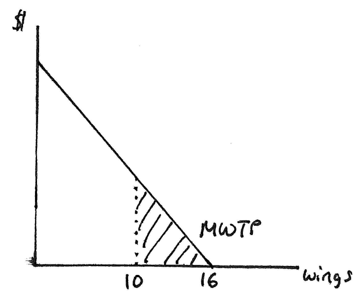
(e) *Using the parameter values you just derived (with $A = B = \$100$ million), what is our total willingness to pay for teachers and star basketball players?*

Answer: The triangle in panel (a) of the graph would then be $(99,950,000(4,000,000))/2 = 199,900,000,000,000 = \$199,900$ billion. To this, we add what we actually pay for teachers — $200 billion — and we get a total willingness to pay for teachers of $200,100 billion or approximately $200 *trillion*! The analogous triangle in panel (b) is $80,000,000(10)/2 = 400,000,000$. Adding what we actually pay for star basketball players, we get a total willingness to pay of $600,000,000 or $600 million. (That would mean we value our teachers over 300,000 times as much as our star basketball players.)

10.6 Everyday Application: *Ordering Appetizers*: *I recently went out to dinner with my brother and my family. We decided we wanted wings for an appetizer and had a choice of getting 10 wings for $4.95 or 20 wings for $7.95. I thought we should get 10; my brother thought we should get 20 and prevailed.*

A: *At the end of the meal, we noticed that there were 4 wings left. My brother then commented: "I guess I am vindicated — it really was the right decision to order 20 rather than 10 wings."*

(a) *Is this a correct assessment; i.e. is the evidence of 4 wings at the end of the meal sufficient to conclude that my brother was right?*



Graph 10.6: Wings Left Over

Answer: It is not necessarily a correct assessment. What we know from the evidence that 4 wings were left at the end is that the $MWTP$ curve must cross the horizontal intercept at approximately 16 wings — i.e. the marginal value of additional wings is zero since we could have eaten them without paying anything more but chose not to. (Whether we left the wings or not, our total bill was the same since restaurants don't refund us for food we leave behind.) The second 10 wings cost us $3 — so in order for my brother to be correct, it must be that the total value we received from the 6 additional wings we ate was at least $3. That total value is represented as the shaded triangle in Graph 10.6 — and without knowing the height of that triangle, we can't be sure what the area of the triangle is. The height of the triangle is the marginal willingness to pay for the 10th wing. (Even if we did know that, we still would not know the shape of the $MWTP$ curve in between 10 and 16 — so even then we could not really be sure unless we assumed the linear shape that is depicted in the graph).

(b) *What if no wings were left at the end of the meal?*

Answer: Even if no wings had been left, I cannot interpret this as evidence that we in fact got more than $3 of value from the additional 10 wings. Before, we at least knew that the marginal value of the 17th wing was zero — now we do not know the marginal value of any of the wings. We might, for instance, have eaten them because we valued them each at one cent — which would mean we only got 10 cents of enjoyment from them. Or we might have valued them each at one dollar, in which case we would have gotten $10 worth of enjoyment.

(c) *What if 10 wings were left?*

Answer: Now we could be sure that my brother was wrong — because we would know that the marginal value of the 11th wing (and each one after that) was zero.

(d) *In order for us to leave wings on the table, which of our usual assumptions about tastes must be violated?*

Answer: Monotonicity — or at least the strict version of monotonicity. We could have eaten additional wings for free but chose not to. More is not better — just at least as good.

B: *Suppose that our $MWTP$ for wings (x) can be approximated by the function $MWTP = A - \alpha x$.*

(a) *Given that 4 wings were left at the end of the meal, what must be the relationship between α and A?*

Answer: Since the $MWTP$ goes to zero at the 16th wing, it must be that $A - 16\alpha = 0$ or $\alpha = A/16$.

(b) *Suppose $A = 8/3$. Was my brother right to want to order 20 instead of 10 wings?*

Answer: In order to answer this, we need to calculate the size of the shaded triangle in Graph 10.6 that depicts the total willingness to pay for 6 additional wings (beyond the first 10). When $A = 8/3$, it must be (given our answer to (a) above) that $\alpha = (8/3)/16 = 1/6$. This implies that the marginal willingness to pay of the 10th wing is $8/3-(1/6)10 = 1$. The triangle therefore has height of 1 — which means its area is $(1)(6)(1/2) = 3$. Thus, the value of the additional 6 wings is \$3 — exactly the amount we paid for them. So both my brother and I were right — our consumer surplus did not increase or decrease from the decision to order 20 instead of 10 wings.

(c) *Suppose instead that $A = 2$. Does your answer change? What if $A = 4$?*

Answer: If $A < 8/3$, the relevant triangle will shrink below \$3, and if $A > 8/3$, it will increase above \$3. Thus, in the former case my brother was wrong, in the latter he was right. (To be more precise, if $A = 2$, then we know $\alpha = 2/16 = 1/8$. Thus, the marginal value of the 10th wing is $2-(1/8)10 = 3/4$. This implies that the area of the triangle is $(3/4)(6)(1/2) = 9/4$ — i.e. the value of the additional wings is only \$2.25. If $A = 4$, on the other hand, $\alpha = 4/16 = 1/4$. This implies that the marginal value of the 10th wing is $4 - (1/4)10 = 3/2$ — which in turn implies that the area of the relevant triangle is $(3/2)(6)(1/2) = 9/2$. In that case, the value of the additional wings is therefore \$4.50.)

(d) *If our tastes were Cobb-Douglas, could it ever be the case that we leave wings on the table?*

Answer: Under Cobb-Douglas tastes, the MRS never goes to zero no matter how much of a good one consumes. Thus, the $MWTP$ never falls to zero — which implies that, when wings are sitting on the table for anyone at the table to consume, they will always be consumed. So no wings should be left on the table under Cobb-Douglas tastes. (Caveat: One could extend this answer a bit further and make Cobb-Douglas tastes consistent with some wings being left on the table if one introduced an additional constraint. Suppose, for instance, that people at the table observe a per-meal calorie constraint — in that case, the constraint could cause some wings to be left.)

10.7 Everyday Application: *To Trade or Not to Trade Pizza Coupons: Exploring the Difference between Willingness to Pay and Willingness to Accept*: *Suppose you and I are identical in every way — same exogenous income, same tastes over pizza and "other goods". The only difference between us is that I have a coupon that allows the owner of the coupon to buy as much pizza as he/she wants at 50% off.*

A: *Now suppose you approach me to see if there was any way we could make a deal under which I would sell you my coupon. Below you will explore under what conditions such a deal is possible.*

(a) *On a graph with pizza on the horizontal axis and "other goods" on the vertical, illustrate (as a vertical distance) the most you are willing to pay me for my coupon. Call this amount P.*

Answer: This is illustrated in the top graph in panel (a) of Graph 10.7. The steeper budget is your (without-coupon) budget, while the shallower budget is my (with coupon) budget that has half the price for pizza. Without a coupon, your optimal bundle is A — and you reach utility level u^A. Getting the coupon means getting the shallower slope for yourself — but buying it means that you are giving up income. In deciding the most you are willing to pay for a coupon, you therefore have to decide how much you are willing to shift the shallower budget in — and the most you are willing to shift it is an amount that will get you the same utility you can get without the coupon. Thus, the most you are willing to shift the coupon budget in is the amount that creates the tangency at B with u^A and the dashed budget (that has the same slope as the with-coupon budget. The vertical distance between my (with-coupon) budget and the dashed budget is the most you are willing to pay me for the coupon — a distance that can be measured anywhere between the two parallel lines. It is indicated as distance P in the graph.

(b) *On a separate but similar graph, illustrate (as a vertical distance) the least I would be willing to accept in cash to give up my coupon. Call this amount R.*

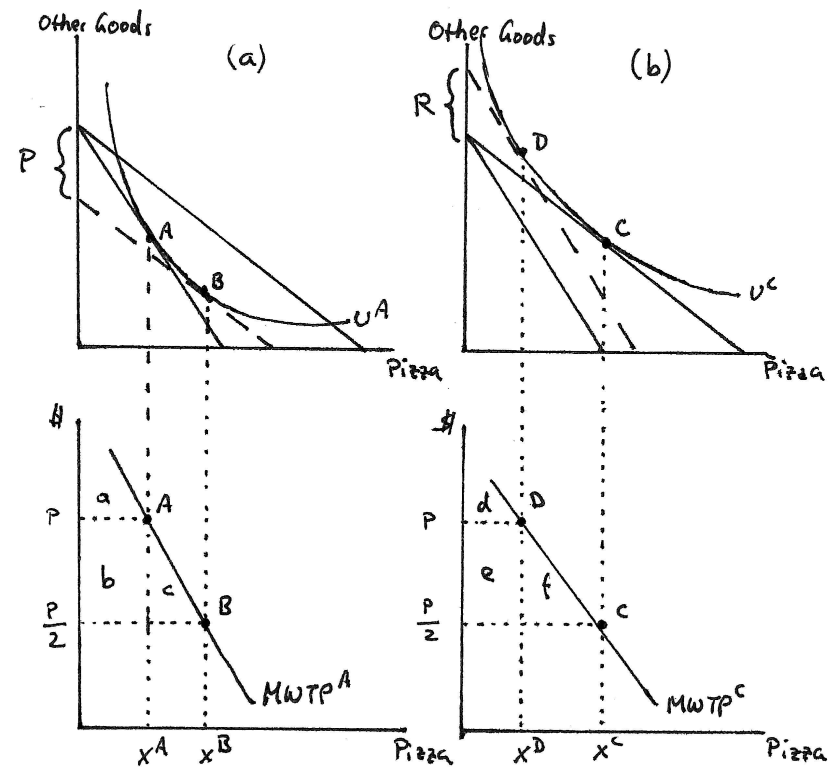
Answer: The top graph in panel (b) illustrates this. By just using the coupon, I will optimize at bundle C — and will reach the indifference curve u^C. Since I can get to that utility level without selling you the coupon, I will not be willing to make a deal that gets me less utility. By selling you the coupon, I will face a steeper budget but I will have received cash from you — i.e. I will face a budget with the steeper (no-coupon) slope but further out than your initial no-coupon budget. The least I am willing to accept for the coupon is an amount that will make me just as well off as I am with the coupon. I can determine that amount by taking your initial budget and shifting it out until it is tangent to my original optimal indifference curve u^C — which would land me at bundle D in the graph. The amount you have to give me in cash to get me to D is the vertical distance between your original (no-coupon) budget and the dashed budget in the graph. That distance, labeled R, is the least I am willing to accept for the coupon.

(c) *Below each of the graphs you have drawn in (a) and (b), illustrate the same amounts P and R (as areas) along the appropriate marginal willingness to pay curves.*

Answer: In the lower graph of panel (a), the marginal willingness to pay curve is derived from the indifference curve u^A. In the absence of a coupon, you will buy x^A in pizza at the no-coupon price p. This gives you consumer surplus of a. If you end up buying the coupon from me at the maximum price you are willing to pay (P), you will buy x^B in pizza and attain consumer surplus of $a+b+c$. Since A and B lie on the same indifference curve, you are equally happy attaining consumer surplus a without having paid me anything for the coupon or consumer surplus $a+b+c$ after paying me P for the coupon. In order for you to be truly indifferent between these two options, it must therefore be the case that $P=b+c$.

In the lower graph of panel (b), the marginal willingness to pay curve is derived from my indifference curve u^C. In the absence of selling my coupon, I buy x^C pizza — and get consumer surplus of $d+e+f$. If I sell the coupon at the lowest price R that I am willing to accept, I end up buying x^D pizza and get consumer surplus of just d. Since I am equally happy in both cases, it must be that I am indifferent between getting consumer surplus of $d+e+f$ without receiving any cash from you or getting consumer surplus d and getting R in cash. Thus, $R=e+f$.

(d) *Is P larger or smaller than R? What does your answer depend on? (*Hint*: By overlaying your lower graphs that illustrate P and R as areas along marginal willingness to pay curves, you should be able to tell whether one is bigger than the other or whether they are the same size depending on what kind of good pizza is.)*



Graph 10.7: Trading Pizza Coupons

Answer: Asking if P is larger or smaller than R is then the same as asking if $b+c$ is larger or smaller than $e+f$. Suppose first that pizza is a quasilinear good for us. Then if I transferred the indifference curve u^C onto the top graph of panel (a), the tangency C would lie vertically above B — because a move from the dashed budget in panel (a) to my original (with-coupon) budget is simply an increase in income without a price change. Such an increase in income would not change consumption of pizza when pizza is quasilinear. Similarly, if we transferred the indifference curve u^A onto panel (b), the tangency at A would lie vertically below D. This is because the move from the dashed budget in panel (b) to your (no-coupon) budget is a simple decrease in income without a price change — which causes to change in consumption of pizza when pizza is quasilinear. This implies that, in the lower graphs, A lies at exactly the same place as D and B lies at exactly the same place as C. Put differently, when pizza is a quasilinear good, $MWTP^A$ lies exactly on top of $MWTP^C$ — which implies $b+c=e+f$ or $P=R$. The most you are willing to pay me for the coupon is then exactly equal to the least I am willing to accept.

Now suppose that pizza is an inferior good. Then the same logic we just went through implies that D will lie to the left of A and C will lie to the left of B — which implies that $b+c>e+f$ or $P>R$. Thus, when pizza is an inferior good, the most you are willing to pay is greater than the least I am willing to accept for the coupon. If, on the other hand, pizza is a normal good, then the same logic implies that D lies to the right of A and C lies to the right of B — which further implies that $b+c<e+f$ of $P<R$. Thus, when pizza is a normal good for us, then the most you are willing to pay is less than the least I am willing to accept for the coupon.

(e) True or False*: You and I will be able to make a deal so long as pizza is not a normal good. Explain your answer intuitively.*

Answer: This is true. We have just concluded that when pizza is an inferior good, you are willing to pay me more than the least I am willing to accept — so there is room for us to make a deal and both become better off. When pizza is quasilinear (i.e. borderline between normal and inferior), then the least I am willing to accept is exactly the most you are willing to pay — so in principle we can make a deal but neither one of us will be better or worse off for it. But when pizza is a normal good for us, the least I am willing to accept is more than them most you are willing to pay — so there is no way we will be able to strike a deal.

Intuitively, this makes sense in the following way: We began by saying that you and I are identical in every way — except there is one way in which we are not identical: I have a coupon and you do not. Thus, I am in essence richer than you are to begin with. If pizza is a normal good, then richer people will buy more pizza than poorer people — and so the coupon has more value to richer people because they would use it more. It is for this reason that we can't make a deal if pizza is normal for us. But if pizza is an inferior good, then being richer means I will want less pizza — and so I have less use for the coupon than you do. As a result, you will be willing to pay more than the least I am willing to accept. And if pizza is quasilinear, rich and poor buy the same amount of pizza — and thus make the same use of the coupon. Thus, if pizza is quasilinear, the coupon is worth the same to us.

B: *Suppose your and my tastes can be represented by the Cobb-Douglas utility function* $u(x_1,x_2) = x_1^{0.5}x_2^{0.5}$*, and suppose we both have income* $I=100$*. Let pizza be denoted by* x_1 *and "other goods" by* x_2*, and let the price of pizza be denoted by* p*. (Since "other goods" are denominated in dollars, the price of* x_2 *is implicitly set to 1.)*

(a) *Calculate our demand functions for pizza and other goods as a function of* p*.*

Answer: Solving the problem

$$\max_{x_1,x_2} x_1^{1/2}x_2^{1/2} \text{ subject to } px_1+x_2=100, \tag{10.10}$$

we get $x_1=50/p$ and $x_2=50$.

(b) *Calculate our compensated demand for pizza* (x_1) *and other goods* (x_2) *as a function of* p *(ignoring for now the existence of a coupon).*

Answer: Solving the problem

$$\min_{x_1,x_2} px_1+x_2 \text{ subject to } u=x_1^{1/2}x_2^{1/2}, \tag{10.11}$$

we get $x_1=u/(p^{1/2})$ and $x_2=p^{1/2}u$.

(c) *Suppose* $p=10$ *and the coupon reduces this price by half (to 5). Assume again that I have a coupon but you do not. How much utility do you and I get when we make optimal decisions?*

Answer: Using our (uncompensated) demand functions, we can calculate that your pizza consumption is $x_1=50/10=5$ while mine is $x_1=50/5=10$. This corresponds to x_A and x_D in Graph 10.7. Both of us consume 50 in other goods. Thus, your utility is $u(5,50)=5^{1/2}50^{1/2}\approx 15.81$ and my utility is $u(10,50)=10^{1/2}50^{1/2}\approx 22.36$.

(d) *How much pizza will you consume if you pay me the most you are willing to pay for the coupon? How much will I consume if you pay me the least I am willing to accept?*

Answer: If you pay me the most you are willing to pay, you will remain at the same utility level but will pay only half as much (i.e. \$5 instead of \$10). The compensated demand function therefore tells us that you will consume $x_1=15.81/(5^{1/2})\approx 7.07$. If you pay me the least I am willing to accept, I will end up with the same utility as before but with a price that is twice as high (i.e. \$10 instead of \$5). Thus, the compensated demand function tells me that $x_1=22.36/(10^{1/2})\approx 7.07$. In terms of the graphs in Graph 10.7, this implies that $x_B=7.07=x_D$.

(e) *Calculate the expenditure function for me and you.*

Answer: To get the expenditure function, we substitute the compensated demands back into the objective function of the minimization problem — i.e. we substitute $x_1 = u/(p^{1/2})$ and $x_2 = p^{1/2}u$ into $px_1 + x_2$ to get

$$E(p,u) = p\left(\frac{u}{p^{1/2}}\right) + p^{1/2}u = 2p^{1/2}u. \tag{10.12}$$

(f) *Using your answers so far, determine R — the least I am willing to accept to give up my coupon. Then determine P — the most you are willing to pay to get a coupon.(*Hint*: Use your graphs from A(a) to determine the appropriate values to plug into the expenditure function to determine how much income I would have to have to give up my coupon. Once you have done this, you can subtract my actual income $I = 100$ to determine how much you have to give me to be willing to let go of the coupon. Then do the analogous to determine how much you'd be willing to pay, this time using your graph from A(b).)*

Answer: The budget required for me to be just as happy without the coupon (i.e. when $p = 10$) is the expenditure necessary for me to reach utility level 22.36 (which is u^C in our graph) at $p = 10$ — i.e. $E(10, 22.36) = 2(10^{1/2})22.36 \approx 141.42$. Since I started out with an income of \$100, this implies that you would have to give me approximately \$41.42 for the coupon in order for me to be just as happy; i.e. $R = 41.42$. The budget required for you to be just as happy with the coupon as you were without is the expenditure necessary to get you to utility level 15.81 (which is u^A in our graph) at the with-coupon price of 5 — i.e. $E(5, 15.81) = 2(5^{1/2})15.81 \approx 70.71$. Since you started with an income of \$100, this means you would be willing to pay me as much as $100 - 70.71 = 29.29$ to get the coupon; i.e. $P = 29.29$.

(g) *Are we able to make a deal under which I sell you my coupon? Make sense of this given what you found intuitively in part A and given what you know about Cobb-Douglas tastes.*

Answer: No, we are not able to make a deal since the most you are willing to pay me (\$29.29) is less than the least I am willing to accept (\$41.42). This is consistent with what we concluded in part A where we said that we would not be able to strike a deal if pizza is a normal good for us. Cobb-Douglas tastes are tastes over normal goods — so under the tastes represented by the utility function we have been working with, pizza is in fact a normal good.

(h) *Now suppose our tastes could instead be represented by the utility function $u(x_1, x_2) = 50\ln x_1 + x_2$. Using steps similar to what you have just done, calculate again the least I am willing to accept and the most you are willing to pay for the coupon. Explain the intuition behind your answer given what you know about quasilinear tastes.*

Answer: Solving the problem

$$\max_{x_1,x_2} 50\ln x_1 + x_2 \text{ subject to } px_1 + x_2 = 100, \tag{10.13}$$

we get the (uncompensated) demands $x_1 = 50/p$ and $x_2 = 50$. Thus, both you and I consume 50 in other goods, but I consume 10 pizzas while you only consume 5 because I face a with-coupon price of \$5 per pizza while you face a without-coupon price of \$10 per pizza.

Plugging $(x_1, x_2) = (10, 50)$ into the utility function, we get my utility (equivalent to u^C in the graph) of 165.13. Plugging $(x_1, x_2) = (5, 50)$ into the utility function for you, we get your utility (equivalent to u^A in our graph) as 130.47.

Solving the minimization problem

$$\min_{x_1,x_2} px_1 + x_2 \text{ subject to } u = 50\ln x_1 + x_2, \tag{10.14}$$

we can derive the compensated demands $x_1 = 50/p$ and $x_2 = u - 50\ln(50/p)$. (Note that the compensated and uncompensated demands for the quasilinear good x_1 are the same — which makes sense since there are no income effects to make the two demands different.) Next, we can find the expenditure function by just plugging the compensated demands into the objective function of the minimization problem to get

$$E(p,u) = p\left(\frac{50}{p}\right) + u - 50\ln\left(\frac{50}{p}\right) = u + 50\left(1 - \ln\left(\frac{50}{p}\right)\right). \tag{10.15}$$

To determine the expenditure necessary for me to get to my current utility level in the absence of the coupon (i.e. when $p = 10$ instead of $p = 5$), we calculate $E(10, 165.13) \approx 134.66$. Since I start with an income of \$100, that means the least I am willing to accept for the coupon is $R = 134.66 - 100 = \$34.66$. To calculate the expenditure necessary to get you to your current utility level in the presence of a coupon (i.e. when $p = 5$ instead of $p = 10$), we calculate $E(5, 130.47) \approx 65.34$. Since you also start with an income of \$100, this means that the most you are willing to pay for the coupon is $P = 100 - 65.34 = \$34.66$. Thus $P = R$ as we concluded it has to be when pizza is a quasilinear good.

(i) *Can you demonstrate, using the compensated demand functions you calculated for the two types of tastes, that the values for P and R are in fact areas under these functions (as you described in your answer to A(c)? (*Note*: This part requires you to use integral calculus.)*

Answer: The compensated demand function for pizza is $x_1 = 50/p$ (as calculated in the previous part). The area in the graph is the integral under this function between the with-coupon and the without-coupon prices. (Note: In our graphs, we are graphing the inverse of the compensated demand functions — which is why the area appears as an area to the left of the curve rather than an area under the curve.) Thus, the least I am willing to accept (R) is

$$R = \int_5^{10} \frac{50}{p}\, dp = 50 \ln p|_5^{10} = 50 \ln 10 - 50 \ln 5 = 34.66. \qquad (10.16)$$

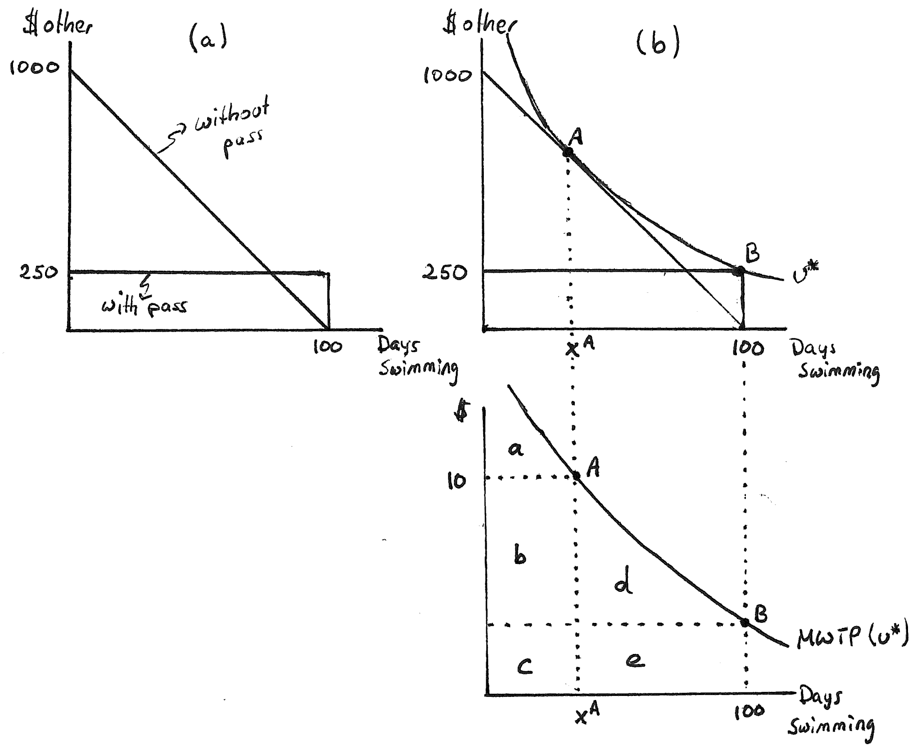
Since, because of the quasilinearity of pizza, our compensated demand functions are the same (because u does not appear in the functions), the same holds for P. Thus, $P = R = 34.66$, exactly as we concluded above.

10.8 Application: *To Join or Not to Join the Local Pool: Where I live, most people do not have swimming pools despite the fact that it gets very hot in the summers. Thus, families, especially those with children, try to find swimming pools in the area. Our local swimming pool offers two ways in which we can get by the entrance guard: We can either purchase a "family pass" for the whole season, or we can pay an entrance fee for the family every time we want to go swimming.*

A: *Suppose we have $1,000 to spend on activities to amuse ourselves during the summer, and suppose that there are exactly 100 days during the summer when the swimming pool is open and usable. The family pass costs $750, while the daily passes cost $10 each (for the whole family).*

(a) *With "days swimming" on the horizontal axis and "dollars spent on other amusements" on the vertical, illustrate our budget constraint if we choose not to buy the season pass.*

Answer: This is illustrated in panel (a) of Graph 10.8 — the opportunity cost of spending a day at the pool is $10 — and if we spend 100 days at the pool, we will have exhausted our entire budget for summer amusements.



Graph 10.8: Buying Access to Swimming Pools

(b) *On the same graph, illustrate the budget constraint we face if we choose to purchase the season pass.*

Answer: This is illustrated on the same panel (a). If we buy the pass, we will have $250 left over for other amusements, and the opportunity cost of spending days at the pool (up to 100 days) is zero (since we will now have free access to the pool.)

(c) *After careful consideration, we decided that we really did not prefer one option over the other — so we flipped a coin with "heads" leading to the season pass and "tails" to no season pass.*

The coin came up "tails" — so we did not buy the season pass. Would we have gone swimming more or less had the coin come up "heads" instead? Illustrate your answer on your graph.

Answer: This is illustrated in panel (b) of the graph where the same indifference curve is "tangent" to the two budget lines (since we are indifferent between buying and not buying a pass). Given that we did not buy the pass, our optimal bundle is A. Had we bought the pass, our optimal bundle would have been B. Thus, had the coin come up "heads" instead, we would have spent more days at the pool than we will given that the coin came up "tails" and we did not buy a season pass.

(d) *My brother bought the season pass. After the summer passed by, my mother said: "I just can't understand how two kids can turn out so differently. One of them spends all his time during the summer at the swimming pool, while the other barely went at all." One possible explanation for my mother's observation is certainly that I am very different than my brother. The other is that we simply faced different circumstances but are actually quite alike. Could the latter be true without large substitution effects?*

Answer: No, the latter explanation requires a large substitution effect — i.e. a relatively flat indifference curve that results in A and B being relatively far apart.

(e) *On a separate graph, illustrate the compensated (Hicksian) demand curve that corresponds to the utility level u^* that my family reached during the summer. Given that we paid \$10 per day at the pool, illustrate the consumer surplus we came away with from the summer experience at the pool.*

Answer: This is done in the panel immediately below panel (b). Since we cannot tell what the slope of the indifference curve is at B, we cannot be sure what our marginal rate of substitution (or our $MWTP$ at that point is. However, we can tell that we would have been willing to pay $a+b+c$ for using the pool x^A number of days, but we actually only paid $b+c$. Thus, our consumer surplus is a.

(f) *Since we would have had to pay no entrance fee had we bought the season pass, can you identify the consumer surplus we would have gotten? (*Hint*: Keep in mind that, once you have the season pass, the price for going to the pool on any day is zero. The cost of the season pass is therefore not relevant for your answer to this part.)*

Answer: The consumer surplus is now the entire area under the marginal willingness to pay curve up to B; i.e. $a+b+c+d+e$.

(g) *Can you identify an area in the graph that represents how much the season pass was?*

Answer: Since we are equally happy at A and B, the cost of the season pass must be such that, when subtracted from our consumer surplus under the season pass, we are left with a, our consumer surplus without the season pass. Thus, the cost of the season pass must have been $b+c+d+e$.

B: *Suppose that my tastes can be represented by the utility function $u(x_1, x_2) = x_1^{\alpha} x_2^{(1-\alpha)}$ with x_1 denoting days of swimming and x_2 denoting dollars spent on other amusements.*

(a) *In the absence of the possibility of a season pass, what would be the optimal number of days for my family to go swimming in the summer? (Your answer should be in terms of α.)*

Answer: We need to solve the problem

$$\max_{x_1, x_2} x_1^{\alpha} x_2^{(1-\alpha)} \text{ subject to } 10x_1 + x_2 = 1000. \tag{10.17}$$

this gives us

$$x_1 = 100\alpha \text{ and } x_2 = 1000(1-\alpha). \tag{10.18}$$

Thus, we would spend 100α days at the pool.

(b) *Derive my indirect utility as a function of α.*

Answer: My indirect utility is then simply derived by substituting the optimal values for x_1 and x_2 into my utility function; i.e.

$$V(\alpha) = (100\alpha)^{\alpha} (1000(1-\alpha))^{(1-\alpha)}. \tag{10.19}$$

(c) *Suppose $\alpha = 0.5$. How much utility do I get out of my \$1,000 of amusement funds? How often do I go to the swimming pool?*

Answer: Substituting 0.5 for α in my indirect utility function $V(\alpha)$, we get

$$V(0.5) = (100(0.5))^{0.5}(1000(0.5))^{0.5} \approx 158.11. \tag{10.20}$$

I would go to the swimming pool for 50 days during the summer.

(d) *Now suppose I had bought the season pass instead (for \$750). How much utility would I have received from my \$1,000 amusement funds?*

Answer: I would then have gone to the pool for 100 days and consumed \$250 in other amusements; i.e. $x_1 = 100$ and $x_2 = 250$. This gives me utility of

$$u(100, 250) = 100^{0.5}250^{0.5} \approx 158.11. \tag{10.21}$$

(e) *What is my marginal willingness to pay for days at the pool if I am going 100 times?*

Answer: The slope of the indifference curves is given by

$$MRS = -\frac{\partial u/\partial x_1}{\partial u/\partial x_2} = -\frac{\alpha x_2}{(1-\alpha)x_1}. \tag{10.22}$$

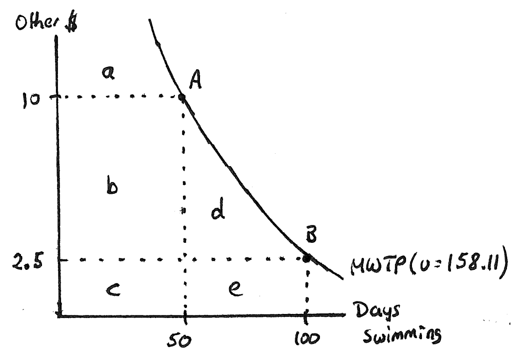
Substituting $\alpha = 0.5$, $x_1 = 100$ and $x_2 = 250$, this gives us

$$MRS = -\frac{0.5(250)}{(1-0.5)(100)} = -2.5. \tag{10.23}$$

Thus, at the consumption bundle $B = (100, 250)$ (which is where I consume with the season pass), my marginal willingness to pay for additional days at the pool is \$2.50.

(f) *On a graph with the compensated demand curve corresponding to my utility this summer, label the horizontal and vertical components of the points that correspond to me taking the season pass and the ones corresponding to me paying a per-use fee.*

Answer: This is done in Graph 10.9 and is based on our calculations above.



Graph 10.9: Buying Access to Swimming Pools: Part 2

(g) *Derive the expenditure function for this problem in terms of p_1, p_2 and u (with $\alpha = 0.5$).*

Answer: To do this, we first need to calculate the compensated demand functions from the expenditure minimization problem

$$\min_{x_1, x_2} E = p_1 x_1 + p_2 x_2 \text{ subject to } u = x_1^{0.5} x_2^{0.5} \tag{10.24}$$

to get

$$x_1 = u\left(\frac{p_2}{p_1}\right)^{0.5} \text{ and } x_2 = u\left(\frac{p_1}{p_2}\right)^{0.5}. \tag{10.25}$$

Plugging these into the objective function of the minimization problem, we get

$$E = p_1 u\left(\frac{p_2}{p_1}\right)^{0.5} + p_2 u\left(\frac{p_1}{p_2}\right)^{0.5} = 2u(p_1 p_2)^{0.5}. \tag{10.26}$$

(h) *In (g) of part A, you identified the area in the $MWTP$ graph that represents the cost of the season pass. Can you now verify mathematically that this area is indeed equal to \$750? (*Hint*: If you have drawn and labeled your graph correctly, the season pass fee is equal to an area composed of two parts: a rectangle equal to 2.5 times 100, and an area to the left of the compensated demand curve between 2.5 and 10 on the vertical axis. The latter is equal to the difference between the expenditure function $E(p_1, p_2, u)$ evaluated at $p_1 = 2.5$ and $p_1 = 10$ (with $p_2 = 1$ and u equal to the correct utility value associated with the indifference curve in your earlier graph.))*

Answer: The area we are trying to calculate is $b + c + d + e$ in Graph 10.9. The area $c + e$ is simply equal to 2.5(100)=250. The area $b + d$ is equal to

$$\begin{aligned} E(10,1,158.11) - E(2.5,1,158.11) &= 2(10(1))^{0.5}(158.11) - 2(2.5(1))^{0.5}(158.11) \\ &\approx 1000 - 500 = 500. \end{aligned} \tag{10.27}$$

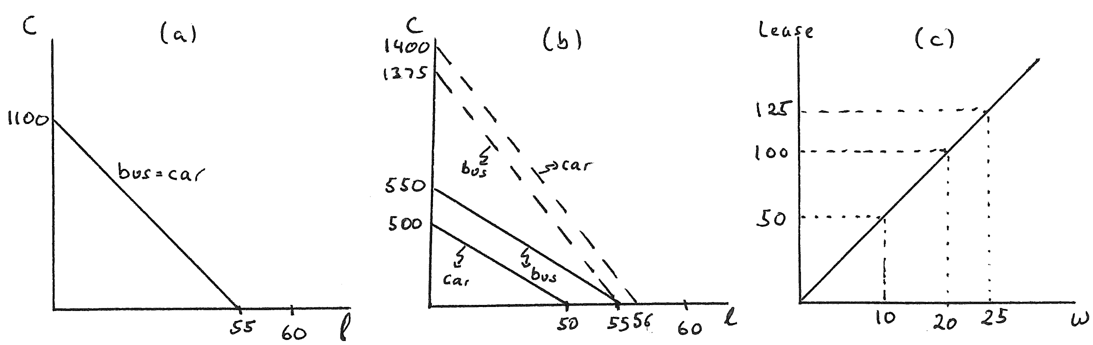
Adding these two areas together, we get that the season pass must have cost \$750.

10.9 Everyday Application: *To Take, or not to Take, the Bus: After you graduate, you get a job in a small city where you have taken your sister's offer of living in her apartment. Your job pays you $20 per hour and you have up to 60 hours per week available. The problem is you also have to get to work.*

A: *Your sister's place is actually pretty close to work — so you could lease a car and pay a total (including insurance and gas) of $100 per week to get to work, spending essentially no time commuting. Alternatively, you could use the city's sparse bus system — but unfortunately there is no direct bus line to your place of work and you would have to change buses a few times to get there. This would take approximately 5 hours per week.*

(a) *Now suppose that you do not consider time spent commuting as "leisure" — and you don't consider money spent on transportation as "consumption". On a graph with "leisure net of commuting time" on the horizontal axis and "consumption dollars net of commuting costs" on the vertical, illustrate your budget constraint if you choose the bus and a separate budget constraint if you choose to lease the car.*

Answer: If you choose to take the bus, you reduce your leisure time from 60 to 55 hours per week — and therefore can earn up to \$20(55)=\$1,100 per week. If you choose to lease a car, you can work up to 60 hours per week — but you have to pay \$100 no matter how much you work. If you work for 60 hours, you earn \$25(60)=\$1,200, but since you have to pay \$100 for the lease, the most you can consume is \$1,100. In order to pay for the lease, you have to work at least 5 hours. Thus, the two budget constraints are exactly the same. They are illustrated in panel (a) of Graph 10.10.



Graph 10.10: To Take, or not to Take, the Bus

(b) *Do you prefer the bus to the car?*

Answer: Since the two choices result in the same effective budget constraint, you are indifferent between taking the bus and leasing the car.

(c) *Suppose that before you get to town you find out that a typo had been made in your offer letter and your actual wage is $10 per hour instead of $20 per hour. How does your answer change?*

Answer: If you choose the bus, the loss of leisure remains unchanged — leaving you again with 55 hours per week. At the lower wage, that allows you to consume at most \$10(55)=\$550 per week. If you choose the lease, it will take you 10 hours just to come up with the payment for the lease. Even if you work all 60 hours, you will therefore be left with consumption corresponding to only 50 hours of work — or \$500. The lease budget is therefore strictly lower than the bus budget, as illustrated by the two solid budget lines in panel (b) of Graph 10.10. You would therefore choose the bus.

(d) *After a few weeks, your employer discovers just how good you are and gives you a raise to $25 per hour. What mode of transportation do you take now?*

Answer: If you choose the bus, you again will take the same 5 hour reduction in your leisure — leaving you with at most \$25(55)=\$1,375 in consumption. If you choose the lease, you can pay for the lease with just 4 hours of work — leaving you with up to 56 hours that you can devote to generating consumption. Now the lease budget therefore strictly dominates the bus budget, as illustrated by the dashed budgets in panel (b) of the graph. You will therefore go by car.

(e) *Illustrate in a graph (not directly derived from what you have done so far) the relationship between wage on the horizontal axis and the most you'd be willing to pay for the leased car.*

Answer: We know from (a) that, when $w = 20$, a lease payment of \$100 per week makes you exactly indifferent. This is because, at a wage of \$20 per hour, it takes 5 hours to make enough money to pay for the lease — which is exactly the same number of hours as it takes to ride the bus. Thus, to get the maximum lease payment Y that you would be willing to pay at wage w, you simply multiply the number of hours required for the bus by the wage — i.e. $Y = 5w$. Put differently, the value of your time on the bus determines the maximum lease you are willing to pay. This is illustrated in panel (c) of Ggraph 10.10.

(f) *If the government taxes gasoline and thus increases the cost of driving a leased cars (while keeping buses running for free), predict what will happen to the demand for bus service and indicate what types of workers will be the source of the change in demand.*

Answer: Demand for bus service will increase, with the increase in demand determined by the lowest wage workers that previously chose to lease cars.

(g) *What happens if the government improves bus service by reducing the time one needs to spend to get from one place to the other?*

Answer: If the government improves bus service, a person with a wage that previously made him indifferent between the bus and leasing a car will now take the bus because the most he would be willing to pay for a lease will fall. Thus, lower wage workers will switch to the improved bus system.

B: *Now suppose your tastes were given by $u(c,\ell) = c^{\alpha}\ell^{(1-\alpha)}$, where c is consumption dollars net of commuting expenses and ℓ is leisure consumption net of time spent commuting. Suppose your leisure endowment is L and your wage is w.*

(a) *Derive consumption and leisure demand assuming you lease a car that costs you \$$Y$ per week which therefore implies no commuting time.*

Answer: We need to solve the problem

$$\max_{c,\ell} c^{\alpha}\ell^{(1-\alpha)} \text{ subject to } w(L-\ell) = Y + c. \tag{10.28}$$

This gives us

$$c = \alpha(wL - Y) \text{ and } \ell = \frac{(1-\alpha)(wL - Y}{w}. \tag{10.29}$$

(b) *Next, derive your demand for consumption and leisure assuming you take the bus instead, with the bus costing no money but taking T hours per week from your leisure.*

Answer: Now we need to solve the problem

$$\max_{c,\ell} c^{\alpha}\ell^{(1-\alpha)} \text{ subject to } w(L-T-\ell) = c. \tag{10.30}$$

This gives us

$$c = \alpha w(L-T) \text{ and } \ell = (1-\alpha)(L-T). \tag{10.31}$$

(c) *Express the indirect utility of leasing the car as a function of Y.*

Answer: To get the indirect utility from leasing a car, we simply have to plug the optimal values for c and ℓ from equation (10.29) into the utility function — which gives

$$V(Y) = \frac{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}(wL - Y)}{w^{(1-\alpha)}}. \tag{10.32}$$

(d) *Express your indirect utility of taking the bus as a function of T.*

Answer: Now we need to take the optimal levels of c and ℓ when taking the bus (from equation (10.31)) into the utility function to get

$$V(T) = \alpha^{\alpha}(1-\alpha)^{(1-\alpha)}\, w^{\alpha}(L-T). \tag{10.33}$$

(e) *Using the indirect utility functions, determine the relationship between Y and T that would keep you indifferent between taking the bus and leasing the car. Is your answer consistent with the relationship you illustrated in A(e) and your conclusions in A(f) and A(g)?*

Answer: Setting $V(Y)$ equal to $V(T)$, i.e.

$$\frac{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}(wL-Y)}{w^{(1-\alpha)}} = \alpha^{\alpha}(1-\alpha)^{(1-\alpha)}\, w^{\alpha}(L-T), \tag{10.34}$$

we get, after canceling some terms, $Y = wT$. This is consistent with what we found in part A where we said that $Y = 5w$ when the time to take the bus is 5 hours per week. It is also consistent with our conclusion that the lowest wage workers who previously leased cars will switch to buses if the government increases the cost of leasing a car by taxing gasoline — because as Y increases, the equation $Y = wT$ implies that w will also have to increase to keep someone indifferent between leasing a car and taking the bus. It is similarly consistent with the conclusion that a more efficient bus system — i.e. a lower T — will cause the same types of workers to switch to the bus.

(f) *Could you have skipped all these steps and derived this relationship directly from the budget constraints? Why or why not?*

Answer: Yes, you could have. The two budget constraints can be expressed as

$$c = w(L-\ell) - Y \;\text{ and }\; c = w(L-T-\ell). \tag{10.35}$$

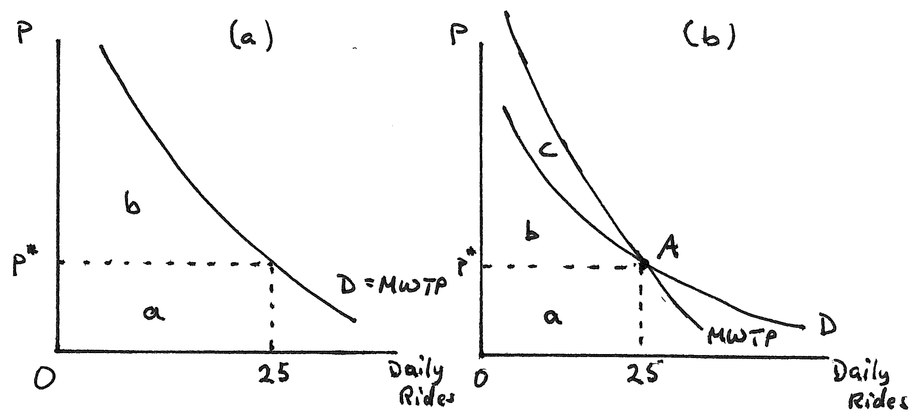
Setting these equal to each other, you get $Y = wT$. This works because the only factors operating in this problem are wealth effects — there are no substitution effects. It is analogous to what we did in part A where we simply focused on the relationship between the budget constraints to determine what the consumer would prefer — we did not have to actually draw any indifference curves.

10.10 Business Application: *Pricing at Disneyland: In the 1970s, Disneyland charged an entrance fee to get into the park and then required customers to separately buy tickets for each ride once they were in the park. In the 1980s, Disneyland switched to a different pricing system that continues to this day. Now, customers simply pay an entrance fee and then all rides in the park are free.*

A: *Suppose you own an amusement park with many rides (and assume, for the sake of simplicity, that all rides cost the same to operate.) Suppose further that the maximum number of rides a customer can take on any given day (given how long rides take and how long the average wait times are) is 25. Your typical vacationing customer has some exogenous daily vacation budget I to allocate between rides at your park and other forms of entertainment (that are, for purposes of this problem) bought from vendors other than you. Finally, suppose tastes are quasilinear in amusement park rides.*

(a) *Draw a demand curve for rides in your park. Suppose you charge no entrance fee and only charge your customers per ride. Indicate the maximum price per ride you could charge while insuring that your consumer will in fact spend all her day riding rides (i.e. ride 25 times).*

Answer: This is illustrated in panel (a) of Graph 10.11 where p^* indicates the maximum price per ride such that the consumer will choose the maximum number of 25 rides per day.



Graph 10.11: Pricing at Disneyland

(b) *On your graph, indicate the total amount that the consumer will spend.*

Answer: This is indicated in panel (a) as area a — the price p^* times the quantity 25.

(c) *Now suppose that you decide you want to keep the price per ride you have been using but you'd also like to charge a separate entrance fee to the park. What is the most you can charge your customer?*

Answer: Since tastes are quasilinear in x_1, the demand curve is also the $MWTP$ curve. We can therefore measure the consumer's total willingness to pay for rides as the area underneath the demand curve up to 25. Since the consumer is already spending $\$25p^* = a$, the remaining amount we could charge as an entrance fee is equal to the area b.

(d) *Suppose you decide that it is just too much trouble to collect fees for each ride — so you eliminate the price per ride and switch to a system where you only charge an entrance fee to the park. How high an entrance fee can you charge?*

Answer: If we lower the price per ride to zero (and there is still a maximum of 25 rides per day that the consumer has time for), we can charge an entrance fee equalto the entire area below the $MWTP$ curve — i.e. the area $a + b$.

(e) *How would your analysis change if x_1, amusement part rides, is a normal good rather than being quasilinear?*

Answer: In that case, the $MWTP$ curve is no longer the same as the demand curve. If x_1 is a normal good, the $MWTP$ that crosses the demand curve at A in panel (b) of Graph 10.11 is steeper than the demand curve — because the income effect that is included in the demand curve points in the same direction as the substitution effect that is included in both curves. Thus, the total willingness to pay is $a+b+c$ — which means we can charge an entrance fee of $b+c$ if we continue to charge a per-ride fee of p^* and we can charge an entrance fee of $a+b+c$ if we charge no price per ride.

B: *Consider a consumer on vacation who visits your amusement park for the day. Suppose her tastes can be summarized by the utility function $u(x_1,x_2)=10x_1^{0.5}+x_2$ where x_1 represents daily rides in the amusement park and x_2 represents dollars of other entertainment spending. Suppose further that her exogenous daily budget for entertainment is \$100.*

(a) *Derive the uncompensated and compensated demand functions for x_1 and x_2.*

Answer: To solve for uncompensated demands, we solve

$$\max_{x_1,x_2} 10x_1^{0.5}+x_2 \text{ subject to } px_1+x_2=100, \tag{10.36}$$

which gives us

$$x_1=\frac{25}{p^2} \text{ and } x_2=100-\frac{25}{p}. \tag{10.37}$$

To solve for compensated demands, we solve

$$\min_{x_1,x_2} px_1+x_2 \text{ subject to } u=10x_1^{0.5}+x_2, \tag{10.38}$$

which gives us

$$x_1=\frac{25}{p^2} \text{ and } x_2=u-\frac{50}{p}. \tag{10.39}$$

(b) *Suppose again there is only enough time for a customer to ride 25 rides a day in your amusement park and suppose that congestion and wear-and-tear on equipment in the park is not a problem. Suppose then that you'd like your customer to ride as much as possible so he can spread the word on how great your rides are. What price will you set per ride?*

Answer: You would like to set a price such that your consumer demands exactly 25 rides per day. Your consumer's demand function is $x_1=25/(p^2)$. Setting $x_1=25$, we can then solve for the price per ride of $p=1$.

(c) *How much utility will your consumer attain under your pricing?*

Answer: Your consumer will consume $x_1=25$ and $x_2=100-(25/1)=75$. Plugging this bundle into the utility function, we get $u(25,75)=10(25^{0.5})+75=125$.

(d) *Suppose you can also charge an entrance fee to your park — in addition to charging the price per ride you calculated above. How high an entrance fee would you charge? (*Hint*: You should be evaluating an integral, which draws on some of the material from the appendix.)*

Answer: The total willingness to pay is an area under the appropriate compensated demand curve — i.e. the compensated demand curve that corresponds to the indifference curve at the bundle at which the consumer consumes. Since tastes are quasilinear in x_1, the compensated demand function is identical to the uncompensated demand function for x_1. Thus, we simply need to evaluate the area under the demand curve up to the quantity $x_1=25$ that is consumed. The demand curve, with p on the vertical axis, is the inverse of the demand function $x_1=25/(p^2)$ — i.e. $p=5/(x_1^{0.5})$. We therefore need to evaluate

$$\int_0^{25}\frac{25}{x_1^{0.5}}dx_1=10x_1^{0.5}|_0^{25}=10(25^{0.5})-10(0^{0.5})=50. \tag{10.40}$$

This means that the consumer's total willingness to pay for the 25 rides is \$50. By charging \$1 per ride for 25 rides, we are charging the consumer \$25. Thus, we can charge an entrance fee of an additional \$25.

(e) *Now suppose you decide to make all rides free (knowing that the most rides the consumer can squeeze into a day is 25) and you simply charge an entrance fee to your park. How high an entrance fee will you now charge to your park? (Note: This part is not computationally difficult — it is designated with ** only because you have to use information from the previous part.)*

Answer: If you no longer charge per ride, you are in essence charging the consumer $p = 0$. Since the consumer chooses to take the maximum 25 rides even at a price of $p = 1$, she will still take all 25 rides. But now she incurs no cost per ride once she is in the park — and we calculated above that she values 25 rides at $50. Thus, we would charge her an entrance free of $50.

(f) *How does your analysis change if the consumer's tastes instead were given by $u(x_1, x_2) = (3^{-0.5})x_1^{0.5} + x_2^{0.5}$?*

Answer: When x_1 was quasilinear, the compensated and uncompensated demand curves were the same (because there are no income effects). Now, the two will differ. In particular, solving the maximization problem, we get uncompensated demands (assuming again a daily budget of $100)

$$x_1 = \frac{100}{p(1+3p)} \text{ and } x_2 = \frac{300p}{1+3p}. \tag{10.41}$$

Solving the expenditure minimization problem, we get compensated demands

$$x_1 = 3\left(\frac{u}{1+3p}\right)^2 \text{ and } x_2 = \left(\frac{3pu}{1+3p}\right)^2. \tag{10.42}$$

In order for the consumer to demand exactly 25 rides at a per-ride price p, we simply set the uncompensated demand for x_1 equal to 25 and solve for p; i.e. we solve

$$25 = \frac{100}{p(1+3p)} \tag{10.43}$$

which gives us $p = 1$, identical to our result from before.[1] At $p = 1$, the consumer will consume the bundle $(x_1, x_2) = (25, 75)$ which gives utility $u = (3^{-0.5})(25^{0.5}) + 75^{0.5} \approx 11.547$.

To calculate the consumer's total willingness to pay, we need to calculate the appropriate area under the *compensated* demand curve that corresponds to this utility level — which is the inverse of the compensated demand function

$$x_1 = 3\left(\frac{11.547}{1+3p}\right)^2; \tag{10.44}$$

i.e. it is this function solved for p —

$$p = \frac{20}{3x_1^{0.5}} + \frac{1}{3}. \tag{10.45}$$

Taking the integral of this and evaluating it from $x_1 = 0$ to $x_1 = 25$, we get the total willingness to pay for 25 rides as

$$\int_0^{25}\left(\frac{20}{3x_1^{0.5}} - \frac{1}{3}\right)dx_1 = \frac{40x_1^{0.5} - x_1}{3}\Big|_0^{25} = \frac{175}{3} \approx 58.33. \tag{10.46}$$

Under the price per ride of $1 per ride, the consumer is already paying $25. If you were to charge an entrance fee plus a price of $1 per ride, you would therefore be able to charge a fee of $58.33-$25=$33.33. If you abandoned the policy of charging per ride and only charged an entrance fee, you would be able to set the entrance fee at $58.33.

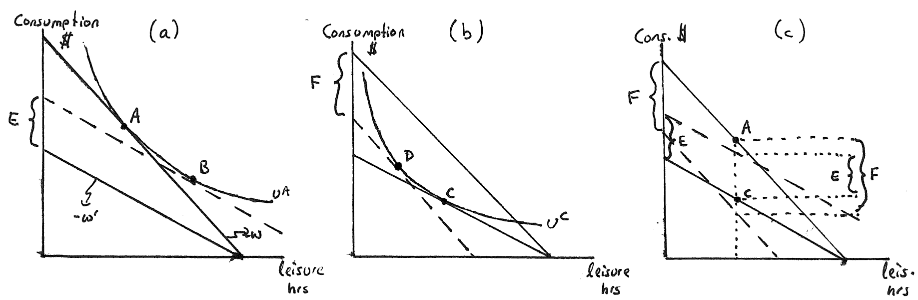
[1] We have to solve the equation $3p^2 + p - 4 = 0$ using the quadratic formula — which gives us two solutions: $p = 1$ and $p = -4/3$. Since the latter is negative, it is not economically meaningful.

10.11 Business Application: *Negotiating an Endorsement Deal and a Bribe: Suppose you are an amateur athlete and your uncle owns the cereal company "Wheaties." Your uncle offers you a job working for his company at a wage of w per hour. After looking around for other jobs, you find that the most you could make elsewhere is w', where $w' < w$. You have a weekly leisure endowment of L and can allocate any amount of that to work. Given the higher wage at Wheaties, you accept your uncle's job offer.*

A: *Then you win a gold medal in the Olympics. "Greeties", the makers of grits, ask you for an endorsement. As part of the deal, they will pay you some fixed weekly amount to appear on their boxes of grits. Unfortunately, your uncle (who hates his competitor "Greeties" with the white hot intensity of a thousand suns) will fire you if you accept the deal offered by "Greeties". Therefore, if you accept the deal, your wage falls to w'.*

(a) *On a graph with Consumption on the vertical and Leisure on the horizontal axis, graph your budget constraint before the "Greeties" offer.*

Answer: This is graphed in panel (a) of Graph 10.12 as the steeper of the two budget lines.



Graph 10.12: Endorsement Deals and Bribes

(b) *On the same graph, illustrate your budget if you worked for someone other than your uncle prior to your success in the Olympics.*

Answer: This is illustrated as the shallower of the two budget lines in panel (a) of the graph.

(c) *Illustrate the minimum amount that "Greeties" would have to pay you (weekly) for your endorsement in order for you to accept the deal. Call this amount E.*

Answer: The optimal bundle at Wheaties is indicated as bundle A — yielding utility level u^A. You would not be willing to enter any endorsement deal that does not at least get you to that same utility level. The Greeties endorsement check must therefore be sufficient to get you to u^A. Such a deal does not change your wage at Greeties — it just causes your Greeties budget to shift out in a parallel way. If the shift is sufficient to get you to bundle B, it is the lowest possible amount you would accept. The is indicated in dollar terms on the vertical axis as E.

(d) *How does this amount E compare to the amount necessary to get you to be able to consume bundle A under a Greeties endorsement deal?*

Answer: The minimum amount E that is acceptable to you is smaller than what is necessary to get to bundle A. You can tell it is less because A lies outside the budget constraint that emerges when Greeties offers E. The difference emerges because of a substitution effect.

(e) *Now suppose that you accepted the endorsement deal from "Greeties" but, unfortunately, the check for the endorsement bounces because "Greeties" goes bankrupt. Therefore the deal is off, but your angry uncle has already fired you. Deep down inside your uncle still cares about you and will give you back your old job if you come back and ask him for it. The problem is that you have to get past his greedy secretary who has full control over who gets to see your uncle. When you get to the "Wheaties" office, she informs you that you have to commit to pay her a weekly bribe if you want access to your uncle. On a new graph, illustrate the largest possible (weekly) payment you would be willing to make. Call this F.*

Answer: Once the endorsement deal falls through, you will optimize at bundle C in panel (b) of Graph 10.12 — along the budget formed by the lower market wage w' in the absence of any lump sum payments. This would give you utility level u^C. Getting employment at Wheaties implies getting the higher wage w (and thus the steeper budget), but paying the weekly bribe means that this budget shifts inward in a parallel way. The most you would be willing to pay to get your old job back is an amount that makes you just as well off as you are without getting back to Wheaties — i.e. an amount that gets you to utility level u^C. This is equivalent to a shift in the steeper budget that gets you to the new optimal bundle D. The amount of the weekly bribe is then indicated in dollar terms on the vertical axis as F.

(f) *If your uncle's secretary just asks you for a weekly bribe that gets you to the bundle C that you would consume in the absence of returning to Wheaties, would you pay her such a bribe?*

Answer: Yes, you would — because the shift in the steeper budget required to get you to C is less than the shift induced by F. You can tell that this is so by the fact that C lies outside the budget that is created by the highest possible bribe F you'd be willing to pay. The reason you are willing to pay more than the amount that would make C affordable is a substitution effect.

(g) *Suppose your tastes are such that the wealth effect from a wage change is exactly offset by the substitution effect — i.e. no matter what the wage, you will always work the same amount (in the absence of receiving endorsement checks or paying bribes). In this case, can you tell whether the amount E (i.e. the minimum endorsement check) is greater than or equal to the amount F (i.e. the maximum bribe)?*

Answer: Yes — we will conclude that $F > E$. Here is how we can tell: If wealth and substitution effects exactly offset as described, then A (in panel (a) of Graph 10.12) lies exactly above C (from panel (b) of Graph 10.12) — because A is optimal at the high wage (in the absence of a bribe) and C is optimal at the low wage (in the absence of a bribe). This is illustrated in panel (c) of the graph where A and C are plotted at the same level of leisure. We then transfer the dashed lines from panel (a) and (b) — to give us the vertical distances E and F. The key is that we concluded that the dashed line in panel (a) must lie *below* A, and the dashed line in panel (b) must lie *below* C. But it is then shown in panel (c) of the graph that this implies F — the vertical distance between the steeper lines — must be larger than E — the vertical distance between the shallower lines. You would therefore be willing to bribe more to get back your job than you required in an endorsement to give it up.

B: *Suppose that your tastes over weekly consumption c and weekly leisure ℓ can be represented by the utility function* $u(c,\ell) = c^{0.5}\ell^{0.5}$ *and your weekly leisure endowment is* $L = 60$.

(a) *If you accept the initial job with Wheaties, how much will you work?*

Answer: To answer this, we need to solve the problem

$$\max_{c,\ell} c^{0.5}\ell^{0.5} \text{ subject to } w(60-\ell) = c. \quad (10.47)$$

This results in

$$c = 30w \text{ and } \ell = 30. \quad (10.48)$$

Thus, you will work for 30 hours at Wheaties.

(b) *Suppose you accept a deal from Greeties that pays you a weekly amount* $\overline{E}$. *How much will you work then? Can you tell whether this is more or less than you would work at Wheaties?*

Answer: We now have to solve the problem

$$\max_{c,\ell} c^{0.5}\ell^{0.5} \text{ subject to } \overline{E} + w'(60-\ell) = c. \quad (10.49)$$

This solves to

$$c = 0.5\overline{E} + 30w' \text{ and } \ell = \frac{0.5\overline{E} + 30w'}{w'} = \frac{0.5\overline{E}}{w'} + 30. \quad (10.50)$$

You will therefore take more than the 30 hours of leisure you took at Wheaties — which implies you will work fewer hours.

(c) *Suppose that the wage w at Wheaties is \$50 per hour and the wage w' at Greeties (or any other potential employer other than Wheaties) is \$25 per hour. What is the lowest possible value for E — the weekly endorsement money from Greeties — that might get you to accept the endorsement deal?*

<u>Answer</u>: First, we have to figure out how much utility you can assure yourself of by just working at Wheaties — because you would not accept an endorsement deal from Greeties that results in less utility than that. At Wheaties, we calculate that $c = 30w$ and $\ell = 30$ — which implies you will consume a bundle $A = (c,\ell) = (1500,30)$ at $w = 50$. This gives utility of $u^A = 1500^{0.5}30^{0.5} \approx 212.13$. From the results in equation (10.50), we know that your consumption bundle at Greeties with an endorsement deal $\overline{E}$ results in the bundle $B = (c,\ell) = (0.5\overline{E} + 750, (0.5\overline{E} + 750)/25)$ when $w' = 25$. Since E would result in the same utility at B as at A, we could then simply set the utility at the consumption bundle with the endorsement deal equal to 212.13 and solve for E — i.e. we could solve

$$(0.5E + 750)^{0.5}\left(\frac{(0.5E + 750)}{25}\right)^{0.5} = 212.13. \tag{10.51}$$

This gives us $E = 612.30$.

Alternatively, we could use the expenditure function approach. To do so, we need to solve the expenditure minimization problem

$$\min_{c,\ell} w\ell + c \;\text{ subject to }\; u = c^{0.5}\ell^{0.5} \tag{10.52}$$

to solve for the compensated demands

$$c = w^{0.5}u \;\text{ and }\; \ell = w^{-0.5}u. \tag{10.53}$$

Plugging these back into the expenditure equation $w\ell + c$, we get that the expenditure necessary to get to utility level u at wage w is

$$E(w,u) = 2w^{0.5}u. \tag{10.54}$$

The value of your endowment at a wage of \$25 is 25(60)=1500. The expenditure necessary to get to utility level $u^A = 212.13$ at $w' = 25$ is $E(25, 212.13) = 2(25^{0.5})(212.13) = 2121.30$. The minimum necessary weekly endorsement check to get you to accept the Greeties offer is then the difference between these; i.e. $E = 2121.30 - 1500 = 612.30$. (Note that the E that refers to this minimum endorsement check is not the same as the *expenditure function* $E(w,u)$.)

(d) *How much will you work if you accept this endorsement deal E?*

<u>Answer</u>: Your optimal leisure (from the equation (10.50)) is then

$$\ell = \frac{0.5\overline{E}}{w'} + 30 = \frac{0.5(612.30)}{25} + 30 \approx 42.25. \tag{10.55}$$

Thus, you will work approximately $60 - 42.25 = 17.75$ hours per week at Greeties.

(e) *Suppose you have accepted this deal but Greeties now goes out of business. What is the highest possible weekly bribe F you'd be willing to pay your uncle's secretary in order to get your job at Wheaties back?*

<u>Answer</u>: First, we have to calculate the utility level you would get if you do not get your old job at Wheaties back. With $E = 0$, our results from equation (10.50) become identical to those from equation (10.48) — $c = 30w'$ and $\ell = 30$. When $w' = 25$, this implies an optimal consumption bundle $C = (c,\ell) = (750,30)$ — which gives utility $u^C = 750^{0.5}30^{0.5} = 150$.

We can then solve for F in one of two ways. Using the expenditure function $E(w,u) = 2w^{0.5}u$, we can calculate the minimum expenditure necessary at the Wheaties wage $w = 50$ to get to the the utility level u^C — i.e. $E(50,150) = 2(50^{0.5})(150) = 2121.32$. Your leisure endowment at the Wheaties wage of $w = 50$ is worth $50(60) = 3000$. Thus, you would be willing to pay $3000 - 2121.32 = 878.68$ in a weekly bribe to get your Wheaties job back.

You can also calculate this by directly setting the utility from paying the bribe at the higher wage equal to $u^C = 150$ (in a way analogous to what we did in the previous part). This would give you the equation

$$(0.5F + 1500)^{0.5}\left(\frac{(0.5F + 1500)}{50}\right)^{0.5} = 150. \tag{10.56}$$

When solved for F, this gives $F = -878.68$; i.e. you would be willing to pay this amount which is the same as calculated through the expenditure function approach.

(f) *How much would you work assuming that the secretary has successfully extracted the maximum amount you are willing to pay to get your Wheaties job back?*

Answer: We can use our answer in equation (10.50) since F is just like a negative endorsement deal that gets to the higher wage.

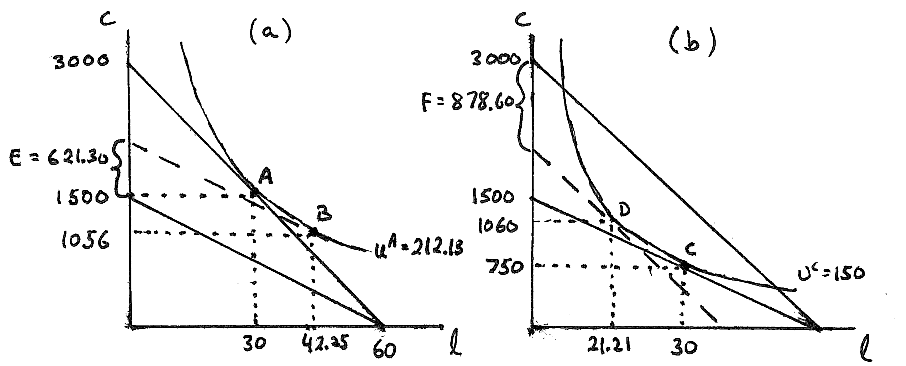
$$\begin{aligned} c &= 0.5(-F) + 30w = 0.5(-878.68) + 30(50) \approx 1060.66 \text{ and} \\ \ell &= \frac{0.5(-F)}{w} + 30 = \frac{0.5(-878.68)}{50} + 30 \approx 21.21. \end{aligned} \tag{10.57}$$

Thus, you will take 21.21 hours of leisure — which means you will work for 60-21.21=38.79 hours.

(g) *Re-draw your graphs from part A but now label all the points and intercepts in accordance with your calculations. Does your prediction from A(g) about the size of the maximum bribe relative to the size of the minimum endorsement hold true?*

Answer: This is done in Graph 10.13.

The prediction from part A(g) holds true. Here we have a case where the wealth and substitution effects exactly offset one another in the absence of bribes or endorsements — which results in leisure of 30 hours at both the high wage (point A) and the low wage (point C). We predicted in part A(g) that this should lead to $F > E$ — which holds for this numerical example where $F = 1060.66 > 612.30 = E$.



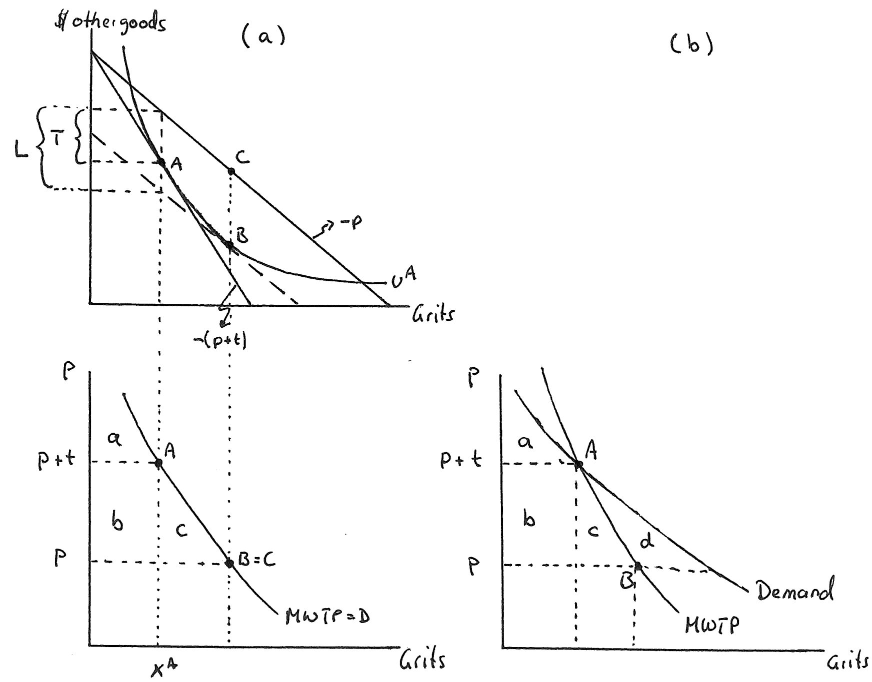
Graph 10.13: Endorsement Deals and Bribes: Part 2

10.12 Policy Application: *Distortionary Taxes: Suppose that you have tastes for grits and "other goods" (where the price of "other goods" is normalized to 1). Assume throughout (unless otherwise stated) that your tastes are quasilinear in grits.*

A: *The government decides to place a tax on grits — thus raising the price of grits from p to $p+t$.*

(a) *On a graph with grits on the horizontal axis and "other goods" on the vertical, illustrate the before and after tax budget.*

Answer: These two budgets are illustrated in the top graph in panel (a) of Graph 10.14, with $-p$ indicating the slope of the before-tax budget and $-(p+t)$ indicating the slope of the after-tax budget.



Graph 10.14: Taxing Grits

(b) *Illustrate your optimal consumption bundle after the tax is imposed — then indicate how much tax revenue T the government collects from you.*

Answer This is also illustrated in the top graph of panel (a) — with A indicating the optimal after-tax bundle and T on the vertical axis indicating the distance representing the tax revenues raised from the tax on grits.

(c) *Illustrate the most L you would be willing to pay to not have the tax.*

Answer: Again, this is illustrated in the top graph of panel (a). The most you are willing to pay to avoid the tax is an amount that shifts your original (before-tax) budget parallel until it is tangent to the after tax indifference curve u^A. This tangency occurs at B — and the vertical distance between the two parallel budgets is equal to L. This is measured on the vertical axis.

(d) *Does your answer depend on the fact that you know your tastes are quasilinear in grits?*

Answer: None of our answers so far have depended on the quasilinearity of grits — because all the answers have been derived from two budgets (that have nothing to do with tastes) and a single indifference curve u^A. Quasilinearity is a property of the relationship of indifference curves to one another — it is consistent with large or small substitution effects.

(e) *On a graph below the one you have drawn, derive the regular demand curve as well as the $MWTP$ curve.*

Answer: The $MWTP$ curve is derived simply from A and B in the top graph. The demand curve requires us to know where the optimal bundle C on the pre-tax budget lies — but since we know that tastes are quasilinear in grits, we know that C must lie directly above B. (Under quasilinear tastes, changes in income that are not accompanied by changes in opportunity costs do not change the optimal consumption level of the quasilinear good — thus, the consumption of grits does not change between the two parallel budgets.) Both C and B happen at the price p — and thus fall at the same point in the lower graph. Thus, the demand curve is identical to the $MWTP$ curve. This makes sense since the only difference between demand and $MWTP$ curves are income effects — and these go to zero with quasilinearity.

(f) *Illustrate T and L on your lower graph and indicate where in the graph you can locate the deadweight loss from the tax.*

Answer: The actual tax revenue from the grits tax is t times the amount of grits the consumer chooses to consume (which is labeled x^A in the graph). This forms the rectangle $t(x^A)$ which is equal to the area b. Thus, $T = b$. The consumer surplus in the grits market after the tax is imposed is just area a, while the consumer surplus in the absence of grits assuming that L has been paid is $a+b+c$. Since the consumer is equally happy at points A and B, the amount L that had to be paid to avoid the tax and keep prices at p must exactly offset the increased consumer surplus in the grits market; i.e. $L = b + c$. The deadweight loss is $L - T$ — i.e. $DWL = c$.

(g) *Suppose you only obseserved the demand curve in the lower graph — and you knew nothing else about tastes. If grits were actually a normal good (rather than a quasilinear good), would you under- or over-estime that deadweight loss by assuming grits are quasilinear?*

Answer: If grits were a normal good, the demand curve would be shallower than the $MWTP$ curve because it now gives rise to an income effect that points in the same direction as the substitution effect. The point A lies on both the $MWTP$ curve that corresponds to u^A as well as the regular demand curve — i.e. the two curves cross at A. This results in two distinct curves as pictured in panel (b) of Graph 10.14.

If you now assumed that the demand curve which you observe is the same as the $MWTP$ curve (which must mean you think grits are quasilinear even though they are not), then you would estimate L as the area $b + c + d$ when in fact $L = b + c$. Thus, you would overestimate the deadweight loss by the area d. This may seem like a small area, but you can see in the graph how it can easily be large relative to the actual size of the deadweight loss c — leading to a substantial overestimate of the deadweight loss.

B: *Suppose that your tastes could be represented by the utility function $u(x_1, x_2) = 10x_1^{0.5} + x_2$, with x_1 representing weekly servings of grits and x_2 representing dollars of other breakfast food consumption. Suppose your weekly (exogenous) budget for breakfast food is \$50.*

(a) *Derive your uncompensated and compensated demand for grits.*

Answer: To solve for uncompensated demands, we solve

$$\max_{x_1,x_2} 10x_1^{0.5} + x_2 \text{ subject to } px_1 + x_2 = 50, \tag{10.58}$$

which gives us

$$x_1 = \frac{25}{p^2} \text{ and } x_2 = 50 - \frac{25}{p}. \tag{10.59}$$

To solve for compensated demands, we solve

$$\min_{x_1,x_2} px_1 + x_2 \text{ subject to } u = 10x_1^{0.5} + x_2, \tag{10.60}$$

which gives us

$$x_1 = \frac{25}{p^2} \text{ and } x_2 = u - \frac{50}{p}. \tag{10.61}$$

(b) *Suppose the tax on grits raises its price from \$1 to \$1.25 per serving. How does your consumption of grits change?*

Answer: Your consumption changes from $x_1 = 25/(1^2) = 25$ to $x_1 = 25/(1.25^2) = 16$ servings of grits per week.

(c) *How much tax revenue T does the government collect from you per week?*

Answer: After the tax is imposed, you consume 16 servings per week. For each serving, the government collects \$0.25 — so the total tax collected per week is $T = \$4$.

(d) *Use the expenditure function for this problem to determine how much L you would have been willing to pay (per week) to avoid this tax?*

Answer: We get the expenditure function by substituting the compensated demands from equation (10.61) into the expenditure equation $px_1 + x_2$ to get

$$E(p,u) = u - \frac{25}{p}. \tag{10.62}$$

In order to determine how much you would be willing to pay (per week) to avoid the tax, we need to calculate the expenditure necessary to get to the post-tax utility level at before tax prices. We know from what we did above that you will consume (after the tax) $x_1 = 16$ (at a per-serving price of \$1.25) — which leaves \$30 for other goods; i.e. $x_2 = 30$. The consumption bundle $(x_1, x_2) = (16, 30)$ then yields utility of $u = 10(16^{0.5}) + 30 = 70$. The expenditure necessary to get to utility level $u = 70$ at before-tax prices (i.e. $p = 1$) is then

$$E(1,70) = 70 - \frac{25}{1} = 45. \tag{10.63}$$

You start with a weekly "income" of \$50 — but you only need \$45 to get to the post-tax utility. Thus, you would be willing to pay \$5 per week to avoid the tax; i.e. $L = 5$.

(e) *Verify your answer about L by checking that it is equal to the appropriate area on the $MWTP$ curve. (For this you need to take an integral, using material from the appendix).*

Answer: The compensated demand function for x_1 is given in equation (10.61) is our inverse $MWTP$ curve along which we measured L in part A of the question. This can then be expressed as

$$L = \int_1^{1.25} \frac{25}{p^2}\,dp = -\frac{25}{p}\Big|_1^{1.25} = 5. \tag{10.64}$$

(f) *How large is the weekly deadweight loss?*

Answer: The deadweight loss is the difference between L — what we could have raised in a lump sum way — and T — what we actually paid in taxes. Given our calculations above, $DWL = L - T = 5 - 4 = \$1$ per week.

(g) *Now suppose that my tastes were represented by $u(x_1, x_2) = x_1^{0.5} + x_2^{0.5}$. How would your answers change?*

Answer: Solving the utility maximization problem, we would get uncompensated demands

$$x_1 = \frac{50}{p(1+p)} \text{ and } x_2 = \frac{50p}{1+p}. \tag{10.65}$$

Solving the expenditure minimization problem, we would get compensated demands

$$x_1 = \left(\frac{u}{1+p}\right)^2 \text{ and } x_1 = \left(\frac{pu}{1+p}\right)^2. \tag{10.66}$$

At the before-tax price of \$1, this results in consumption of $x_1 = 50/(1(1+1)) = 25$ and $x_2 = 50(1)/(1+1) = 25$ — which is the same pre-tax consumption bundle as under the previous tastes. After the tax raises the price to \$1.25, the consumption bundle changes to

$x_1 = 50/(1.25(1+1.25)) \approx 17.78$ and $x_2 = 50(1.25)/(1+1.25) \approx 27.78$. The after-tax utility is therefore $u(17.78, 27.78) = (17.78)^{0.5} + 27.78^{0.5} \approx 9.487$.

The expenditure function, which we get by substituting the compensated demands from equation (10.66) into the expenditure equation $px_1 + x_2$, is now

$$E(p,u) = p\left(\frac{u}{1+p}\right)^2 + \left(\frac{pu}{1+p}\right)^2 = \frac{pu^2}{1+p}. \quad (10.67)$$

Thus, the expenditure necessary to get to post-tax utility of 9.487 at the pre-tax price of $1 is

$$E(1, 9.487) = \frac{(1)(9.487)^2}{1+1} = 45. \quad (10.68)$$

Given that you started with $50, the fact that it only costs $45 to get to post-tax utility at pre-tax prices means you are willing to pay $5 per week to avoid the tax on grits; i.e. $L = 5$. If the tax on grits (that raises the price to $1.25) goes into effect, then you will pay $0.25 in tax on each of the 17.18 servings of post-tax grits — which means you will pay approximately $4.44 in taxes per week; i.e. $T = 4.44$. This implies a deadweight loss of $DWL = L - T = 5 - 4.44 =$ $0.56 per week.

You can also verify L with the compensated demand curve by taking the integral between the before and after-tax prices; i.e.

$$L = \int_1^{1.25} \left(\frac{9.487}{1+p}\right)^2 dp = -\frac{9.487^2}{1+p}\Big|_1^{1.25} = 5. \quad (10.69)$$

(h) *Under these new tastes, suppose you only observed the regular demand curve and then used it to calculate deadweight loss while incorrectly assuming it was the same as the $MWTP$ curve. By what percentage would you be overestimating the deadweight loss? (*Hint*: You again need to evaluate an integral. Note that the integral of* $1/(p(1+p))$ *with respect to p is* $\ln p - \ln(1+p)$.*)*

Answer: If you assumed that that the uncompensated demand for x_1 was the $MWTP$ curve, you would evaluate the area (between the before and after-tax prices) on this uncompensated demand curve to get your approximation for L — i.e.

$$\int_1^{1.25} \left(\frac{50}{p(1+p)}\right) dp = 50\left(\ln p - \ln(1+p)\right)\Big|_1^{1.25} \approx 5.27. \quad (10.70)$$

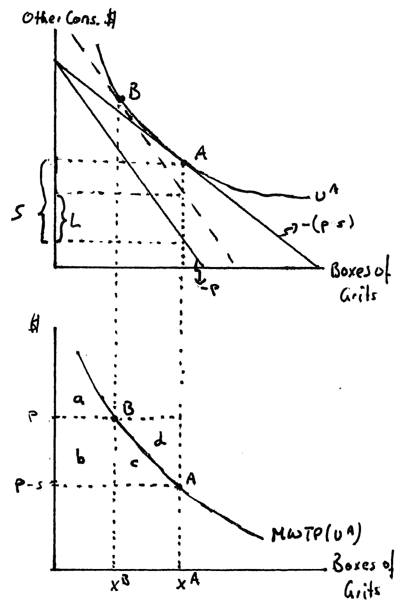
Since the true L is $5, you would be overestimating the weekly deadweight loss by $0.27. Put differently, you would get an approximate deadweight loss of 5.27-4.44=$0.83 per week when the true deadweight loss is $0.56 per week — an overestimate of almost 50%.

10.13 Policy Application: *Price Subsidies: Suppose the government decides to subsidize (rather than tax) consumption of grits.*

A: *Consider a consumer that consumes boxes of grits and "other goods".*

(a) *Begin by drawing a budget constraint (assuming some exogenous income) with grits on the horizontal axis and "other consumption" on the vertical. Then illustrate a new budget constraint with the subsidy — reflecting that each box of grits now costs the consumer less than it did before.*

Answer: This is illustrated in the top graph of Graph 10.15, with $(p-s)$ indicating the price with the subsidy and p indicating the price without.



Graph 10.15: Subsidizing Grits

(b) *Illustrate the optimal consumption of grits with an indifference curve tangent to the after-subsidy budget. Then illustrate in your graph the amount that the government spends on the subsidy for you. Call this amount S.*

Answer: The optimal consumption bundle is illustrated as bundle A. The vertical intercept of that bundle indicates how much in other consumption the consumer is able to afford given that grits are subsidized. Had they not been subsidized, a much lower amount (read off the no-subsidy budget) would be available for other consumption. The difference is S — the amount the government paid for this consumer under the price subsidy policy.

(c) *Next, illustrate how much the government could have given you in a lump sum cash payment instead and made you just as happy as you are under the subsidy policy. Call this amount L.*

Answer: The government could have chosen not to alter the price of grits (and thus not alter the slope of the no-subsidy budget line) but instead simply shift that budget out in a parallel way by giving a cash subsidy. The amount in cash the government could have given to make the consumer just as happy as she is under the price subsidy is then an amount that creates the dashed budget which is tangent to the post-subsidy indifference curve u^A.

This tangency occurs at bundle B, and the cost of this cash subsidy is simply the vertical difference between the dashed budget and the parallel no-subsidy budget. That distance can be measured anywhere (since the lines are parallel) and is indicated as the distance L.

(d) *Which is bigger — S or L?*

Answer: S is bigger than L because of the substitution effect from A to B. You don't have to give someone as much in unrestricted cash as you would have to spend in a subsidy that is restricted to the purchase of one good.

(e) *On a graph below the one you have drawn, illustrate the relevant MWTP curve and show where S and L can be found on that graph.*

Answer: The graph below the top graph derives the $MWTP$ or compensated demand curve that corresponds to utility level u^A. Under the price subsidy, the consumer consumes at A — which gives consumer surplus of $a+b+c$. The government is paying the difference between p and $(p-s)$ for each of the x^A boxes of grits the consumer buys — which means that the cost of the price subsidy is $S = b+c+d$.

Under the cash subsidy, the consumer faces the higher price p (rather than $(p-s)$) and buys x^B rather than x^A assuming she receives the cash subsidy L. This leaves her with consumer surplus of a in the grits market — but she is equally happy since both A and B lie on the same indifference curve. The only way she can be equally happy is if the cash subsidy was enough to make up for the loss in consumer surplus in the grits market — i.e. $L = b+c$.

(f) *What would your tastes have to be like in order for S to be equal to L.*

Answer: The substitution effect that creates the difference between S and L would have to disappear — which happens only if there is a sharp kink in the indifference curve at A (such as if grits and other goods are perfect complements). In that case, $A = B$ and $L = S$. In the lower graph, this implies that B lies directly above A — leading to a perfectly vertical $MWTP$ curve and the disappearance of the area d.

(g) *True or False: For almost all tastes, price subsidies are inefficient.*

Answer: This is true — so long as there aren't sharp kinks in just the right places of indifference curves — i.e. so long as goods are somewhat substitutable at the margin, $S > L$ which leaves the difference as a deadweight loss. If the substitutability goes away, so does the deadweight loss triangle d as the $MWTP$ curve becomes vertical.

B: *Suppose the consumer's tastes are Cobb-Douglas and take the form $u(x_1,x_2) = x_1^{\alpha}x_2^{(1-\alpha)}$ where x_1 is boxes of grits and x_2 is a composite good with price normalized to 1. The consumer's exogenous income is I.*

(a) *Suppose the government price subsidy lowers the price of grits from p to $(p-s)$. How much S will the government have to pay to fund this price subsidy for this consumer?*

Answer: We need to solve the problem

$$\max_{x_1,x_2} x_1^{\alpha}x_2^{(1-\alpha)} \text{ subject to } px_1 + x_2. \tag{10.71}$$

This solves to

$$x_1 = \frac{\alpha I}{p} \text{ and } x_2 = (1-\alpha)I. \tag{10.72}$$

When the government lowers the price to $(p-s)$, demand is $x_1 = \alpha I/(p-s)$. For each box of grits, the consumer pays $(p-s)$ while the government pays s. Thus, the government's expense is

$$S = \frac{s\alpha I}{p-s}. \tag{10.73}$$

(b) *How much utility does the consumer attain under this price subsidy?*

Answer: Under the price subsidy, the consumer chooses the bundle $(x_1,x_2) = (\alpha I/(p-2),(1-\alpha)I)$. Substituting this into the utility function, we get the indirect utility function

$$V(p,s) = \left(\frac{\alpha I}{p-s}\right)^{\alpha}((1-\alpha)I)^{(1-\alpha)} = \frac{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}{(p-s)^{\alpha}}I. \tag{10.74}$$

(c) *How much L would the government have had to pay this consumer in cash to make the consumer equally happy as she is under the price subsidy?*

Answer: Given that we know how much utility the consumer gets under the price subsidy, we now have to ask what expenditure (in cash) would have been necessary to get to the same utility level at the non-subsidized price p. We can derive the expenditure function by solving the expenditure minimization problem

$$\min_{x_1,x_2} px_1 + x_2 \text{ subject to } u = x_1^{\alpha} x_2^{(1-\alpha)}, \tag{10.75}$$

and then plug the compensated demands into the objective $px_1 + x_2$, or we can simply invert the indirect utility function. Either way, we get

$$E(p,u) = \frac{p^{\alpha}}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}. \tag{10.76}$$

We are interested in knowing the expenditure necessary at p to get to utility level $V(p,s)$ from equation (10.74); i.e. we are interested in

$$E(p,V(p,s)) = \left(\frac{p^{\alpha}}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}\right)\left(\frac{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}{(p-s)^{\alpha}} I\right) = \left(\frac{p}{p-s}\right)^{\alpha} I. \tag{10.77}$$

This is the total expenditure necessary to get to the price-subsidy utility level $V(p,s)$. Since the consumer starts with an income I, the amount of cash L the consumer would have to get to be equally happy as under the price subsidy would therefore be

$$L = E(p,V(p,s)) - I = \left[\left(\frac{p}{p-s}\right)^{\alpha} - 1\right] I. \tag{10.78}$$

(d) *What is the deadweight loss from the price subsidy?*

Answer: The deadweight loss is then just the difference between what the government spends under the price subsidy (S) and what the government could have spent in a lump sum way (L) to make the consumer just as well off. This is

$$DWL = S - L = \left[1 + \frac{s\alpha}{p-s} - \left(\frac{p}{p-s}\right)^{\alpha}\right] I. \tag{10.79}$$

(e) *Suppose $I = 1000$, $p = 2$, $s = 1$ and $\alpha = 0.5$. How much grits does the consumer buy before any subsidy, under the price subsidy and under the utility-equivalent cash subsidy? What is the deadweight loss from the price subsidy?*

Answer: We have calculated that the demand for grits is $x_1 = \alpha I/p$. Thus, when the price is unsubsidized originally, the consumer buys $x_1 = 0.5(1000)/2 = 250$. The price subsidy lowers the effective price for the consumer to 1 — which implies the new quantity demanded is $x_1 = 0.5(1000)/1 = 500$. To calculate the equivalent cash subsidy, we can use equation (10.78) to get

$$L = \left[\left(\frac{2}{2-1}\right)^{0.5} - 1\right](1000) \approx 414.21. \tag{10.80}$$

The consumer's income under the cash subsidy would therefore rise to \$1,414.21 but the price would remain at $p = 2$. The consumer's demand would therefore be $x_1 = 0.5(1414.21)/2$ which is approximately 354.

Finally, the deadweight loss is simply $(S - L)$. The cost of the price subsidy, given that the consumer will demand 500 units of x_1 and the cost of the subsidy is \$1 per unit, is \$500. The deadweight loss is therefore 500-414.21=\$85.79. You can also get this by simply plugging the relevant values into the DWL equation we calculated in equation (10.79); i.e.

$$DWL = \left[1 + \frac{1(0.5)}{2-1} - \left(\frac{2}{2-1}\right)^{0.5}\right](1000) \approx 85.79. \tag{10.81}$$

(f) *Continue with the values from he previous part. Can you calculate the compensated demand curve you illustrated in A(e) and verify that the area you identified as the deadweight loss is equal to what you have calculated? (*Hint*: You need to take an integral and use some of the material from the appendix to answer this.)*

Answer: The compensated demand curve arises from the expenditure minimization problem

$$\min_{x_1,x_2} px_1 + x_2 \text{ subject to } u = x_1^{0.5}x_2^{0.5}. \quad (10.82)$$

From this, we get compensated demands

$$x_1 = \frac{u}{p^{0.5}} \text{ and } x_2 = p^{0.5}u. \quad (10.83)$$

Using the indirect utility function $V(p,s)$ from equation (10.74), we can determine the utility the consumer gets under the price subsidy as

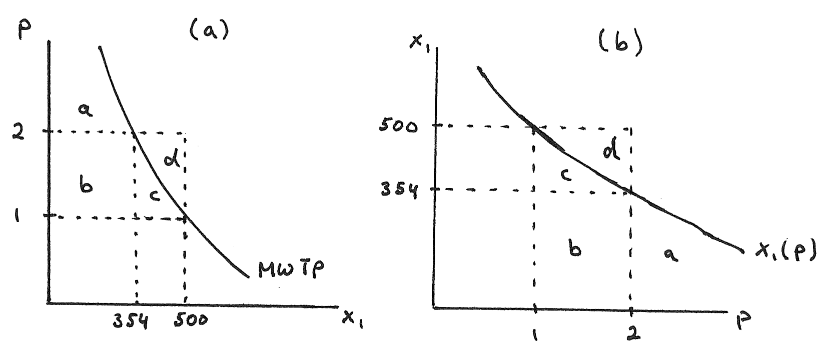
$$V(2,1) = \left(\frac{0.5^{0.5}0.5^{0.5}}{(2-1)^{0.5}}\right)(1000) = 500. \quad (10.84)$$

The appropriate compensated (or $MWTP$) curve is then $x_1 = 500/(p^{0.5})$. The inverse of this function is sketched in panel (a) of Graph 10.16 — it is similar to the lower graph in Graph 10.15 where we indicated that $S = b+c+d$, $L = b+c$ and $DWL = d$. Panel (b) of Graph 10.16 then simply inverts panel (a), placing x_1 on the vertical (rather than the horizontal) and p on the horizontal (rather than the vertical) axes.

The area $L = c + b$ is then simply the integral under the function $x_1 = 500/(p^{0.5})$ evaluated from $p = 1$ to $p = 2$. This is

$$\int_1^2 \frac{500}{p^{0.5}}\,dp = 2(500)p^{0.5}|_1^2 = 1000(2^{0.5}-1) \approx 414.21. \quad (10.85)$$

This is exactly what we calculated for L in the previous part. The area $S = b+c+d$ is simply 500. Thus, the deadweight loss is $DWL = 500-414.21 = 85.79$, again exactly as we calculated before.

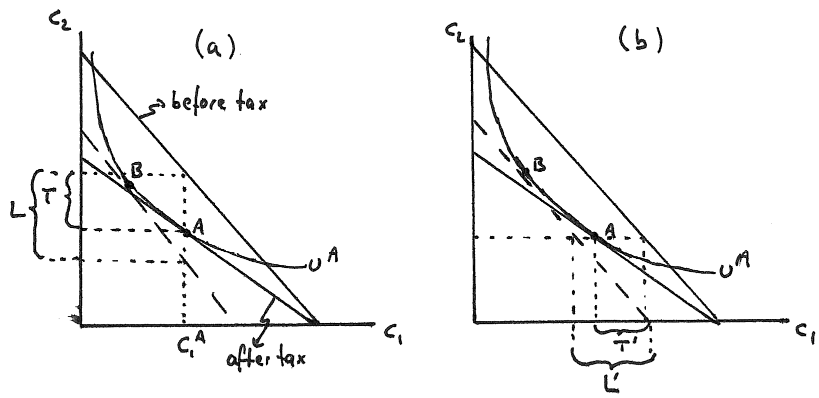


Graph 10.16: Subsidizing Grits: Part 2

10.14 Policy Application: *Taxing Interest on Savings*: *Suppose I care only about consumption this year and consumption next year, and suppose I earn an income this year but do not expect to earn an income next year.*

A: *The government announces an increase in the tax on interest income. Illustrate my before and after tax intertemporal budget constraint.*

Answer: The before and after-tax budgets are illustrated in panel (a) of Graph 10.17.



Graph 10.17: Tax on Interest Income

(a) *Suppose I save 50% of my income after the new tax is imposed. Illustrate the amount of the tax the government will collect from me next year. Call this T.*

Answer: This is also illustrated in panel (a) of the graph. Given the consumption level c_1^A that I consume this period after the tax, the height of A illustrates the amount I will have left over for consumption next period. Had I consumed c_1^A prior to the tax increase, I could have consumed all the way up to the before-tax budget. The difference is how much I will pay in tax next period when the government taxes my interest income. This is labeled T on the vertical axis.

(b) *Illustrate the most I would be willing to pay next year to keep the government from imposing this tax on interest income. Call this amount L.*

Answer: To determine this, we have to ask where I would consume if the interest rate remained at the before-tax levels but I simply shifted the budget in until it was tangent to u^A. This occurs at B in panel (a) of Graph 10.17. The most I would pay next year to get rid of the tax is therefore an amount that makes me just as well off as I would be under the tax. It is the difference between the two parallel lines — indicated as L on the vertical axis.

(c) *Is L larger or smaller than T? What does your answer depend on?*

Answer: L is larger than T because of the substitution effect from A to B. The difference increases as consumption this period becomes more substitutable with consumption next period — and the difference does not disappear unless we have a sharp kink point on the indifference curve at A (such as if consumption across the two periods were perfectly complementary).

(d) *If consumption is always a normal good, will I consume more or less next year if the tax on interest income is removed?*

Answer: I will consume more. This is because, in terms of consumption next period, both the substitution and the wealth effect point in the direction of greater consumption next year when the interest rate increases. (The substitution effect from A to B points to greater consumption next period, and the additional wealth effect due to a shift from the dashed budget in the graph to the before-tax budget also points toward more consumption next period so long as consumption is a normal good.)

(e) *If consumption is always a normal good, will I consume more or less* this *year if the tax on interest income is eliminated?*

Answer: It depends on the size of the substitution effect relative to the wealth effect. An increase in the effective interest rate causes an unambiguous substitution effect in the direction of consuming less this year (at B versus A in the graph). But the wealth effect from the dashed budget to the before-tax budget points in the opposite direction when consumption is a normal good.

(f) *Can you re-draw your graph but this time indicate how much T' you are paying in taxes in terms of* this *year's consumption — and how much L' you would be willing to pay to avoid the tax in terms of* this *year's consumption?*

Answer: This is done in panel (b) of Graph 10.17.

B: *Now suppose that my tastes over consumption now, c_1, and consumption next period, c_2, can be captured by the utility function $u(c_1,c_2) = c_1^{\alpha} c_2^{(1-\alpha)}$.*

(a) *Suppose the interest rate is r. What does α have to be in order for me to optimally save 50% of my income this year?*

Answer: We first need to solve

$$\max_{c_1,c_2} c_1^{\alpha} c_2^{(1-\alpha)} \text{ subject to } (1+r)(I-c_1) = c_2 \tag{10.86}$$

where I is this year's income.[2] Solving this, we get

$$c_1 = \alpha I \text{ and } c_2 = (1-\alpha)(1+r)I. \tag{10.87}$$

In order for it to be optimal to save half my current income, it has to be optimal for me to consume half my current income now; i.e. it has to be the case that $c_1 = 0.5I$. Thus, it must be that $\alpha = 0.5$.

(b) *Assume from now on that α is as you calculated above and suppose that my current income is \$200,000. Suppose the interest rate before the tax increase was 10% and the after-tax interest rate after the tax increase is 5%. How much tax revenue T does the government collect from me next period? What is the present value of that this period?*

Answer: My consumption this period is then $c_1 = 0.5(200{,}000) = 100{,}000$ regardless of the interest rate. Next period, my consumption is $c_2 = 0.5(1+r)200{,}000 = 100{,}000(1+r)$. At the after tax interest rate, that's $100{,}000(1+0.05) = 105{,}000$, and at the before tax interest rate, it is $100{,}000(1+0.1) = 110{,}000$. Thus, the government collected \$5,000 from me in the second period; i.e. $T = 5{,}000$.

The present value of \$5,000 next period is $\$5{,}000/1.1 \approx \$4{,}545$.

(c) *What is the most (L) I would be willing to pay to avoid this tax increase (in either today's dollars or in next period's dollars)?*

Answer: To know what the most is that I would be willing to pay to avoid the tax, I first have to know how much utility I will get if the tax is put in place. If the tax does go into effect, we have calculated that I will consume the bundle $(c_1,c_2) = (100000, 105000)$ which gives utility $u = (100{,}000)^{0.5}(105{,}000)^{0.5} \approx 102{,}469.5$. Next, we need to calculate the expenditure necessary to get to that utility level at an interest rate of 10%. We could express that expenditure in two ways — in terms of dollars in period 2 or in terms of dollars in period 1. If we do the latter, the problem becomes

$$\min_{c_1,c_2} c_1 + \frac{c_2}{(1+0.1)} \text{ subject to } c_1^{0.5} c_2^{0.5} = 102469.5, \tag{10.88}$$

[2] You could also specify the constraint in today's dollars rather than next period's dollars. In that case, the constraint would be $I - c_1 = c_2/(1+r)$. Your answer will come out the same way regardless of which way you specify the constraint.

and if we do the former, it becomes

$$\min_{c_1,c_2} (1+0.1)c_1 + c_2 \text{ subject to } c_1^{0.5}c_2^{0.5} = 102469.5, \quad (10.89)$$

Solving either problem gives us compensated consumption demands of

$$c_1 = 102469.5(1+0.1)^{-0.5} \approx 97,001 \text{ and } c_2 = 102469.5(1+0.1)^{0.5} \approx 107,471. \quad (10.90)$$

Plugging this into the objective function in problem (10.88) — i.e. $c_1 + c_2/(1+0.1)$, we get approximately \$195,402. Thus, the expenditure necessary to get to the after-tax utility level in today's dollars is \$195,402 — which is \$4,598 less than the \$200,000 in current income I have. Put differently, I would be willing to pay \$4,598 *today* in order to avoid the tax on interest income.

Alternatively, we can plug the compensated demands into the objective function from problem (10.89) — i.e. $(1+0.1)c_1 + c_2$ — to get approximately \$214,942. Our current income of \$200,000 is worth \$220,000 in next period's dollars (given that the interest rate is 10%) — which implies that in next period's dollars, I would be willing to pay \$5,058 to avoid the tax on interest income.

Thus, I would be willing to pay \$4,598 now or \$5,058 next period to avoid this tax. Notice that this is exactly the same answer — because \$4,598 now is worth \$4,598(1.1)=\$5,058 next period given that the interest rate is 10%.

(d) *Does the amount that I save today change as a result of the tax increase?*

Answer: We calculated in (a) that my current consumption is αI and thus does not depend on the interest rate. Given that my income is \$200,000 and $\alpha = 0.5$, that means I will consume \$100,000 now and save \$100,000 for the future both before and after the tax increase.

(e) *Is the tax efficient? If not, how big is the deadweight loss?*

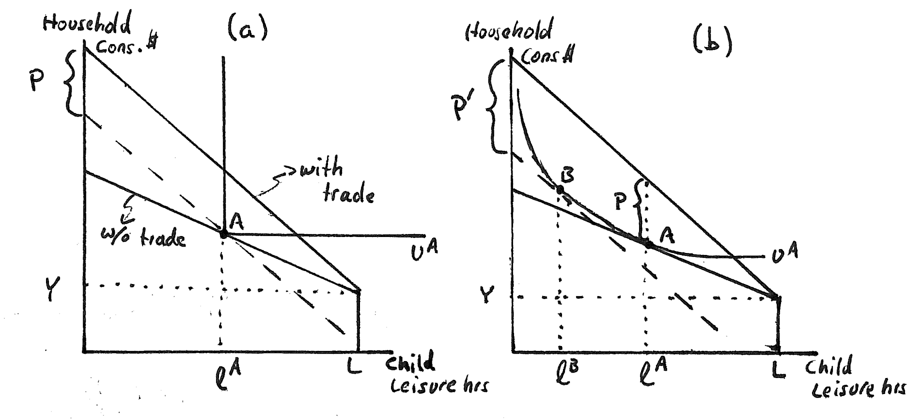
Answer: No, the tax is inefficient. We calculated above that I would be willing to pay \$5,058 next period in order to avoid the tax, but we also calculated that the government only collects \$5,000 in tax next period. Thus, I would be willing to pay \$58 more (in terms of next period dollars) than I end up paying under the tax on interest income — creating a deadweight loss of \$58 in terms of next period dollars. Alternatively, we calculated that I would be willing to pay \$4,598 right now to avoid the tax. The actual tax I have to pay in terms of current dollars is $\$5000/(1.1) \approx \$4,545$. Thus, I would be willing to pay about \$53 more now to avoid the tax in today's dollars. Note that the present discounted value of \$58 next period is $\$58/1.1 \approx \$53$. Thus, the deadweight loss can be expressed either as \$53 now or \$58 in terms of next period's dollars.

10.15 Policy Application: *International Trade and Child Labor: Consider again the end-of-chapter problem 8.9 about the impact of international trade on child labor in the developing world.*

A: *Suppose again that households have non-child income Y, that children have a certain weekly time endowment L, and that child wages are w in the absence of trade and $w' > w$ with trade.*

(a) *On a graph with child leisure hours on the horizontal axis and household consumption on the vertical, illustrate the before and after trade household budget constraints.*

Answer: This is illustrated in panel (a) of Graph 10.18.



Graph 10.18: Child Leisure and Trade

(b) *Suppose that tastes over consumption and child leisure were those of perfect complements. Illustrate in your graph how much a household would be willing to pay to permit trade — i.e. how much would a household be willing to pay to increase the child wage from w to w'?*

Answer: This is also illustrated in panel (a) of Graph 10.18. The household would be willing to give up an amount P that makes it just as well off as it would be in the absence of trade. In the absence of trade, the household optimizes at A — leaving the child with leisure of ℓ^A. Paying a lump sum amount to get the higher wage is equivalent to shifting the after-trade (steeper) budget inward in a parallel way — and to figure out the highest payment the household is willing to make, we shift this budget until it just barely reaches the original indifference curve u^A. Given the shape of the perfect complements indifference curve, this means we shift the after-trade budget until it intersects at A. We can then read the size of the payment P as the vertical distance between the parallel lines. This distance is indicated on the vertical axis of the graph.

(c) *If the household paid the maximum it was willing to pay to cause the child wage to increase, will the child work more or less than before the wage increase?*

Answer: The child will continue to work the same amount — i.e. it will receive leisure of ℓ^A which implies that it's labor supply has not changed.

(d) *Re-draw your graph, assume that the same bundle (as at the beginning of part (b)) is optimal, but now assume that consumption and leisure are quite (though not perfectly) substitutable. Illustrate again how much the household would be willing to pay to cause the wage to increase.*

Answer: This is illustrated in panel (b) of Graph 10.18. We can now shift the after-trade budget inward until it is tangent to B — which puts it below A. The amount that the household is willing to pay is indicated as the vertical distance between the parallel lines on the vertical axis.

(e) *If the household actually had to pay this amount to get the wage to increase, will the child end up working more or less than before trade?*

Answer: The child will now work more — because it's leisure time has fallen from ℓ^A to ℓ^B — which means it's labor supply has increased by the same amount. (This is due to the emergence of the substitution effect that was absent in panel (a)).

(f) *Does your prediction of whether the child will work more or less if the household pays the maximum bribe to get the higher wage depend on how substitutable consumption and child leisure are?*

Answer: No — it does not depend on how substitutable. Even the slightest degree of substitutability in the indifference curve will cause B to lie to the left of A — which implies an increase in the child's labor supply. This goes away only if the substitutability is assumed away completely — and it never goes in the other direction (because substitution effects always point in the same direction).

(g) *Can you make a prediction about the relative size of the payment the household is willing to make to get the higher child wage as it relates to the degree of substitutability of consumption and child leisure? Are "good" parents willing to pay more or less?*

Answer: The size of the payment a household is willing to make to get the higher wages for the child increases with the degree of substitutability of household consumption with child leisure. You can see this by comparing P to P' in the graphs. In panel (b) of the graph, P is illustrated as the vertical difference between A and the after-trade budget (which is the same vertical distance found in (a) under perfect complementarity). This is unambiguously smaller than P' because B unambiguously falls to the left of A when there is some substitutability between consumption and child leisure. We concluded in the chapter 8 exercise that "good parents" (i.e. those that reduce child labor as child wages increase in the absence of a required payment to make that increase happen) are those who view consumption and child leisure as not very substitutable — parents that find it difficult to think of the household being better off if the child is not also working less. We are now finding that such "good parents" (i.e. those more like what is graphed in panel (a)) are not willing to pay as much of a bribe to get child wages to increase as "bad parents" are.

B: *Suppose that the household's tastes over consumption and leisure can be represented by the CES utility function* $u(c,\ell) = (\alpha c^{-\rho} + (1-\alpha)\ell^{-\rho})^{-1/\rho}$.

(a) *Derive the optimal household consumption and child leisure levels assuming the household has non-child weekly income Y, the child has a weekly time endowment of L, and the child wage is w.*

Answer: We have to solve the problem

$$\max_{c,\ell} \left(\alpha c^{-\rho} + (1-\alpha)\ell^{-\rho}\right)^{-1/\rho} \text{ subject to } Y + w(L-\ell) = c. \tag{10.91}$$

This gives us

$$c = \left(\frac{\alpha w}{(1-\alpha)}\right)^{1/(\rho+1)} (Y+wL)\left[w + \left(\frac{\alpha w}{(1-\alpha)}\right)^{1/(\rho+1)}\right]^{-1} \tag{10.92}$$

and

$$\ell = (Y+wL)\left[w + \left(\frac{\alpha w}{(1-\alpha)}\right)^{1/(\rho+1)}\right]^{-1}. \tag{10.93}$$

(b) *Verify your conclusion from end-of-chapter problem 8.9 that parents are neither "good" nor "bad" when $Y = 0$ and $\rho = 0$; i.e. parents will neither increase nor decrease child labor when w increases.*

Answer: When $Y = 0$ and $\rho = 0$, the child leisure expression becomes

$$\ell = wL\left[w + \left(\frac{\alpha w}{(1-\alpha)}\right)\right]^{-1} = wL\left[\frac{w}{(1-\alpha)}\right]^{-1} = (1-\alpha)L. \tag{10.94}$$

Child leisure and thus child labor therefore do not depend on w, which implies an increase in child wages will result in neither an increase nor a decrease in child labor.

(c) *If international trade raises household income Y, what will happen to child labor in the absence of any change in child wages? Does your answer depend on how substitutable c and ℓ are?*

Answer: Since Y enters the leisure demand function above positively, chid leisure will increase — i.e. child labor will fall. The answer does not qualitatively depend on the value of ρ — and thus does not depend on the elasticity of substitution.

(d) *When $\alpha = 0.5$ and $w = 1$, does your answer depend on the household elasticity of substitution between consumption and child leisure?*

Answer: When $\alpha = 0.5$ and $w = 1$, the expressions for optimal consumption and leisure reduce to

$$c = \frac{Y+L}{2} \text{ and } \ell = \frac{Y+L}{2}. \tag{10.95}$$

The answer therefore does not depend on the household elasticity of substitution.

(e) *How much utility will the household get when $\alpha = 0.5$ and $w = 1$?*

Answer: The utility is then given by

$$u\left(\frac{Y+L}{2}, \frac{Y+L}{2}\right) = \left[0.5\left(\frac{Y+L}{2}\right)^{-\rho} + 0.5\left(\frac{Y+L}{2}\right)^{-\rho}\right]^{-1/\rho} = \frac{Y+L}{2}. \tag{10.96}$$

(f) *Derive the expenditure function for this household as a function of w and u. What does this reduce to when $\alpha = 0.5$? (Hint: You can assume $Y = 0$ for this part.)*

Answer: We now need to solve the problem

$$\min_{c,\ell} \; w\ell + c \text{ subject to } u = (\alpha c^{-\rho} + (1-\alpha)\ell^{-\rho})^{-1/\rho}. \tag{10.97}$$

This gives compensated demands

$$c = \left(\frac{\alpha w}{(1-\alpha)}\right)^{1/(\rho+1)} \left[\alpha\left(\frac{\alpha w}{(1-\alpha)}\right)^{-\rho/(\rho+1)} + (1-\alpha)\right]^{1/\rho} u \tag{10.98}$$

and

$$\ell = \left[\alpha\left(\frac{\alpha w}{(1-\alpha)}\right)^{-\rho/(\rho+1)} + (1-\alpha)\right]^{1/\rho} u. \tag{10.99}$$

Plugging these into the expenditure expression $w\ell + c$, we get the expenditure function

$$E(w,u) = \left[w + \left(\frac{\alpha w}{(1-\alpha)}\right)^{1/(\rho+1)}\right] \left[\alpha\left(\frac{\alpha w}{(1-\alpha)}\right)^{-\rho/(\rho+1)} + (1-\alpha)\right]^{1/\rho} u. \tag{10.100}$$

When $\alpha = 0.5$, this reduces to

$$E(w,u) = \left(w + w^{1/(\rho+1)}\right)\left(0.5 w^{-\rho/(\rho+1)} + 0.5\right)^{1/\rho} u. \tag{10.101}$$

(g) *Suppose non-child income $Y = 0$, child time is $L = 100$, $\alpha = 0.5$, $\rho = 1$ and w is initially 1. Then international trade raises w to 2. How does the household respond in its allocation of child leisure?*

Answer: We calculated in (d) that, when $\alpha = 0.5$ and $w = 1$, consumption and leisure are both $(Y + L)/2$ regardless of ρ. When $Y = 0$ and $L = 100$, this implies that initial household consumption and child leisure will both be equal to 50, leaving the child to work 50 hours per week. When $\alpha = 0.5$, $Y = 0$, $L = 100$ and $\rho = 1$, the optimal household consumption from equation (10.92) and the optimal child leisure from equation (10.93) become

$$c = \frac{100w^{3/2}}{w + w^{1/2}} \text{ and } \ell = \frac{100w}{w + w^{1/2}}. \tag{10.102}$$

Plugging in $w = 2$, we then get $c \approx 82.84$ and $\ell \approx 58.58$. Household allocation of child leisure therefore increased by 8.58 hours, and child labor falls from 50 to 41.42.

(h) *Using your expenditure function, can you determine how much the household would be willing to pay to cause child wages to increase from 1 to 2? If it did in fact pay this amount, how would it change the amount of child labor?*

Answer: To determine how much the household would be willing to pay to induce an increase in child wages from 1 to 2, we first have to know household utility at the original wage $w = 1$. From our answer to (e), we see that household utility when $\alpha = 0.5$ and $w = 1$ is independent of ρ and equal to $(Y + L)/2$ which reduces to 50 when $Y = 0$ and $L = 100$. The most the household would be willing to pay to cause an increase in child wages from 1 to 2 is therefore an amount that would, once the wages increase, leave the household with utility of 50.

The expenditure necessary to attain utility of 50 at $w = 2$ is given by the expenditure function. In our answer to (f), we calculated this for the case when $\alpha = 0.5$ in equation (10.101). When $\rho = 1$, this reduces to $E(w,u) = (0.5w + w^{0.5} + 0.5)u$. Evaluating this at $w = 2$ and $u = 50$, we get $E(2,50) \approx 145.71$. The value of the household's endowment — which is just the value of the child's leisure since $Y = 0$ — at $w = 2$ is 2(100)=200. The most the household is willing to pay for the child wage to increase from 1 to 2 is therefore 200-145.71=54.29.

If the household did in fact pay 54.29 to cause child wages to increase from 1 to 2, we can again use equation (10.93) to determine the new optimal level of child leisure. When $\alpha = 0.5$ and $\rho = 1$, this equation becomes

$$\ell = (Y + wL)\left(w + w^{1/2}\right)^{-1}. \tag{10.103}$$

The payment of 54.29 is then a lump sum negative amount that can be inserted in place of Y. Substituting this, and substituting $w = 2$ and $L = 100$, we get

$$\ell = (-54.29 + 2(100))\left(2 + 2^{1/2}\right)^{-1} \approx 42.68. \tag{10.104}$$

This implies that child labor would increase to 100-42.68=57.32 hours.

(i) *Repeat the two previous steps for the case when $\rho = -0.5$ instead of 1.*

Answer: None of our answers for $w = 1$ change when ρ changes as illustrated in previous parts of the question. Thus, initially the household will again choose $c = 50$, $\ell = 50$ and attain utility $u = 50$, with the child working 50 hours per week.

To calculate how the household decision will change when wage increases to 2, we can again use equations (10.92) and (10.93). When $\alpha = 0.5$, $L = 100$, $Y = 0$ and $\rho = -0.5$, these reduce to

$$c = \frac{100w^2}{1 + w} \text{ and } \ell = \frac{100}{1 + w}, \tag{10.105}$$

and when $w = 2$, this implies $c = 133.33$ and $\ell = 33.33$. Thus, child labor *increases* from 50 when $w = 1$ to $100 - 33.33 = 66.66$ when $w = 2$.

To determine how much the household would be willing to pay to cause wages to increase in this way, we can again use the expenditure function. When $\alpha = 0.5$ and $\rho = -0.5$, this reduces to

$$E(w,u) = \left(w + w^2\right)(0.5w + 0.5)^{-2}\, u. \tag{10.106}$$

Since the household's utility at $w = 1$ is 50, we need to calculate

$$E(2,50) = \left(2 + 2^2\right)(0.5(2) + 0.5)\,(50) = 133.33. \tag{10.107}$$

Since the value of the household endowment is 2(100)=200 when $w = 2$, the household would therefore be willing to pay up to $200 - 133.33 = 66.67$ in order to get the wage to increase from 1 to 2.

Finally, we can determine the amount of child labor (if the household actually paid 66.67 to cause an increase in the wage) by returning to equation (10.93) which, when $\alpha = 0.5$ and $\rho = -0.5$, reduces to

$$\ell = \frac{Y + wL}{w + w^2}. \tag{10.108}$$

Since the payment of 66.67 is equivalent to a negative household income, we can set Y to -66.67. Substituting this, and letting $L = 100$, we then get $\ell = 22.22$. Thus, child labor would increase to $100 - 22.22 = 77.78$ hours per week.

(j) *Are your calculations consistent with your predictions in (f) and (g) of part A of the question?*

Answer: Yes. In (f), we predicted the following: When parents pay the most they are willing to pay to open trade and raise child wages, children will work more than they did before — and the amount they will work more increases the more substitutable consumption and child leisure are. When ρ changes from 1 to -0.5, it causes the elasticity of substitution to increase (from 0.5 to 2). We have shown that at $\rho = 1$ (when the elasticity of substitution is low), child labor increases from 50 to 57.32 when parents have to pay the maximum bribe to get the higher child wage; and at $\rho = -0.5$ (when the elasticity of substitution is high), child labor increases from 50 to 77.78.

In (g), we predicted that the size of the payment a household is willing to make to get higher child wages increases as consumption and child leisure become more substitutable. We have shown that this payment increases from 54.29 to 66.67 when ρ falls from 1 to -0.5 — i.e. when the elasticity of substitution increases from 0.5 to 2.

10.16 Policy Application: *Efficient Land Taxes*: *We have argued in this chapter that it is difficult to find taxes that are efficient — i.e. taxes that do not give rise to a deadweight loss. Economists have long pointed to one exception to this proposition: taxation of land.*

A: *Suppose a particular plot of commercial land generates approximately $10,000 in income for its owner each year into the foreseeable future.*

(a) *Assuming an annual interest rate of 10%, what is the most that you would be willing to pay for this land? (*Hint*: Recall from our Chapter 3 exercises that the present discounted value of an annual stream of income of y is y/r where r is the annual interest rate.)*

Answer: The most you would be willing to pay for this land is the net present value of the future income; i.e. $10000/(0.1) = \$100,000$.

(b) *Now suppose the government announces that, from now on, it will impose a 50% tax on all income derived from land. How does your answer regarding how much you would be willing to pay for this plot of land change?*

Answer: This changes the net-of-tax annual income from $10,000 to $5,000. The value of the land for you will therefore fall to $5000/(0.1) = \$50,000$.

(c) *If you currently own this land, how are you affected by this tax? Is there any way you can change your behavior and avoid some portion of the tax; i.e. are there any substitution effects that might arise to create a deadweight loss?*

Answer: If you currently own this land, the value of what you own has just fallen from $100,000 to $50,000. Since the value falls immediately (because buyers will immediately re-calculate the value of the land given they know that they will have to pay taxes on future income derived from the land), the tax on future land income therefore implies an immediate loss in wealth for landowners. In effect, the land owner immediately incurs the present value of future tax payments. There is no way for you as the owner of the land to avoid this: If you sell the land, you will sell it for $50,000 less than you could have sold it before the land tax was announced; if you hold onto the land, you will, in present value terms, end up paying $50,000 in taxes on income as you derive it from the land. If it was optimal before the tax to use the land to generate $10,000 in annual income, that remains optimal even though you now only get to keep half of it — half of the most you can make from the land is still the best you can do. So there is simply nothing you can do to avoid paying the $50,000 tax.

(d) *If you currently don't own this land but are about to buy it, how are you affected by the imposition of this land tax?*

Answer: You are not affected. Before the tax, you would have had to pay $100,000 to buy the land. Now you only have to pay the owner $50,000, but you also take on future tax obligations whose present value is $50,000. Thus, you are still paying $100,000 overall to buy the land.

(e) *True or False*: *Regardless of whether the current owner of the land keeps it or sells it to me (after the announcement of the tax), the current owner effectively pays all future taxes associated with income from this land.*

Answer: This is true — the current owner incurs an immediate loss of $50,000 if he sells the land to me, or alternatively he assumes tax obligations whose present value is $50,000. (The same is true in present value terms if he holds onto the land for a few more years and then sells it.)

(f) *In light of your answers above, how is this an example of an efficient lump sum tax?*

Answer: This is equivalent to a lump sum tax of $50,000 on the current owner of the land — because no matter what the owner does after the tax is imposed, he will end up with a reduction of $50,000 (in today's dollars) in terms of his wealth. No opportunity costs are affected — it is simply a loss in wealth.

B: *Consider the more general case where a particular plot of land yields* $\$y$ *in annual income.*

(a) *What is the value of this land assuming an interest rate of r?*

Answer: This implies that land value is equal to y/r.

(b) *Now suppose the government announces a tax rate t (with $0 < t \leq 1$) that will be levied on income obtained purely from land. What happens to the value of the plot of land?*

Answer: This changes the land value to $(1-t)y/r$.

(c) *Who is affected by this — current land owners or future land owners?*

Answer: Only current land owners are affected — the value of what they own declines from y/r to $(1-t)y/r$ — i.e. they lose ty/r in wealth. While future land owners now have to pay less for land when they purchase it, they end up taking on a stream of tax obligations whose present value is ty/r — the exact amount that they have to pay less for obtaining the land.

(d) *Suppose the government decides to set $t=1$ — i.e. it announces that it will from now on tax income from land at 100%. What happens to the price of land?*

Answer: The price of land falls to $(1-1)y/r=0$.

(e) *Defend the following statement: A 100% tax on income from land is equivalent to the government confiscating land and asking for annual rental payments — with the present value of all future rental payments equal to the previous price of the land.*

Answer: First, the imposition of a 100% tax on income from land drives the value (or price) of land to zero. Thus, current landowners lose the entire portion of their wealth that lies in land — i.e. the government has confiscated their land wealth. Whoever owns the land — whether previous landowners or new ones that "purchase" the land for a price of zero, is now obligated to pay the government an annual amount y — the entire income derived from the land that year. Thus, those who own land in essence rent it from the government at y per year. The present value of these future tax payments is y/r — exactly the amount that land was priced at before the tax was imposed.