SOLUTIONS

11

One Input and One Output: A Short-Run Producer Model

Solutions for *Microeconomics: An Intuitive Approach with Calculus*

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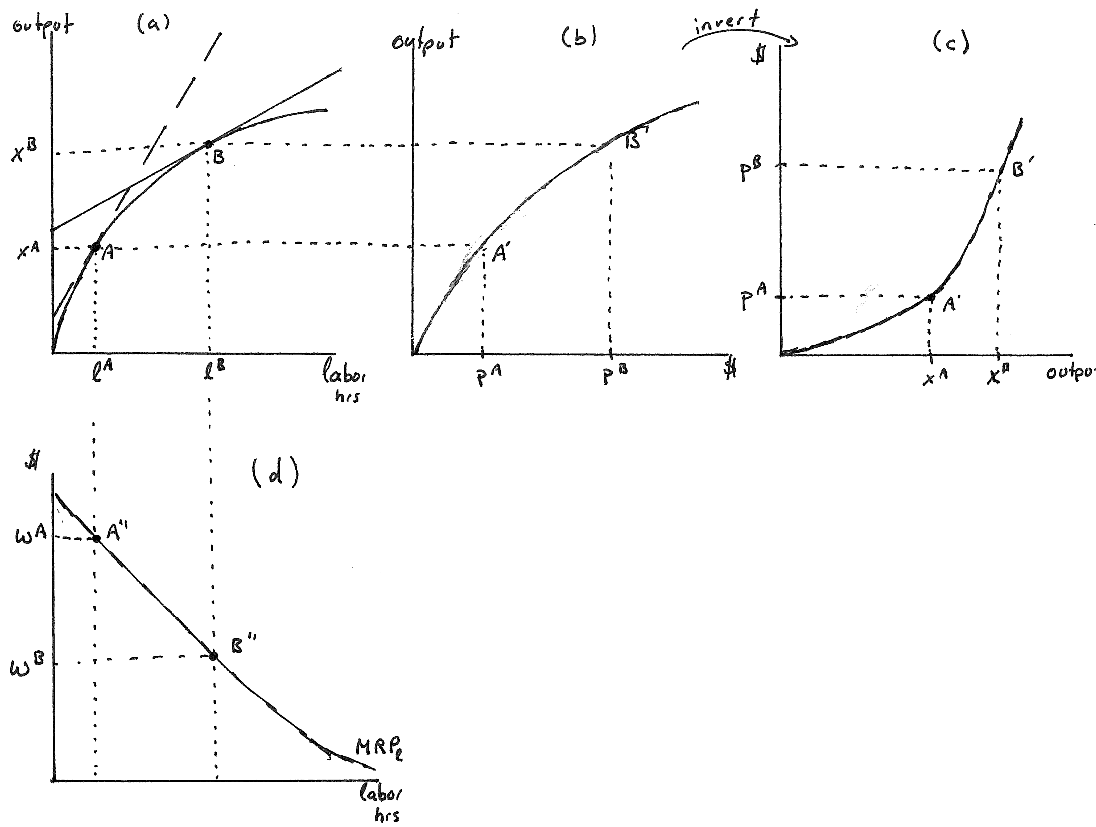
- Each end-of-chapter exercise begins on a new page. This is to facilitate maximum flexibility for instructors who may wish to share answers to some but not all exercises with their students.
- If you are assigning only the A-parts of exercises in *Microeconomics: An Intuitive Approach with Calculus*, you may wish to instead use the solution set created for the companion book *Microeconomics: An Intuitive Approach*.
- *Solutions to Within-Chapter Exercises are provided in the student* Study Guide.

11.1 *Throughout part A of the text, we used the technology we called more "realistic" in panel (b) of Graph 11.1.*

A: *Suppose now that the producer choice set was instead strictly convex everywhere.*

(a) *Illustrate what such a technology would look like in terms of a production frontier.*

Answer: This is illustrated in panel (a) of Graph 11.1.



Graph 11.1: Production when Choice Set is Convex

(b) *Derive the output supply curve with price on the vertical and output on the horizontal axis (in graphs analogous to those in Graph 11.9) for this technology.*

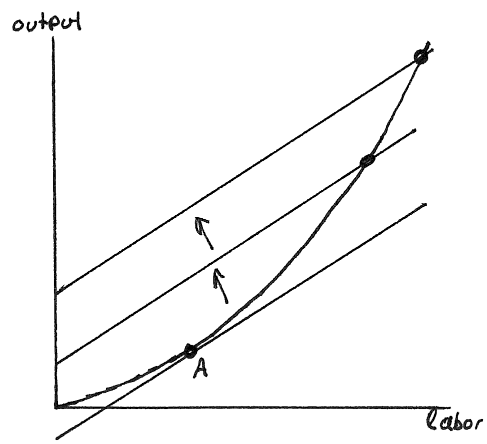
Answer: In panel (a) of Graph 11.1, two isoprofits corresponding to a low and a high price level are illustrated tangent to the producer choice set. The dashed one corresponds to the low output price p^A while the solid one corresponds to the higher output price p^B. The profit maximizing production plans at these two prices are then given by A and B. Panel (b) of the graph then plots the output levels x^A and x^B from these optimal production plans for the two price levels p^A and p^B on the horizontal axis. Panel (c) then simply flips the axes — giving us an upward sloping output supply curve.

(c) *Derive the labor demand curve for such a technology.*

Answer: This is done in panel (d) of Graph 11.1 where the marginal revenue product — which is derived from the slope of the production frontier — is downward sloping throughout (because the slope of the production frontier is becoming shallower throughout).

(d) *Now suppose the technology were instead such that the marginal product of labor is always increasing. What does this imply for the shape of the producer choice set?*

Answer: This implies that the production frontier gets steeper throughout — which in turn implies that the producer choice set is non-convex as drawn in Graph 11.2.



Graph 11.2: Increasing Marginal Product of Labor

(e) *How much should the firm produce if it is maximizing its profits in such a case? (*Hint*: Consider corner solutions.)*

Answer: If the firm is a price taker, then it should produce an infinite amount. This is illustrated in Graph 11.2 where three parallel isoprofits are drawn. The lowest is tangent to the production frontier, but this is not a profit maximizing production plan because we can still go to higher isoprofits that contain production plans which are technologically feasible. In fact, we can keep going to higher and higher isoprofits that continue to intersect the production frontier — and thus can keep going up.

B: *Suppose that the production function a firm faces is $x = f(\ell) = 100\ell^{\alpha}$.*

(a) *For what values of α is the producer choice set strictly convex? For what values is it non-convex?*

Answer: The producer choice set is strictly convex as long as $\alpha < 1$ and non-convex if $\alpha > 1$.

(b) *Suppose $\alpha = 0.5$. Derive the firm's output supply and labor demand function.*

Answer: The production function in this problem takes the same form as the one used in the last part of the chapter within the text. Using the results from there, and plugging 100 in for A and 0.5 for α, we get

$$\ell(p,w) = \left(\frac{w}{50p}\right)^{-2} = \frac{2500p^2}{w^2} \text{ and } x(p,w) = 100\left(\frac{w}{50p}\right)^{-1} = \frac{5000p}{w} \tag{11.1}$$

(c) *How much labor will the firm hire and how much will it produce if $p = 10$ and $w = 20$?*

Answer: Plugging these into our expressions for labor demand and output supply, we get $\ell = 625$ and $x = 2,500$.

(d) *How does labor demand and output supply respond to changes in w and p?*

Answer: The derivative of the labor demand function is negative with respect to w and positive with respect to p — which implies the firm will less labor as wage increases and more as output price increases. Similarly, the derivative of the output supply function is positive with respect to price and negative with respect to wage — which implies that supply increases in price and decreases in wage.

(e) *Suppose that* $\alpha = 1.5$. *How do your answers change?*

Answer: If we used the functions that arise from our mathematical optimization problem in the chapter, we would get the incorrect answer that

$$\ell(p,w) = \left(\frac{w}{150p}\right)^2 \text{ and } x(p,w) = 100\left(\frac{w}{150p}\right)^3. \tag{11.2}$$

The answer is wrong because it is actually a local minimum, not a maximum — equivalent to the tangency at A in Graph 11.2. To see this, you can check to see whether profit is positive by plugging the "optimal" labor and output values into the equation $\pi = xp - w\ell$ for profit; i.e.

$$\pi = 100\left(\frac{w}{150p}\right)^3 p - w\left(\frac{w}{150p}\right)^2 = \frac{2}{3}\left(\frac{w^3}{150^2p^2}\right) - \frac{w^3}{150^2p^2} = -\frac{1}{3}\left(\frac{w^3}{150^2p^2}\right) < 0. \tag{11.3}$$

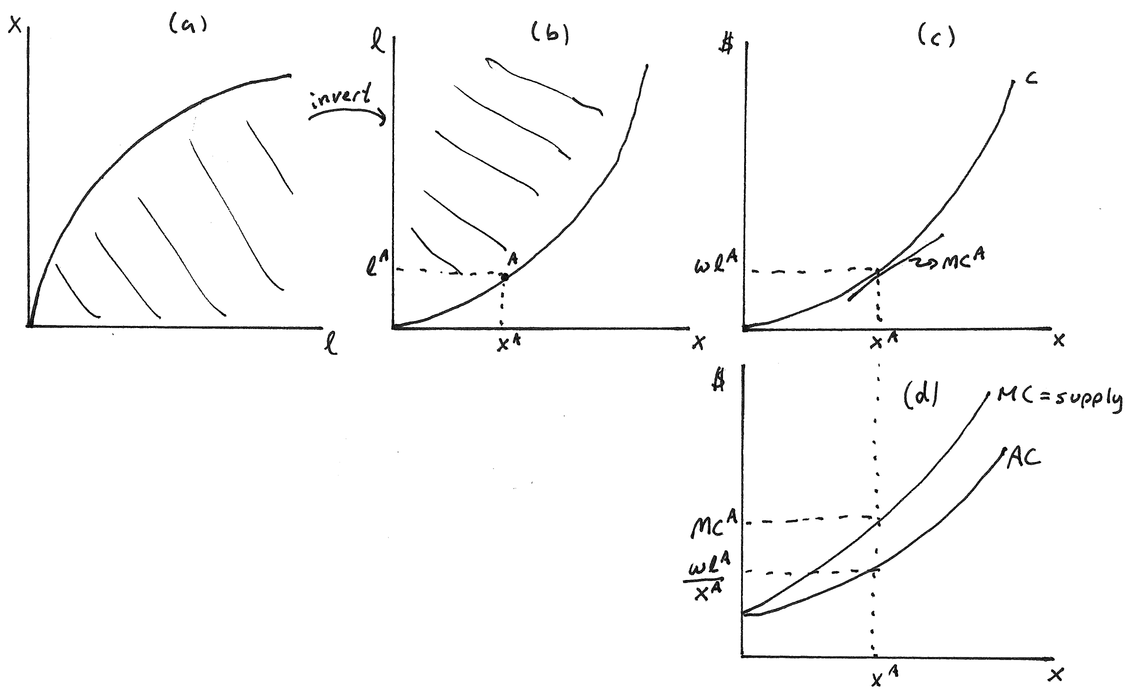
You can also tell that the "answer" can't possibly be right by checking how labor demand and output supply would respond to changes in w and p — both would increase with increases in wages and decreases in output prices. The correct answer, of course, is that a price taking firm under these conditions would produce an infinite amount of output. (Of course that is not realistic — such a firm would have market power and would thus not be a price taker.)

11.2 *Below, we will investigate the profit maximizing choice in the two steps that first involve a strict focus on the cost side.*

A: *Consider again (as in the previous exercise) a production process that gives rise to a strictly convex producer choice set.*

(a) *Derive the cost curve from a picture of the production frontier.*

Answer: This is done in panels (a) through (c) of Graph 11.3. Panel (a) illustrates the producer choice set. Panel (b) inverts this, flipping the axes so that x is on the horizontal and ℓ on the vertical. Finally, (c) is derived from (b) by simply multiplying labor input by w to convert the labor input needs of production into the dollar needs for production.



Graph 11.3: Cost Curves and Supply when Choice Set is Convex

(b) *Derive the marginal and average cost curves from the cost curve.*

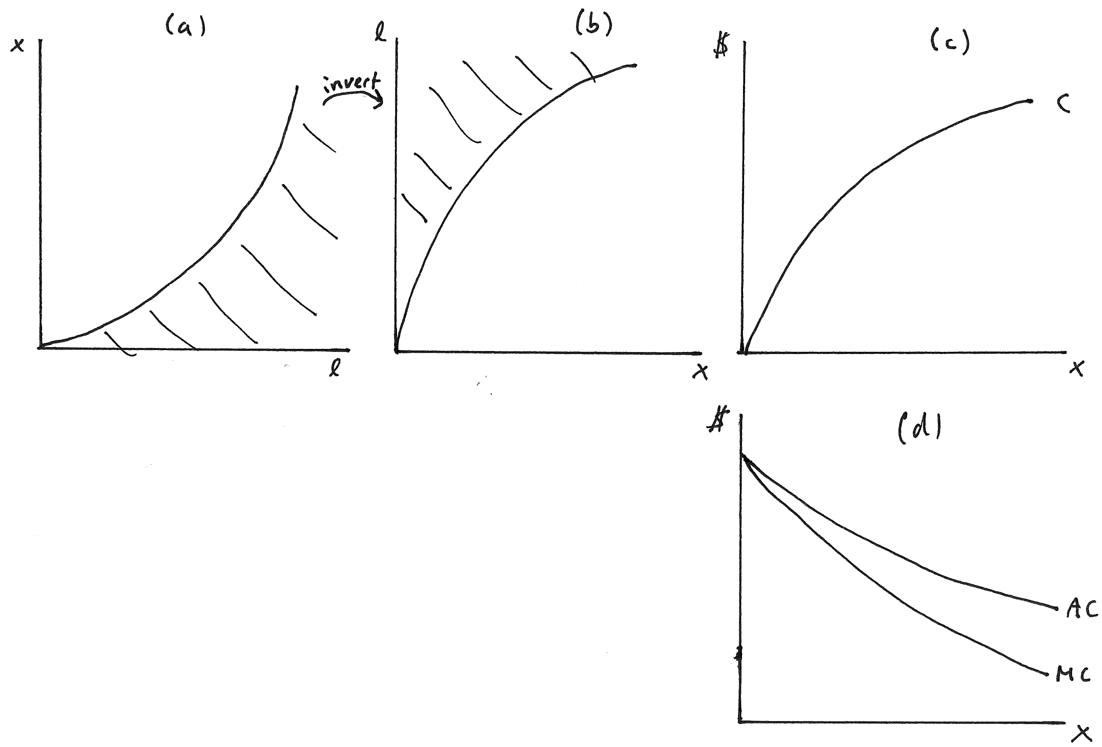
Answer: This is done in panel (d) where the *MC* curve is simply plotting the slope of the cost curve and the average cost curve is simply the total cost divided by x.

(c) *Illustrate the supply curve on your graph. How does it change if the wage rate increases?*

Answer: The supply curve is the part of the *MC* curve that lies above the *AC* curve. Since the entire *MC* curve lies above the *AC* curve in this case, the supply curve is simply equal to the *MC* curve. If wages go up, this rotates up the c curve in panel (c) — causing an increase in its slope at all output levels. As a result, the *MC* curve shifts/rotates up — which implies the supply curve similarly shifts up.

(d) *Now suppose the production process gives rise to increasing marginal product of labor throughout. Derive the cost curve and from it the marginal and average cost curves.*

Answer: This is done in Graph 11.4 (next page) and follows steps analogous to those we just went through in Graph 11.3.



Graph 11.4: Cost Curves and Supply when Choice Set is Non-Convex

(e) *Can you use these curves to derive a supply curve?*

Answer: In this case there is no supply curve — because the supply curve is the *MC* curve above *AC*, and the *MC* curve falls below *AC* throughout.

(f) *The typical production process is one that has increasing marginal product initially but eventually turns to one where marginal product is diminishing. Can you see how the two cases considered in this exercise combine to form the typical case?*

Answer: The case graphed in Graph 11.4(d) represents the initial part of the typical production process — the part where production is getting easier and easier because marginal product is increasing. The case graphed in Graph 11.3 represents the later part of the typical production process — the part where production is getting harder and harder because the marginal product is decreasing. If you were to combine the (d) panels of the two graphs, you would then get precisely the U-shaped marginal and average cost curves that we find in the typical production process.

B: *Consider again (as in the previous problem) the production function $x = f(\ell) = 100\ell^{\alpha}$.*

(a) *Derive the firm's cost function.*

Answer: First, we need to invert the production function. We thus take $x = 100\ell^{\alpha}$ and solve for ℓ to get

$$\ell = \left(\frac{x}{100}\right)^{1/\alpha}. \tag{11.4}$$

Then we multiply this by w to get

$$c(w,x) = w\ell = w\left(\frac{x}{100}\right)^{1/\alpha}. \tag{11.5}$$

(b) *Derive the marginal and average cost functions and determine how their relationship to one another differs depending on α.*

Answer: The marginal cost function is

$$MC(w,x) = \frac{\partial c(w,x)}{\partial x} = \frac{w}{\alpha}\left(\frac{x^{(1-\alpha)}}{100}\right)^{1/\alpha}, \tag{11.6}$$

and the average cost function is

$$AC(w,x) = \frac{c(w,x)}{x} = w\left(\frac{x^{(1-\alpha)}}{100}\right)^{1/\alpha}. \tag{11.7}$$

Notice that $AC = MC$ only when $x = 0$ unless $\alpha = 1$ in which case $MC = AC$ everywhere. This is because when $\alpha = 1$, the production process is linear — which means the marginal cost is constant. When $\alpha < 1$, $MC > AC$ everywhere other than when $x = 0$; and when $\alpha > 1$, $MC < AC$ everywhere other than when $x = 0$. This is because the former represents cases where marginal product of labor diminishes throughout while the latter represents cases where the marginal product of labor increases throughout.

(c) *What is the supply function for this firm when $\alpha = 0.5$? What is the firm's labor demand curve?*

Answer: In this case $MC > AC$ everywhere — so the MC curve is in fact the supply curve. Put differently, we can set p equal to MC and solve for the output supply of

$$x(w,p) = \left[100\left(\frac{\alpha p}{w}\right)^{\alpha}\right]^{1/(1-\alpha)}. \tag{11.8}$$

When $\alpha = 0.5$, this reduces to

$$x(w,p) = \left[100\left(\frac{0.5p}{w}\right)^{0.5}\right]^{1/(1-0.5)} = 5000\frac{p}{w} \tag{11.9}$$

which is exactly the same we derived in the previous end-of-chapter exercise when we directly employed profit maximization. Substituting this into equation (11.4) and setting $\alpha = 0.5$ in that equation, we then get the labor demand as

$$\ell(w,p) = \left(\frac{5000p/w}{100}\right)^{1/(0.5)} = 2500\frac{p^2}{w^2}, \tag{11.10}$$

again exactly the same as we derived through direct profit maximization in the previous end-of-chapter exercise.

(d) *How do your answers change when $\alpha = 1.5$?*

Answer: When $\alpha > 1$, $MC < AC$ throughout — which implies there is no supply curve. The reason, of course, is that in this case we have increasing marginal product of labor throughout — which implies the price-taking firm would wish to produce an infinite quantity of the output.

11.3 *Consider a profit maximizing firm.*

A: *Explain whether the following statements are true or false:*

(a) *For price-taking, profit maximizing producers, the "constraint" is determined by the technological environment in which the producer finds herself while the "tastes" are formed by the economic environment in which the producer operates.*

Answer: The physical constraint that a producer cannot get around arises from the simple laws of physics — you can only get so much x out of the inputs you use. The more sophisticated the technology available to the producer, the more x she can squeeze out — and thus the technology creates the production constraint that tells the producer which production plans are feasible and which are not. In terms of tastes, we typically assume that producers simply care about profit — and that more profit is better than less. To form indifference curves for producers who simply care about profit, we have to find production plans that all result in the same amount of profit. And how easy it is to make profit depends on how high the output price is and how low the input prices are relative to the output price. Thus, the production plans that produce the same level of profit will differ depending on the economic environment — depending on how much the producer can get for her output and how much she has to pay for the inputs in her production plans.

(b) *Every profit maximizing producer is automatically cost-minimizing.*

Answer: This is true. If you are maximizing profit, you must be producing whatever you are producing without wasting any inputs — i.e. you must be producing the output at the minimum cost possible.

(c) *Every cost-minimizing producer is automatically profit maximizing.*

Answer: This is not true. You could be producing some arbitrary quantity without wasting any inputs — i.e. in the least cost way. But that does not mean you are producing the "right" quantity. Cost minimization is only part of profit maximization — it only takes into account input prices as they relate to the cost of production. Only when you profit maximize do you also take into account the output price and what that implies for how much you should produce in a cost minimizing way.

(d) *Price taking behavior makes sense only when marginal product diminishes at least at some point.*

Answer: This is true. If marginal product always increases, then it is getting cheaper and cheaper to produce additional units of output. And if I can sell all my output at the same price (i.e. if I am a price taker), then I should keep producing and drive down my average cost.

B: *Consider the production function $x = f(\ell) = \alpha \ln(\ell + 1)$.*

(a) *Does this production function have increasing or decreasing marginal product of labor.*

Answer: Marginal product for this production function is given by

$$MP_\ell = \frac{\alpha}{\ell - 1}. \tag{11.11}$$

Thus, as ℓ increases, MP_ℓ clearly decreases. You can also see this by simply taking the derivative of marginal product of labor with respect to ℓ —

$$\frac{\partial MP_\ell}{\partial \ell} = \frac{-\alpha}{(\ell - 1)^2}. \tag{11.12}$$

(b) *Set up the profit maximization problem and solve for the labor input demand and output supply functions.*

Answer: The profit maximization problem is

$$\max_{\ell, x} \; px - w\ell \;\text{ subject to }\; x = \alpha \ln(\ell + 1) \tag{11.13}$$

which can also be written as the unconstrained maximization problem

$$\max_{\ell} \; p\alpha \ln(\ell + 1) - w\ell. \tag{11.14}$$

The first order condition for this problem is

$$\frac{p\alpha}{\ell+1} - w = 0 \tag{11.15}$$

which can be solved to get the labor demand function

$$\ell(w,p) = \frac{\alpha p - w}{w}. \tag{11.16}$$

Substituting this into the production function, we then get the output supply function

$$x(w,p) = \alpha \ln\left(\frac{p\alpha - w}{w} + 1\right) = \alpha \ln\left(\frac{\alpha p}{w}\right). \tag{11.17}$$

(c) *Recalling that* $\ln x = y$ *implies* $e^y = x$ *(where* $e \approx 2.7183$ *is the base of the natural log), invert the production function and derive from this the cost function* $c(w,x)$.

Answer: To invert the production function $x = \alpha \ln(\ell+1)$, we first note that this implies

$$e^{x/\alpha} = \ell + 1 \tag{11.18}$$

which can then be solved in terms of ℓ to give us the amount of labor required for each level of output; i.e.

$$\ell = e^{x/\alpha} - 1. \tag{11.19}$$

When multiplied by w, we then get the cost function

$$c(w,x) = w\left(e^{x/\alpha} - 1\right). \tag{11.20}$$

(d) *Determine the marginal and average cost functions.*

Answer: The marginal cost is

$$MC(w,x) = \frac{\partial c(w,x)}{\partial x} = \frac{w}{\alpha} e^{x/\alpha}, \tag{11.21}$$

and the average cost is

$$AC(w,x) = \frac{we^{x/\alpha}}{x} - \frac{w}{x}. \tag{11.22}$$

(e) *Derive from this the output supply and labor demand functions. Compare them to what you derived directly from the profit maximization problem in part (b).*

Answer: To derive the output supply function, we begin by setting p equal to MC; i.e.

$$p = \frac{w}{\alpha} e^{x/\alpha}. \tag{11.23}$$

Multiplying both sides by α/w, taking natural logs and then multiplying both sides by α, we then get

$$x(p,w) = \alpha \ln \frac{\alpha p}{w}. \tag{11.24}$$

Plugging this back into equation (11.19), we get the labor input demand

$$\ell(w,p) = e^{\ln(\alpha p/w)} - 1 = \frac{\alpha p}{w} - 1 = \frac{\alpha p - w}{w}. \tag{11.25}$$

Note that the output supply and labor demand equations are identical to those derived directly through profit maximization earlier in the problem. The equations are correct, however, only for prices above the lowest point of AC.

(f) *In your mathematical derivations, what is required for a producer to be cost minimizing? What, in addition, is required for her to be profit maximizing?*

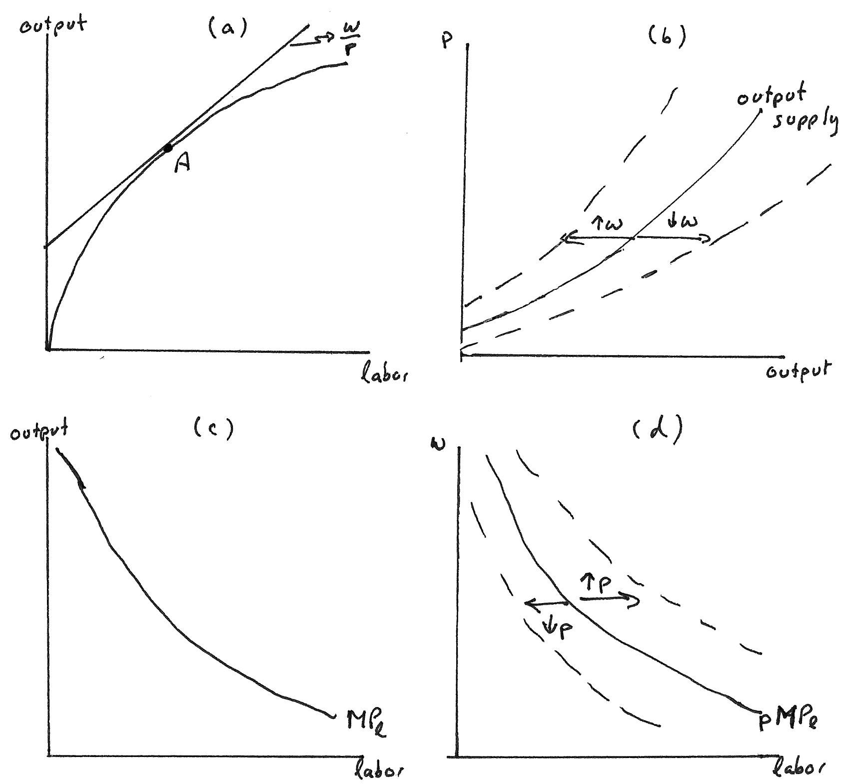
Answer: The only requirement for the producer to be cost minimizing is that she not waste any input — i.e. that she hire labor in accordance with equation (11.19). There is no requirement imposed by cost minimization on *how much* to produce. The additional requirement imposed by profit maximization is that output quantity be chosen such that $p = MC$. Alternatively, we could phrase this same additional requirement as the requirement that $pMP_\ell = w$.

11.4 *In this exercise, we will explore how changes in output and input prices affect output supply and input demand curves.*

A: *Suppose your firm has a production technology with diminishing marginal product throughout.*

(a) *With labor on the horizontal axis and output on the vertical, illustrate what your production frontier looks like.*

Answer: This is illustrated in panel (a) of Graph 11.5 — an initially steep slope to the production frontier (representing initially high marginal product of labor) which becomes shallower as labor input increases.



Graph 11.5: Output Supply and Labor Demand

(b) *On your graph, illustrate your optimal production plan for a given p and w.* True or False*: As long as there is a production plan at which an isoprofit curve is tangent, it is profit maximizing to produce this plan rather than shut down.*

Answer: This is also illustrated in panel (a) where the isoprofit is tangent at the profit maximizing production plan A. As long as such a tangency exists, the isoprofit will have positive intercept — which implies that profit will be positive. Therefore it is better to produce at the tangency than not at all. It is also the case that only one such tangency can exist under this shape of the production frontier — so no other potentially profit maximizing production plan can exist. (Note: It is technically possible for price to be so low or wage to be so high that the optimal production plan is at the origin — but in this case, there is no tangency at a production plan with positive output.)

(c) *Illustrate what your output supply curve looks like in this case.*

Answer: The output supply curve illustrates the relationship between output price on the vertical axis and output quantity on the horizontal. This is illustrated in panel (b) of Graph 11.5. This curve must be upward sloping — because an increase in p causes the isoprofits to become shallower which in turn causes the tangency with the production frontier to move to the right on the production frontier.

(d) *What happens to your supply curve if w increases? What happens if w falls?*

Answer: When w increases, the isoprofits become steeper — which means that the tangency with the production frontier moves to the left even as p remains the same. Thus, each point on the output supply curve shifts to the left. The reverse happens when w decreases.

(e) *Illustrate what your marginal product of labor curve looks like and derive the labor demand curve.*

Answer: The marginal product of labor curve is simply the slope of the production frontier. Since the production frontier starts steep, the marginal product of labor is high for initial labor units, and since the production frontier becomes shallower as labor increases, the marginal product of labor falls. This downward sloping MP_ℓ curve is illustrated in panel (c) of Graph 11.5. The labor demand curve is then simply derived from the marginal revenue product curve — which in turn is simply the marginal product curve multiplied by output price p. This is illustrated in panel (d) of the graph.

(f) *What happens to your labor demand curve when p increases? What happens when p decreases?*

Answer: When p increases, the isoprofit curve becomes shallower — which implies the tangency of the isoprofit with the production frontier moves to the right, resulting in more labor input. Thus, when p increases and w stays the same, more labor is hired — which means the labor supply curve shifts to the right. This can also be seen by simply recognizing that pMP_ℓ increases as p increases. The reverse happens when p decreases.

B: *Suppose that your production process is characterized by the production function $x = f(\ell) = 100\ln(\ell+1)$. For purposes of this problem, assume $w > 1$ and $p > 0.01$.*

(a) *Set up your profit maximization problem.*

Answer: The profit maximization problem is

$$\max_{x,\ell}\ px - w\ell \ \text{ subject to } \ x = 100\ln(\ell+1) \tag{11.26}$$

which can also be written as the unconstrained maximization problem

$$\max_{\ell}\ 100p\ln(\ell+1) - w\ell. \tag{11.27}$$

(b) *Derive the labor demand function.*

Answer: The first order condition for the unconstrained maximization problem above is

$$\frac{100p}{\ell+1} - w = 0. \tag{11.28}$$

Solving for ℓ, we get the labor demand function

$$\ell(p,w) = \frac{100p - w}{w}. \tag{11.29}$$

(c) *The labor demand curve is the inverse of the labor demand function with p held fixed. Can you demonstrate what happens to this labor demand curve when p changes?*

Answer: The labor supply curve we graph (with ℓ on the horizontal and w on the vertical axis) is then

$$w(\ell) = \frac{100p}{\ell+1}. \tag{11.30}$$

When p increases, the right hand side of this equation increases — which implies that the labor supply curve shifts to the right. When p falls, the right hand side decreases — which implies that the labor supply curve shifts to the left.

(d) *Derive the output supply function.*

Answer: To get the output supply function, we simply plug the labor demand into the production function; i.e.

$$x(p,w) = f(\ell(p,w)) = 100\ln\left(\frac{100p - w}{w} + 1\right) = 100\ln\left(\frac{100p}{w}\right). \tag{11.31}$$

(e) *The supply curve is the inverse of the supply function with w held fixed. What happens to this supply curve as w changes? (*Hint*: Recall that $\ln x = y$ implies $e^y = x$, where e is the base of the natural log.)*

Answer: To take the inverse, we first divide both sides of the supply equation by 100 and recognize that this implies

$$e^{(x/100)} = \frac{100p}{w}. \tag{11.32}$$

Solving for p, we get

$$p(x,w) = \frac{we^{(x/100)}}{100}. \tag{11.33}$$

To see what happens as w changes, we simply take the derivative of this supply curve with respect to w — i.e.

$$\frac{\partial p(x,w)}{\partial w} = \frac{we^{(x/100)}}{10000} > 0. \tag{11.34}$$

Thus, as w increases, the supply curve shifts up (and as it decreases, the supply curve shifts down). We could also see this directly by simply looking at the supply function $x(p,w)$. Just take the partial derivative of $x(p,w)$ with respect to w to find that

$$\frac{\partial x(p,w)}{\partial w} = \frac{-100}{w} < 0. \tag{11.35}$$

Thus, as w increases, output decreases — which is equivalent to an upward shift in the supply *curve*.

(f) *Suppose $p = 2$ and $w = 10$. What is your profit maximizing production plan, and how much profit will you make?*

Answer: Plugging these prices into the labor demand $\ell(p,w)$ and output supply $x(p,w)$ equations, we get the production plan $(\ell, x) = (19, 299.57)$; i.e. you will hire 19 workers and produce approximately 300 units of output. Profit is then approximately

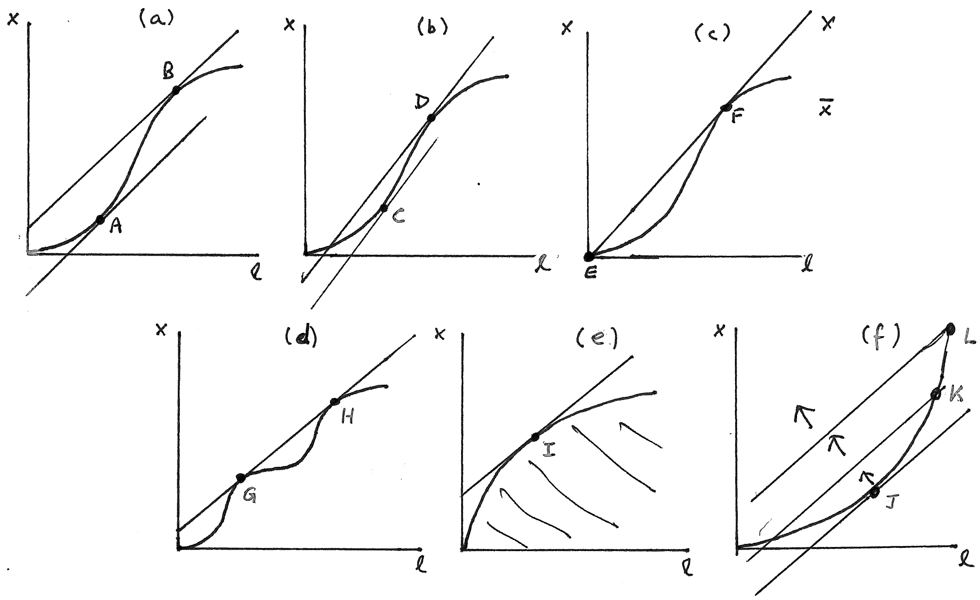
$$\text{Profit} = px - w\ell = 2(300) - 10(19) = 410. \tag{11.36}$$

11.5 *When we discussed optimal behavior for consumers in Chapter 6, we illustrated that there may be two optimal solutions for consumers whenever there are non-convexities in either tastes or choice sets. We can now explore conditions under which multiple optimal production plans might appear in our producer model.*

A: *Consider only profit maximizing firms whose tastes (or isoprofits) are shaped by prices.*

(a) *Consider first the standard production frontier that has initially increasing marginal product of labor and eventually decreasing marginal product of labor.* True or False*: If there are two points at which isoprofits are tangent to the production frontier in this model, the lower output quantity cannot possibly be part of a truly optimal production plan.*

Answer: This is true. Panel (a) of Graph 11.6 illustrates this with two tangencies at A and B. Since the isoprofit tangent at A has a negative intercept, profit at the production plan A is negative.



Graph 11.6: Profit Maximizing under Different Production Sets

(b) *Could it be that neither of the tangencies represents a truly optimal production plan?*

Answer: Yes. This is illustrated in panel (b) of Graph 11.6 where the two tangencies occur at production plans C and D. In this case, both isoprofit curves have negative intercepts — and both therefore involve negative profit. In this case, the profit maximizing production plan is (0,0) — i.e. no labor input is bought and no output is produced.

(c) *Illustrate a case where there are two truly optimal solutions where one of these does not occur at a tangency.*

Answer: This is illustrated in panel (c) of Graph 11.6. The production plan F occurs at a tangency and involves zero profit because the isoprofit curve intersects at the origin. Thus, profit at F is the same as profit at $E = (0,0)$ where no labor is purchased and no output is produced. Thus both E and F are truly profit maximizing production plans.

(d) *What would a production frontier have to look like in order for there to be two truly optimal production plans which both involve positive levels of output? (*Hint*: Consider technologies that involve multiple switches between increasing and decreasing marginal product of labor.)*

Answer: This is illustrated in panel (d) of Graph 11.6 where both G and H are tangencies that lie on the same isoprofit curve (and thus result in the same amount of positive profit).

(e) True or False*: If the producer choice set is convex, there can only be one optimal production plan.*

Answer: This is true. In panel (e) of Graph 11.6, a strictly convex production frontier is illustrated — with only I emerging as a profit maximizing production plan.

(f) *Where does the optimal production plan lie if the production frontier is such that the marginal product of labor is always increasing?*

Answer: This is illustrated in panel (f) of Graph 11.6 where the increasing marginal product of labor results in an ever steepening production frontier. The only tangency occurs at J — but J actually results in negative profit since the isoprofit has negative intercept. The producer can do better by moving to higher isoprofits that intersect the production frontier — as those that intersect at K and L. But it's always possible to go to an even higher isoprofit and mover higher on the production frontier. Thus, the profit maximizing quantity is infinite (which, of course, does not make sense in a world of scarcity — but neither does an ever increasing marginal product of labor.)

(g) *Finally, suppose that the marginal product of labor is constant throughout. What production plans might be optimal in this case?*

Answer: This would result in a production frontier that is simply a straight line. If the ratio w/p happens to be the same as the slope of this production frontier, then every production plan on the production frontier lies on the same isoprofit curve which in turn intersects at the origin. Thus, all production plans on the production frontier yield zero profit — and all are therefore optimal. If the isoprofits are steeper than the production frontier, then all production plans on the frontier other than (0,0) result in negative profit — and only (0,0) is optimal. If, on the other hand, the isoprofits are shallower than the production frontier, the optimal output quantity is infinite for reasons similar to what we described in part (f) where we considered a production frontier with increasing marginal product of labor throughout.

B: *In the text, we used a cosine function to illustrate a production process that has initially increasing and then decreasing marginal product of labor. In many of the end-of-chapter exercises, we will instead use a function of the form $x = f(\ell) = \beta\ell^2 - \gamma\ell^3$ where β and γ are both greater than zero.*

(a) *Illustrate how the profit maximization problem results in two "solutions". (Use the quadratic formula to solve for these.)*

Answer: The profit maximization problem is then

$$\max_{\ell,x} \; px - w\ell \;\text{ subject to }\; x = \beta\ell^2 - \gamma\ell^3 \tag{11.37}$$

which can be written in the form of an unconstrained maximization problem by substituting the constraint into the objective to give us

$$\max_{\ell} \; p\left(\beta\ell^2 - \gamma\ell^3\right) - w\ell. \tag{11.38}$$

The first order condition for this problem is then

$$-3\gamma p\ell^2 + 2\beta p\ell - w = 0. \tag{11.39}$$

For an equation of the form $ax^2 + bx + c = 0$, the quadratic formula gives the solutions

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \;\text{ and }\; x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{11.40}$$

We can therefore solve equation (11.39) for ℓ by applying this formula, letting $a = -3\gamma p$, $b = 2\beta p$ and $c = -w$, to give us

$$\ell = \frac{-2\beta p + \sqrt{4\beta^2 p^2 - 12\gamma p w}}{-6\gamma p} \text{ and } \ell = \frac{-2\beta p - \sqrt{4\beta^2 p^2 - 12\gamma p w}}{-6\gamma p}. \tag{11.41}$$

(b) *Which of your two "solutions" is unambiguously* not *the actual profit maximizing solution?*

Answer: Our two "solutions" can be rewritten as

$$\ell = \frac{\beta}{3\gamma} - \frac{\sqrt{\beta^2 p^2 - 3\gamma p w}}{3\gamma p} \text{ and } \ell = \frac{\beta}{3\gamma} + \frac{\sqrt{\beta^2 p^2 - 3\gamma p w}}{3\gamma p}, \tag{11.42}$$

where the latter is clearly greater than the former. We know from our intuitive graphs that only the higher of the two production plans could be optimal — thus the first solution is definitely not a profit maximizing solution.

(c) *What else would you have to check to be sure that the other "solution" is profit maximizing?*

Answer: You would have to check that the larger of the two solutions results in positive profit.

(d) *Now consider instead a production process characterized by the equation $x = A\ell^{\alpha}$. Suppose $\alpha < 1$. Determine the profit maximizing production plan.*

Answer: The last part of the chapter sets up this optimization problem and solves for the production plan

$$\ell = \left(\frac{w}{\alpha A p}\right)^{\frac{1}{\alpha-1}} \text{ and } x = A\left(\frac{w}{\alpha A p}\right)^{\frac{\alpha}{\alpha-1}} \tag{11.43}$$

(e) *What if $\alpha > 1$?*

Answer: When $\alpha > 1$, the marginal product of labor is increasing throughout — and the "solution" is not a true profit maximum because the firm should produce an infinite amount instead. You can tell that the "solution" makes no sense in this case because, when $\alpha > 1$, it would suggest that you hire more labor and produce more output as wages go up and output prices go down.

(f) *What if $\alpha = 1$?*

Answer: When $\alpha = 1$, the marginal product of labor is constant — and the production frontier is a straight line. The "solution" we calculated is not defined because the exponent $1/(\alpha-1)$ is not defined when $\alpha = 1$. This is because there is no tangency — either all production plans on the frontier yield zero profit and all such plans are therefore profit maximizing, or the true profit maximum involves zero or an infinite level of output.

11.6 *This exercise explores in some more detail the relationship between production technologies and marginal product of labor.*

A: *We often work with production technologies that give rise to initially increasing marginal product of labor that eventually decreases.*

(a) True or False*: For such production technologies, the marginal product of labor is increasing so long as the slope of the production frontier becomes steeper as we move toward more labor input.*

Answer: This is true. The marginal product of labor is in fact equal to the slope of the production frontier — so if the slope of the production frontier is increasing, this must mean the marginal product of labor is increasing.

(b) True or False*: The marginal product of labor becomes negative when the slope of the production frontier begins to get shallower as we move toward more labor input.*

Answer: This is false. When the slope of the production frontier becomes shallower, it is still positive but is now declining. Thus, at that point the marginal product of labor is declining as more labor is hired, but it is still positive.

(c) True or False*: The marginal product of labor is positive so long as the slope of the production frontier is positive.*

Answer: This is true — since the slope of the production frontier and the marginal product of labor is the same thing, one being positive implies the other must be positive.

(d) True or False*: If the marginal product of labor ever becomes zero, we know that the production frontier becomes perfectly flat at that point.*

Answer: This is true — as the production frontier becomes flat, its slope approaches zero, which means the marginal product of labor (which is the same thing as the slope of the production frontier) approaches zero.

(e) True or False*: A negative marginal product of labor necessarily implies a downward sloping production frontier at that level of labor input.*

Answer: True — a downward slope is the same thing as a negative slope. Since the slope of the production frontier is the marginal product of labor, a negative marginal product of labor must imply a downward slope of the production frontier.

B: *We have thus far introduced two general forms for production functions that give rise to initially increasing and eventually decreasing marginal product.*

(a) *The first of these was given as an example in the text and took the general form* $f(\ell) = \alpha(1-\cos(\beta\ell))$ *for all* $\ell \le \pi/\beta \approx 3.1416/\beta$ *and* $f(\ell) = 2\alpha$ *for all* $\ell > \pi/\beta \approx 3.1416/\beta$), *with* α *and* β *assumed to be greater than 0. Determine the labor input level at which the marginal product of labor begins to decline. (*Hint*: Recall that the cosine of* $\pi/2 \approx 1.5708$ *is equal to zero.)*

Answer: The marginal product of labor for this function is

$$MP_\ell = \frac{df}{d\ell} = \frac{\alpha}{\beta}\sin(\beta\ell). \tag{11.44}$$

This function attains a maximum where its derivative is equal to zero; i.e. where

$$\frac{dMP_\ell}{d\ell} = \alpha\cos(\beta\ell) = 0. \tag{11.45}$$

The cosine function is, of course, oscillating — and we use only the first part for our production function (since the function converges to 2α as described in the problem). The first time the function equals zero is where $\beta\ell = \pi/2 \approx 1.5708$. Thus, the marginal product of labor begins to decline at $\ell = \pi/2\beta \approx 1.5708/\beta$.

(b) *Does the marginal product of labor ever become negative? If so, at what labor input level?*

Answer: The marginal product of labor expression becomes zero when

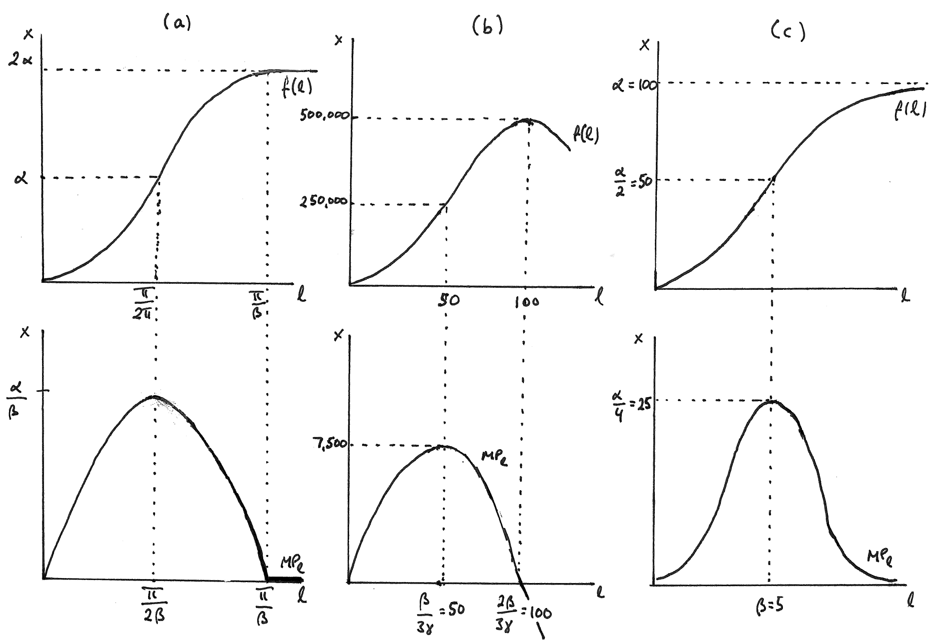
$$MP_\ell = \frac{\alpha}{\beta}\sin(\beta\ell) = 0 \tag{11.46}$$

or, equivalently, when $\sin(\beta\ell) = 0$. This holds for the first time when $\beta\ell = \pi \approx 3.1416$ or $\ell = \pi/\beta$. At that labor input level, the problem specifies that output remains at 2α — which implies that the marginal product of labor is zero from then forward. It never becomes negative.

(c) *In light of what you just learned, can you sketch the production function given in (a)? What does the marginal product of labor for this function look like?*

Answer: This is illustrated in the two graphs of panel (a) of Graph 11.7. In the top panel, the production frontier begins at the origin and reaches its maximum of 2α at $\ell = \pi/\beta$. In the lower panel, the marginal product of labor reaches its maximum at $\ell = \pi/2\beta$. Evaluating the marginal product function at that labor input level, we get

$$MP_\ell = \frac{\alpha}{\beta}\sin\left(\beta\frac{\pi}{2\beta}\right) = \frac{\alpha}{\beta}\sin\left(\frac{\pi}{2}\right) = \frac{\alpha}{\beta}. \quad (11.47)$$



Graph 11.7: Two Production Functions

(d) *The second general form for such a production function was given in exercise 11.5 and took the general form $f(\ell) = \beta\ell^2 - \gamma\ell^3$. Determine the labor input level at which the marginal product of labor begins to decline.*

Answer: The marginal product of labor for this production function is

$$MP_\ell = \frac{df}{d\ell} = 2\beta\ell - 3\gamma\ell^2. \quad (11.48)$$

This function attains a maximum where its derivative is equal to zero; i.e. where

$$\frac{dMP_\ell}{d\ell} = 2\beta - 6\gamma\ell = 0, \quad (11.49)$$

which occurs when $\ell = \beta/3\gamma$.

(e) *Does the marginal product of labor ever become negative? If so, at what labor input level?*

Answer: Yes, it becomes negative when

$$MP_\ell = 2\beta\ell - 3\gamma\ell^2 = 0, \tag{11.50}$$

which occurs when $\ell = 2\beta/3\gamma$.

(f) *Given what you have learned about the function $f(\ell) = \beta\ell^2 - \gamma\ell^3$, illustrate the production function when $\beta = 150$ and $\gamma = 1$. What does the marginal product of labor look like?*

Answer: This is sketched in the two graphs of panel (b) of Graph 11.7. In the top graph, the production function is depicted. We concluded that the marginal product of labor reaches zero at $\ell = 2\beta/3\gamma$ which reduces to $\ell = 100$ in this example. Thus, the production function attains its maximum when $\ell = 100$ — which, when plugged into the production function, implies output of 500,000 units of x. After that, however, the production function has negative slope. This is reflected in the lower panel where we depict the marginal product of labor. It reaches zero when the production function reaches its maximum — and then becomes negative as the production function goes past its peak. We also concluded that the marginal product of labor attains its maximum when $\ell = \beta/3\gamma$ which reduces to $\ell = 50$ in our example. Plugging this into our expression for MP_ℓ, we get that the highest marginal product worker has a marginal product of 7,500 units of x.

(g) *In each of the two previous cases, you should have concluded that the marginal product of labor eventually becomes zero and/or negative. Now consider the following new production technology: $f(\ell) = \alpha/(1 + e^{-(\ell-\beta)})$ where $e \approx 2.7183$ is the base of the natural logarithm. Determine the labor input level at which the marginal product of labor begins to decline.*

Answer: The marginal product of labor for this production function is

$$MP_\ell = \frac{df}{d\ell} = \frac{\alpha e^{-(\ell-\beta)}}{\left(1 + e^{-(\ell-\beta)}\right)^2}. \tag{11.51}$$

This marginal product becomes negative at the point where the derivative of MP_ℓ is zero; i.e. when

$$\frac{dMP_\ell}{d\ell} = \frac{-\alpha e^{-(\ell-\beta)}}{\left(1 + e^{-(\ell-\beta)}\right)^2} + \frac{2\alpha e^{-2(\ell-\beta)}}{\left(1 + e^{-(\ell-\beta)}\right)^3} = 0. \tag{11.52}$$

Dividing this expression by $-\alpha e^{-(\ell-\beta)}$ and multiplying by $\left(1 + e^{-(\ell-\beta)}\right)^3$, this simplifies to

$$\left(1 + e^{-(\ell-\beta)}\right) - 2e^{-(\ell-\beta)} = 0, \tag{11.53}$$

which further simplifies to

$$e^{-(\ell-\beta)} = 1. \tag{11.54}$$

Taking the natural log of both sides, we then get that $-(\ell-\beta) = 0$ which solves to $\ell = \beta$. Thus, when $\ell = \beta$, the marginal product of labor begins to decline.

(h) *Does the marginal product of labor ever become negative? If so, at what labor input level?*

Answer: No, the marginal product of labor never becomes negative. To see this, recall from above that

$$MP_\ell = \frac{df}{d\ell} = \frac{\alpha e^{-(\ell-\beta)}}{\left(1 + e^{-(\ell-\beta)}\right)^2}. \tag{11.55}$$

Were you to set this to zero, it would simplify to

$$e^{-(\ell-\beta)} = 0. \tag{11.56}$$

Taking natural logs, you would then get that $-(\ell - \beta) = \ln(0)$ or $\ell = \beta - \ln(0)$. The natural log of zero is undefined, but $\ln(x)$ approaches $-\infty$ as x approaches zero. The marginal product of labor therefore approaches zero as ℓ approaches infinity. You can easily see that this makes sense by simply taking the limit of the production function as ℓ approaches ∞:

$$\lim_{\ell\to\infty}\left(\frac{\alpha}{1+e^{-(\ell-\beta)}}\right)=\alpha. \tag{11.57}$$

Thus, the production function converges to α as ℓ approaches ∞.

(i) *Given what you have discovered about the production function $f(\ell)=\alpha/(1+e^{-(\ell-\beta)})$, can you sketch the shape of this function when $\alpha=100$ and $\beta=5$? What does the marginal product of labor function look like?*

Answer: This is sketched in the two graphs of panel (c) of Graph 11.7. The top graph represents the production function which we concluded converges to α which is equal to 100 here. The lower graph illustrates the marginal product of labor which we concluded above attains a maximum at $\ell=\beta$ which is equal to 5 in our case. You can then also derive the highest marginal product by plugging $\ell=\beta$ into the equation for MP_ℓ, which gives

$$MP_\ell=\frac{\alpha e^{-(\beta-\beta)}}{\left(1+e^{-(\beta-\beta)}\right)^2}=\frac{\alpha}{(1+1)^2}=\frac{\alpha}{4}. \tag{11.58}$$

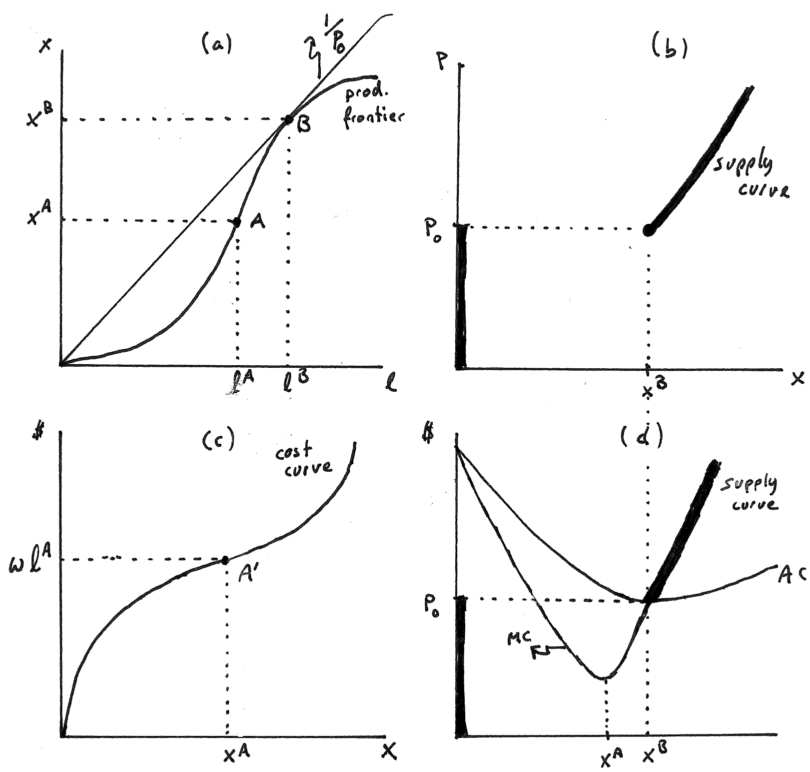
Since $\alpha=100$ in our example, this implies that the marginal product function attains a maximum of 25 units of output. You can similarly pin down that $x=\alpha/2=50$ when $\ell=\beta=5$ in the top graph of the production function.

11.7 *We have shown that there are two ways in which we can think of the producer as maximizing profits: Either directly, or in a two-step process that begins with cost minimization.*

A: *This exercise reviews this equivalence for the case where the production process initially has increasing marginal product of labor but eventually reaches decreasing marginal product. Assume such a production process throughout.*

(a) *Begin by plotting the production frontier with labor on the horizontal and output on the vertical axis. Identify in your graph the production plan $A = (\ell^A, x^A)$ at which increasing returns turns to decreasing returns.*

Answer: This is illustrated in panel (a) of Graph 11.8 where A lies at the point at which the production frontier stops increasing at an increasing rate and starts increasing at a decreasing rate. Put differently, at A the slope stops increasing and starts decreasing.



Graph 11.8: 2 Ways to Maximize Profit

(b) *Suppose wage is $w = 1$. Illustrate in your graph the price p_0 at which the firm obtains zero profit by using a production plan B. Does this necessarily lie above or below A on the production frontier?*

Answer: This is also illustrated in panel (a) where B lies on an isoprofit that is tangent to the production frontier at B *and* intersects the origin (which implies zero profit). The slope of the isoprofit is $1/p_0$ since $w = 1$. It is apparent from the graph that B must lie above A on the production frontier — i.e. it must be the case that the zero-profit price results in production on the decreasing marginal product of labor portion of the production frontier.

(c) *Draw a second graph next to the one you have just drawn. With price on the vertical axis and output on the horizontal, illustrate the amount the firm produces at p_0.*

Answer: This is illustrated in panel (b) of Graph 11.8 where the point (x^B, p_0) illustrates the lowest price at which the firm produces positive output.

(d) *Suppose price rises above p_0. What changes on your graph with the production frontier — and how does that translate to points on the supply curve in your second graph?*

Answer: If price rises above p_0, the isoprofit lines become shallower — which implies the new optimal quantity lies at a tangency higher on production frontier than B. The isoprofit that is tangent at the new profit maximizing production plan also has positive intercept on the vertical axis — implying profit will be positive. Thus, output increases as p rises above p_0 — leading to an upward sloping supply curve (as illustrated in panel (b).)

(e) *What if price falls below p_0?*

Answer: If price falls below p_0, the isoprofit curves become steeper — implying tangencies to the left of B. At those tangencies, however, the intercept on the vertical axis will be negative — implying negative profit. Thus, the firm is better off not producing at all — which is why the supply curve in panel (b) is vertical at zero output level up to the price p_0.

(f) *Illustrate the cost curve on a graph below your production frontier graph. What is similar about the two graphs — and what is different — around the point that corresponds to production plan A.*

Answer: The cost curve, as illustrated in panel (c) of Graph 11.8, has the inverse shape from the production frontier — because when each additional labor unit increases production more than the last (on the increasing marginal product part of the production frontier), the cost of increasing output rises at a decreasing rate (and vice versa). Around A' — the point corresponding to A in panel (a), the cost curve switches from increasing at a decreasing rate to increasing at an increasing rate (just as the switch from increasing at an increasing rate to increasing at a decreasing rate happens on the production frontier.)

(g) *Next to your cost curve graph, illustrate the marginal and average cost curves. Which of these reaches its lowest point at the output quantity x^A? Which reaches its lowest point at x^B?*

Answer: This is illustrated in panel (d) of Graph 11.8. The marginal cost (MC) curve is the slope of the cost curve — so it reaches its lowest point at the output level x^A where the slope of the cost curve begins to get steeper. The average cost curve reaches its lowest point where the MC curve crosses it — which is also where the supply curve begins. This occurs at x^B.

(h) *Illustrate the supply curve on your graph and compare it to the one you derived in parts (c) and (d).*

Answer: The supply curve is the part of the MC curve that lies above the AC curve — with output of zero below that. This is highlighted in panel (d) of Graph 11.8 — with the resulting supply curve being identical to what we derived before in panel (b).

B: *Suppose that you face a production technology characterized by the function $x = f(\ell) = \alpha/(1 + e^{-(\ell-\beta)})$.*

(a) *Assuming labor ℓ costs w and the output x can be sold at p, set up the profit maximization problem.*

Answer: The profit maximization problem is

$$\max_{\ell,x}\ px - w\ell \ \text{ subject to }\ x = \frac{\alpha}{1+e^{-(\ell-\beta)}} \tag{11.59}$$

which can be written as the unconstrained problem

$$\max_{\ell}\ \frac{p\alpha}{1+e^{-(\ell-\beta)}} - w\ell. \tag{11.60}$$

(b) *Derive the first order condition for this problem.*
Answer: The first order condition is

$$\frac{\alpha p e^{-(\ell-\beta)}}{\left[1+e^{-(\ell-\beta)}\right]^2} = w. \tag{11.61}$$

(c) *Substitute $y = e^{-(\ell-\beta)}$ into your first order condition and, using the quadratic formula, solve for y. Then, recognizing that $y = e^{-(\ell-\beta)}$ implies $\ln y = -(\ell-\beta)$, solve for the two implied labor inputs and identify which one is profit maximizing (assuming that an interior production plan is optimal).*
Answer: Substituting $y = e^{-(\ell-\beta)}$, the first order condition reduces to $\alpha p y/(1+y)^2 = w$ which can be written in the form

$$wy^2 + (2w - \alpha p)y + w = 0. \tag{11.62}$$

The quadratic formula then gives two solutions for y:

$$y_1 = \frac{-(2w-\alpha p) + \sqrt{(2w-\alpha p)^2 - 4w^2}}{2w} = \frac{\alpha p - 2w + \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2w} \tag{11.63}$$

and

$$y_2 = \frac{-(2w-\alpha p) - \sqrt{(2w-\alpha p)^2 - 4w^2}}{2w} = \frac{\alpha p - 2w - \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2w}. \tag{11.64}$$

Given that $y = e^{-(\ell-\beta)}$, we can take natural logs of both sides to get $-(\ell-\beta) = \ln y$ or $\ell = \beta - \ln y$. Using the two solutions for y, we therefore get

$$\ell_1 = \beta - \ln\left(\frac{\alpha p - 2w + \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2w}\right) \text{ and}$$
$$\ell_2 = \beta - \ln\left(\frac{\alpha p - 2w - \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2w}\right). \tag{11.65}$$

Since $y_1 > y_2$, we know that $\ell_1 < \ell_2$ — and thus the true profit maximizing labor input (assuming an interior production plan is profit maximizing) is given by

$$\ell(w,r,p) = \beta - \ln\left(\frac{\alpha p - 2w - \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2w}\right). \tag{11.66}$$

(d) *Use your answer to solve for the supply function (assuming an interior solution is optimal).*
Answer: Plugging $\ell = \beta - \ln y_2$ into the production function, we then get

$$x = \frac{\alpha}{1 + e^{-(\beta - \ln y_2 - \beta)}} = \frac{\alpha}{1 + e^{\ln y_2}} = \frac{\alpha}{1 + y_2}. \tag{11.67}$$

Substituting in y_2 from equation (11.64), this then simplifies to the supply function (assuming an interior optimum)

$$x(w,r,p) = \frac{2\alpha w}{\alpha p - \sqrt{\alpha^2 p^2 - 4\alpha w p}}. \tag{11.68}$$

We can also re-write this by multiplying both numerator and denominator by the term $(\alpha p + \sqrt{\alpha^2 p^2 - 4\alpha w p})$ to get

$$x(w,r,p) = \frac{2\alpha w\left(\alpha p + \sqrt{\alpha^2 p^2 - 4\alpha w p}\right)}{\alpha^2 p^2 - (\alpha^2 p^2 - 4\alpha w p)} = \frac{\alpha p + \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2p}. \tag{11.69}$$

(e) *Now use the two-step method to verify your answer. Begin by solving the production function for ℓ to determine how much labor is required for each output level assuming none is wasted.*

Answer: To do the 2-step optimization, we begin by solving the production function $x = \alpha/(1+e^{-(\ell-\beta)})$ for ℓ. We can do this by first multiplying through by $(1+e^{-(\ell-\beta)})$, dividing by x and subtracting 1 from both sides to get

$$e^{-(\ell-\beta)} = \frac{\alpha}{x} - 1 = \frac{\alpha - x}{x} \tag{11.70}$$

which allows us to write

$$-(\ell-\beta) = \ln\left(\frac{\alpha - x}{x}\right) \tag{11.71}$$

which then solves to

$$\ell = \beta - \ln\left(\frac{\alpha - x}{x}\right). \tag{11.72}$$

(f) *Use your answer to derive the cost function and the marginal cost function.*

Answer: The minimum cost at which the firm can produce any level of output x is then simply this inverted production function times the wage; i.e. the cost function is

$$C(w,x) = w\beta - w\ln\left(\frac{\alpha - x}{x}\right). \tag{11.73}$$

From this, we can get the marginal cost function

$$MC(w,x) = \frac{\partial C}{\partial x} = \frac{\alpha w}{(\alpha - x)x}. \tag{11.74}$$

(g) *Set price equal to marginal cost and solve for the output supply function (assuming an interior solution is optimal). Can you get your answer into the same form as the supply function from your direct profit maximization problem?*

Answer: Setting $MC = p$, we can then solve for the supply function; i.e. we set $p = \alpha w/((\alpha - x)x)$, multiply through and write it in the form that allows us to once again apply the quadratic formula:

$$x^2 - \alpha x + \frac{\alpha w}{p} = 0 \quad \text{or equivalently} \quad px^2 - \alpha p x + \alpha w = 0. \tag{11.75}$$

Applying the quadratic formula, we get two "solutions":

$$x_1 = \frac{\alpha p + \sqrt{\alpha^2 p^2 - 4p\alpha w}}{2p} \quad \text{and} \quad x_2 = \frac{\alpha p - \sqrt{\alpha^2 p^2 - 4p\alpha w}}{2p} \tag{11.76}$$

of which the first one is the true solution (since it is the larger of the two). The supply function (assuming an interior solution) is therefore

$$x(w,r,p) = \frac{\alpha p + \sqrt{\alpha^2 p^2 - 4p\alpha w}}{2p} \tag{11.77}$$

which is equivalent to the previous solution we got in equation (11.69) by solving the profit maximization problem directly.

(h) *Use the supply function and your answer from part (e) to derive the labor input demand function (assuming an interior solution is optimal). Is it the same as what you derived through direct profit maximization in part (c)?*

Answer: Plugging the supply function into equation (11.72), we get

$$\ell = \beta - \ln\left[\frac{\alpha - \frac{\alpha p + \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2p}}{\frac{\alpha p + \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2p}}\right] = \beta - \ln\left[\frac{\alpha p - \sqrt{\alpha^2 p^2 - 4\alpha w p}}{\alpha p + \sqrt{\alpha^2 p^2 - 4\alpha w p}}\right]. \tag{11.78}$$

Multiplying both the denominator and numerator within the brackets by the numerator $(\alpha p - \sqrt{\alpha^2 p^2 - 4\alpha w p})$, we can then write this as

$$\begin{aligned} \ell &= \beta - \ln\left[\frac{\alpha^2 p^2 - 2\alpha p\sqrt{\alpha^2 p^2 - 4\alpha w p} + (\alpha^2 p^2 - 4\alpha w p)}{\alpha^2 p^2 - (\alpha^2 p^2 - 4\alpha w p)}\right] \\ &= \beta - \ln\left(\frac{\alpha p - 2w - \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2w}\right) \end{aligned} \tag{11.79}$$

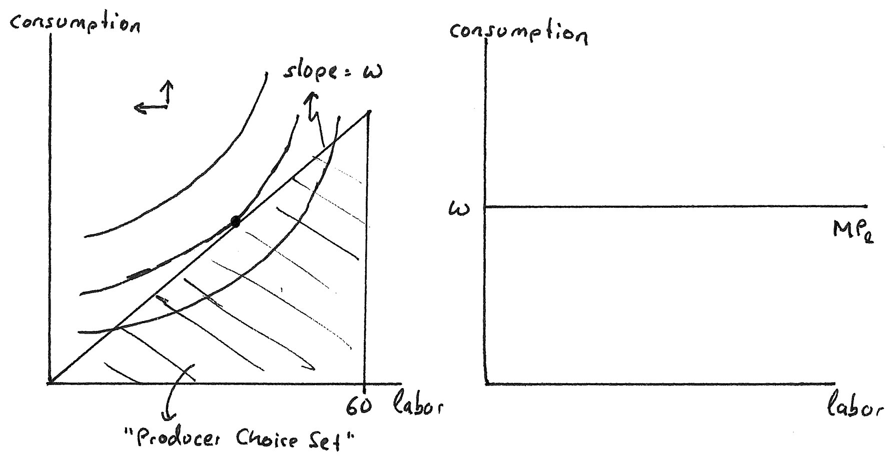
which is exactly equal to the labor demand function we derived in equation (11.66) through direct profit maximization.

11.8 Everyday Application: *Workers as Producers of Consumption: We can see some of the connections between consumer and producer theory by re-framing models from consumer theory in producer language.*

A: *Suppose we modeled a worker as a "producer of consumption" who can sell leisure of up to 60 hours per week at a wage w.*

(a) *On a graph with "labor" as the input on the horizontal axis and "consumption" as the output on the vertical, illustrate what the producer choice set faced by such a "producer" would look like.*

Answer: This is illustrated in panel (a) of Graph 11.9.



Graph 11.9: Workers as "Producers of Consumption"

(b) *How is this fundamentally different from the usual producer case where the producer choice set has nothing to do with prices in the economy?*

Answer: Typically, the producer's choice set is determined by the state of the technology — how easy it is to turn inputs into outputs. It has nothing to do with prices. In this case, however, the way that the worker "produces" consumption is by *selling* his labor — and thus the market wage is the defining characteristic of the "technology" used by the worker.

(c) *What does the marginal product of labor curve look like for this "producer"?*

Answer: This is depicted in panel (b) of Graph 11.9.

(d) *On the graph you drew for part (a), illustrate what "producer tastes" for this producer would look like assuming the worker's tastes over consumption and leisure satisfy the usual five assumptions for tastes we developed in Chapter 4. How is this fundamentally different from the usual producer case where the producer's indifference curves are formed by prices in the economy?*

Answer: Three indifference curves are illustrated in panel (a) of the graph. The worker becomes better off in the direction of the arrows — i.e. as he moves to the north-west in the graph. These indifference curves are derived strictly from the internal tastes of the worker — and have nothing to do with market prices. This is different from the typical producer case where "indifference curves" are simply isoprofits — and where the producer is equally well off as long as profit does not change. Since profit is determined by subtracting costs from revenues — and since costs and revenues are impacted by market prices, these isoprofits typically depend on market prices (with the slope equal to a ratio of input over output prices).

B: *Suppose the worker's tastes over consumption and leisure are Cobb-Douglas with equal weights on the two variables in the utility function.*

(a) *Derive an expression for the production function in this model.*

Answer: The production function is simply $c = w\ell$ where c is consumption and ℓ is labor input.

(b) *Set up the worker's optimization problem similar to a profit maximization problem for producers.*

Answer: Typically, we would set up the problem as a "profit maximization" problem. The worker, however, is not maximizing profit — he is maximizing utility. Let the utility function be given by $u(c,l) = cl$ where c is consumption and l is leisure. (It does not matter what you let the exponents on c and l be as long as they are equal.) Since the worker has 60 hours of leisure he can sell per week, we can express l as $60-\ell$ (where ℓ is labor) — and this permits us to write the utility function as $u(c,\ell) = c(60-\ell)$. The worker's optimization problem is then

$$\max_{c,\ell} \; c(60-\ell) \text{ subject to } c = w\ell. \tag{11.80}$$

We can furthermore substitute the constraint into the objective function to write the problem as

$$\max_{\ell} w\ell(60-\ell). \tag{11.81}$$

(c) *Derive the "output supply" function — i.e. the function that tells us how much consumption the worker will "produce" for different economic conditions.*

Answer: Solving the problem set up above (either using the Lagrange method for the problem as expressed in (11.80) or simply taking a first derivative for the problem as expressed in (11.81), we get

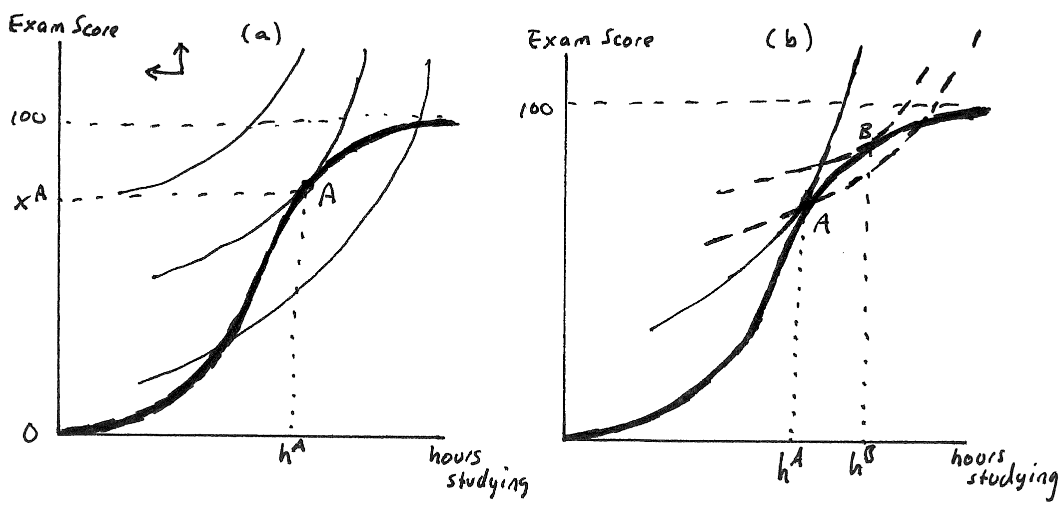
$$\ell = 30 \text{ and } c = 30w. \tag{11.82}$$

11.9 Everyday Application: *Studying for an Exam: Consider the problem you face as a student as you determine how much to study for an exam by modeling yourself as a "producer of an exam score" between 0 and 100.*

A: *Suppose that the marginal payoff to studying for the initial hours you study increases but that this marginal payoff eventually declines as you study more.*

(a) *Illustrate, on a graph with "hours studying for the exam" as an input on the horizontal axis and "exam score" (ranging from 0 to 100) as an output on the vertical axis, what your production frontier will look like.*

Answer: The production frontier is graphed as the (dark) frontier plotted in panel (a) of Graph 11.10.



Graph 11.10: Studying for an Exam

(b) *Now suppose that your tastes over leisure time (i.e. non-study time) and exam scores satisfies the usual five assumptions about tastes that we outlined in Chapter 4. What will your producer tastes look like. (Be careful to recognize that the producer picture has "hours studying" and not leisure hours on the horizontal axis.)*

Answer: Three indifference curves are graphed in panel (a) of Graph 11.10. You become better off as you move to the north-west in the graph — fewer hours of studying, higher exam grades.

(c) *Combining your production frontier with graphs of your indifference curves, illustrate the optimal number of hours you will study.*

Answer: Bundle A in panel (a) of the graph is your optimal "production plan" — resulting in h^A hours of studying and an exam grade of x^A.

(d) *Suppose that you and your friend differ in that your friend's marginal rate of substitution at every possible "production plan" is shallower than yours. Who will do better on the exam?*

Answer: Your friend will do better, as illustrated in panel (b) of the graph. At your optimal production plan A, your friend's indifference curve cuts the producer choice set in such a way that all the "better" plans (that lie to the north-west) involve greater effort than h^A. Your friend's optimal production plan is B.

(e) *Notice that the same model can be applied to anything we do where the amount of effort is an input and how well we perform a task is the output. As we were growing up, adults often told us: "Anything worth doing is worth doing well." Is that really true?*

Answer: Not really. Both you and your friend could have done very well on the exam — even scoring 100 — by putting more time in. But you also have other priorities in life — which is why you would prefer to devote less time to studying. It would not be optimal to devote all your time to one dimension — to doing well on this exam.

B: *Now suppose that you and your friends Larry and Daryl each face the same "production technology" $x = 3\ell^2 - 0.2\ell^3$ where x is the exam grade and ℓ is the number of hours of studying. Suppose further that each of you has tastes that can be captured by the utility function $u(\ell, x) = x - \alpha\ell$.*

(a) *Calculate your optimal hours of studying as a function of α.*

Answer: The problem that needs to be solved is

$$\max_{\ell,x} x - \alpha\ell \text{ subject to } x = 3\ell^2 - 0.2\ell^3 \tag{11.83}$$

which can be written as an unconstrained optimization problem by replacing x in the objective with the constraint — i.e.

$$\max_{\ell} 3\ell^2 - 0.2\ell^3 - \alpha\ell. \tag{11.84}$$

Taking the first derivative and setting it to zero, we get $6\ell - 0.6\ell^2 - \alpha = 0$ or, rearranging terms slightly,

$$-0.6\ell^2 + 6\ell - \alpha = 0. \tag{11.85}$$

For an equation of the form $ax^2 + bx + c = 0$, the quadratic formula gives the solutions

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{11.86}$$

Using this to solve equation (11.85) (where $a = -0.6$, $b = 6$ and $c = -\alpha$), we get

$$\ell = \frac{-6 + \sqrt{36 - 2.4\alpha}}{-1.2} \text{ and } \ell = \frac{-6 - \sqrt{36 - 2.4\alpha}}{-1.2}. \tag{11.87}$$

We know that these two "solutions" are tangencies between the production frontier and two indifference curves — but only the higher is a true optimum. Thus, the actual solution is simply

$$\ell = \frac{-6 - \sqrt{36 - 2.4\alpha}}{-1.2}. \tag{11.88}$$

(b) *Suppose the values for α are 7, 10, and 13 for you, Larry and Daryl respectively. How much time will each of you study?*

Answer Plugging these values into the solution of equation (11.88), we get that you, Larry and Daryl will study approximately 8.65, 7.87 and 6.83 hours respectively.

(c) *What exam grades will each of you get?*

Answer: Plugging these values of hours studied back into the production function, we get that you, Larry and Daryl will receive the following approximate grades respectively: 95, 88.5 and 76.2.

(d) *If each of you had 10 hours available that you could have used to study for the exam, could you each have made a 100? If so, why didn't you?*

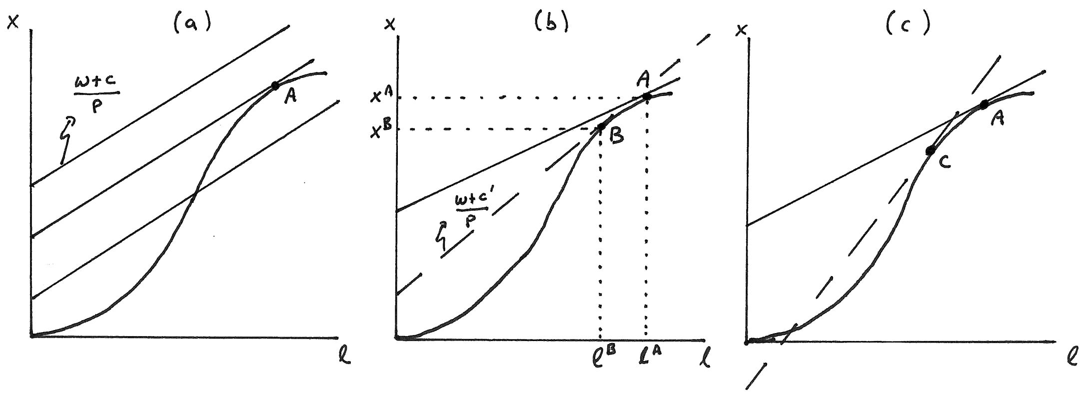
Answer: Yes, each of you could have scored 100 by simply putting 10 hours of effort into preparing for the exam. (The answer is the same for each of you since your "production technology" is the same — just check that when you plug 10 into the production function, you actually get 100.) But you would have had to spend all your time preparing for the exam — which, given your tastes that place value on doing other things, was not worth it.

11.10 Business Application: *Optimal Response to Labor Regulations: Governments often impose costs on businesses in direct relation to how much labor they hire. They may, for instance, require that businesses provide certain benefits like health insurance or retirement plans.*

A: *Suppose we model such government regulations as a cost c per worker in addition to the wage w that is paid directly to the worker. Assume that you face a production technology that has the typical property of initially increasing marginal product of labor that eventually diminishes.*

(a) *Illustrate the isoprofits for this firm and include both the explicit labor cost w as well as the implicit cost c of the regulation.*

Answer: Three isoprofits are illustrated in panel (a) of Graph 11.11. The only difference from the usual case is that we must include c as part of the labor cost to the firm — thus causing the slope of the isoprofit curves to be $(w+c)/p$.



Graph 11.11: Increasing Regulatory Labor Costs

(b) *Illustrate the profit maximizing production plan.*

Answer: This is illustrated as A in panel (a) of the graph.

(c) *Assuming that it continues to be optimal for your firm to produce, how does your optimal production plan change as c increases?*

Answer: When c increases to c', the slope of the isoprofits get steeper. Thus, the optimal production plan in panel (b) of Graph 11.11 changes from A to B — less labor input and lower output.

(d) *Illustrate a case where an increase in c is sufficiently large to cause your firm to stop producing.*

Answer: This is illustrated in panel (c) where the new (dashed) isoprofit has become sufficiently steep as a result of an increase in c such that the "optimal" production plan C lies on an isoprofit with negative vertical intercept — and thus negative profit. As a result, this firm would not produce at C but would simply choose to hire no labor and produce no output.

(e) True or False*: For firms that make close to zero profit, additional labor regulations might cause large changes in behavior.*

Answer: This is true. When profit is high, the regulatory cost associated with labor can be large without causing profit to fall to zero. While this would still cause a change in firm behavior (as illustrated in panel (b)), it would be a marginal change — somewhat less labor and somewhat less output. But if profit initially is close to zero, then even a small increase in regulatory labor costs can cause the firm to shut down completely — and thus cause a dramatic change in behavior.

B: *Suppose that your production technology can be represented by the production function* $x = 100/(1+e^{-(\ell-5)})$ *where* $e \approx 2.7183$ *is the base of the natural logarithm.*

(a) *Suppose* $w = 10$ *and* $p = 1$. *Set up your profit maximization problem and explicitly include the cost of regulation.*

Answer: The profit maximization problem is

$$\max_{x,\ell} \; x-(10+c)\ell \;\text{ subject to }\; x=\frac{100}{1+e^{-(\ell-5)}} \tag{11.89}$$

which can also be written as the unconstrained optimization problem

$$\max_{\ell} \frac{100}{1+e^{-(\ell-5)}}-(10+c)\ell. \tag{11.90}$$

(b) *Calculate the optimal labor demand and output supply as a function of* c. *(*Hint*: Solving the first order condition becomes considerably easier if you substitute* $y=e^{-(\ell-5)}$ *and solve for* y *using the quadratic formula. Once you have a solution for* y*, you know this is equal to* $e^{-(\ell-5)}$. *You can then take natural logs of both sides, recalling that* $\ln e^{-(\ell-5)} = -(\ell-5)$. *This follows the steps in exercise 11.7 where we used an almost identical production function.)*

Answer: The first order condition for the unconstrained problem above is

$$\frac{100e^{-(\ell-5)}}{\left(1+e^{-(\ell-5)}\right)^2}-(10+c)=0. \tag{11.91}$$

Define $y=e^{-(\ell-5)}$. Substituting this into the above, we get

$$\frac{100y}{(1+y)^2}-(10+c)=0. \tag{11.92}$$

Multiplying out the denominator and re-arranging terms, we then get

$$(10+c)y^2+(2c-80)y+(10+c)=0. \tag{11.93}$$

For an equation of the form $ax^2+bx+c=0$, the quadratic formula gives the solutions

$$x=\frac{-b+\sqrt{b^2-4ac}}{2a} \;\text{ and }\; x=\frac{-b-\sqrt{b^2-4ac}}{2a}. \tag{11.94}$$

We can thus solve for y by applying this formula (letting $a=(10+c)$, $b=(2c-80)$ and $c=(10+c)$) to get two candidate solutions:

$$y=\frac{80-2c+\sqrt{(2c-80)^2-4(10+c)^2}}{2(10+c)} \;\text{ and }\; y=\frac{80-2c-\sqrt{(2c-80)^2-4(10+c)^2}}{2(10+c)}. \tag{11.95}$$

Taking natural logs of $y=e^{-(\ell-5)}$, we can write $-(\ell-5)=\ln y$ where y can take one of the two values we solved for above. Solving for ℓ, we get $\ell = 5-\ln y$. The larger of the two solutions that this provides is the profit maximizing quantity (assuming it yields positive profit).

(c) *What is the profit maximizing production plan when* $c=0$?

Answer: When $c=0$, the two solutions for y reduce to $y\approx 7.873$ and $y\approx 0.127$. The corresponding solutions for ℓ are then

$$\ell = 5-\ln(7.873)=2.937 \;\text{ and }\; \ell=5-\ln(0.127)=7.063. \tag{11.96}$$

Of these, the larger is the true profit maximum — and plugging this into the production function, it gives output of $x=88.73$.

(d) *How does your answer change when* $c=2$?

Answer: The two solutions for y are now 6.171 and 0.162 — with the latter resulting in the higher labor input of $\ell=6.82$ and output of $x=86.056$.

(e) *What if $c = 3$? (*Hint*: Check to see what happens to profit.)*

Answer: The two solutions for y are now 5.511 and 0.181 — with the latter resulting in the higher labor input of $\ell = 6.707$ and output of $x = 84.641$. However, when we evaluate profit at this production plan, we get

$$\text{Profit} = 84.641 - (10+3)6.707 = -2.55. \tag{11.97}$$

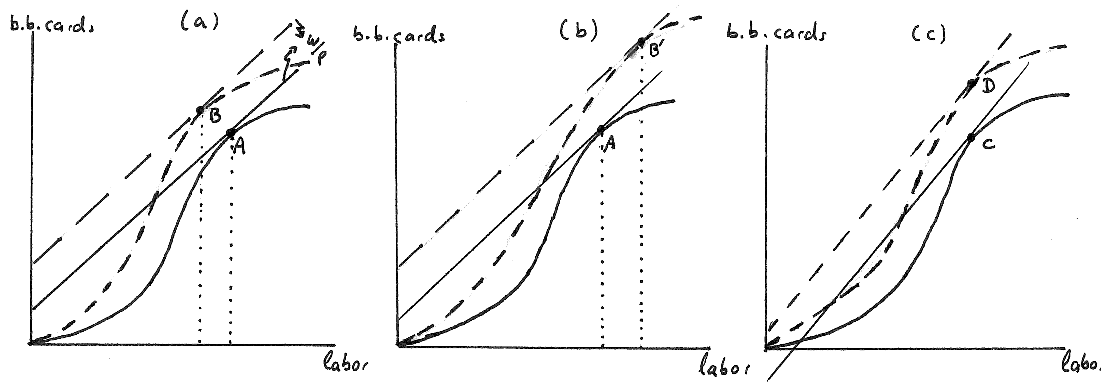
Since profit at the "optimal" production plan is negative, the true optimal production plan is hiring no labor and producing no output (which results in zero profit). (You can check that for $c = 0$ and $c = 2$ in the previous parts, profit for producing the production plans we identified is in fact positive.)

11.11 Business Application: *Technological Change in Production: Suppose you and your friend Bob are in the business of producing baseball cards.*

A: *Both of you face the same production technology which has the property that the marginal product of labor initially increases for the first workers you hire but eventually decreases. You both sell your cards in a competitive market where the price of cards is p, and you hire in a competitive labor market were the wage is w.*

(a) *Illustrate your profit maximizing production plan assuming that p and w are such that you and Bob can make a positive profit.*

Answer: This is shown in the solid lines of panel (a) of Graph 11.12 where A is the profit maximizing production plan given the tangency of the (solid) isoprofit line at A.



Graph 11.12: Technological Innovation

(b) *Now suppose you find a costless way to improve the technology of your firm in a way that unambiguously expands your producer choice set. As a result, you end up producing more than Bob (who has not found this technology). Illustrate how the new technology might have changed your production frontier.*

Answer: Two possibilities are illustrated with the dashed portions of panels (a) and (b) of Graph 11.12. In panel (a), your new profit maximizing production plan is B — which results in more baseball cards. In panel (b), your new profit maximizing production plan is B' which also results in more baseball cards.

(c) *Can you necessarily tell whether you will hire more or less labor with the new technology?*

Answer: No, it is not clear. In panel (a), you end up hiring fewer workers — while in panel (b) you end up hiring more workers.

(d) *Can you say for sure that adopting the new technology will result in more profit?*

Answer: Yes, profit increases unambiguously in both panels (as seen in the higher intercept of the dashed isoprofit curve relative to the solid one.)

(e) *Finally, suppose p falls. Illustrate how it might now be the case that Bob stops producing but you continue to stay in the business.*

Answer: A drop in p causes the slope of the isoprofits (i.e. w/p) to increase. Panel (c) then illustrates a case where Bob's "optimal" production plan C is not actually optimal because the isoprofit that is tangent at C has negative intercept and thus involves negative profit. You, on the other hand, maximize profit at D along an isoprofit with positive intercept (and thus positive profit). So Bob shuts down and you remain open.

B: *You and Bob initially face the production technology $x = 3A\ell^2 - 0.1\ell^3$, and you can sell your output for p and hire workers at a wage w.*

(a) *Derive the marginal product of labor and describe its properties.*

Answer: The marginal product of labor is calculated from the production function $x = f(\ell)$ as

$$MP_\ell = \frac{df}{d\ell} = 6A\ell - 0.3\ell^2. \tag{11.98}$$

This function is initially increasing but eventually decreases. To find the quantity of ℓ at which the marginal product of labor begins to decline, all we have to do is determine where the derivative of MP_ℓ is equal to zero; i.e. we need to solve

$$\frac{dMP_\ell}{d\ell} = 6A - 0.6\ell = 0 \tag{11.99}$$

which solves to $\ell = 10A$. Thus, marginal product of labor increases until $\ell = 10A$ and falls thereafter. You can also check that marginal product of labor becomes negative at $\ell = 20A$.

(b) *Calculate the optimal number of baseball cards as a function of A assuming output price is given by p and the wage is $w = 20$. (Use the quadratic formula.)*

Answer: We need to solve

$$\max_{\ell,x}\; px - 20\ell \;\text{ subject to }\; x = 3A\ell^2 - 0.1\ell^3, \tag{11.100}$$

which can be written as the unconstrained optimization problem

$$\max_{\ell}\; p(3A\ell^2 - 0.1\ell^3) - 20\ell. \tag{11.101}$$

The first order condition for this problem is

$$-0.3p\ell^2 + 6Ap\ell - 20 = 0. \tag{11.102}$$

For an equation of the form $ax^2 + bx + c = 0$, the quadratic formula gives the solutions

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \;\text{ and }\; x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{11.103}$$

We can use this formula to solve equation (11.102) and get the two "solutions"

$$\ell = \frac{-6Ap + \sqrt{36A^2p^2 - 24p}}{-0.6p} \;\text{ and }\; \ell = \frac{-6Ap - \sqrt{36A^2p^2 - 24p}}{-0.6p}, \tag{11.104}$$

of which the latter is the true optimum (since we know from our intuitive treatment of the material that the larger of the two input amounts represents the profit maximum.)

(c) *How much will you each produce if $A = 1$ and $p = 1$, and how much profit do each of you earn?*

Answer: When $p = 1$ and $A = 1$ are plugged into the true optimum we just calculated, we get $\ell \approx 15.77$. (The "false" optimum in the first equation would come out to be $\ell \approx 4.23$.)

We can calculate the profit by first calculating the output you would produce with this much labor. Plugging $\ell = 15.77$ into the production function, we get $x \approx 353.96$. We can then calculate the profit as $\pi = px - w\ell = (1)(353.96) - 20(15.77) \approx 38.5$. (You can also verify that the second "solution" for labor input is not an optimum by calculating the profit you would get under that input level — which turns out to be -38.5.)

(d) *Now suppose you find a better technology — one that changes your production function from one where $A = 1$ to one where $A = 1.1$. How do your answers change?*

Answer: You will now hire $\ell = 18.20$, produce $x = 493.72$ and earn profit $\pi = 126.30$. (You can derive these exactly as you did for the previous case.)

(e) *Now suppose that competition in the industry intensifies and the price of baseball cards falls to* $p = 0.88$. *How will you and Bob change your production decisions?*

Answer: Bob still operates under $A = 1$. Plugging in $A = 1$ and $p = 0.88$ into our solution for the optimal labor input, we get $\ell \approx 14.92$. Plugging this into the production function, this would give us $x \approx 335.77$, and using these to calculate profit, we get a profit of $\pi \approx -2.99$. Since the highest profit Bob can get by producing is negative, he will no longer produce.

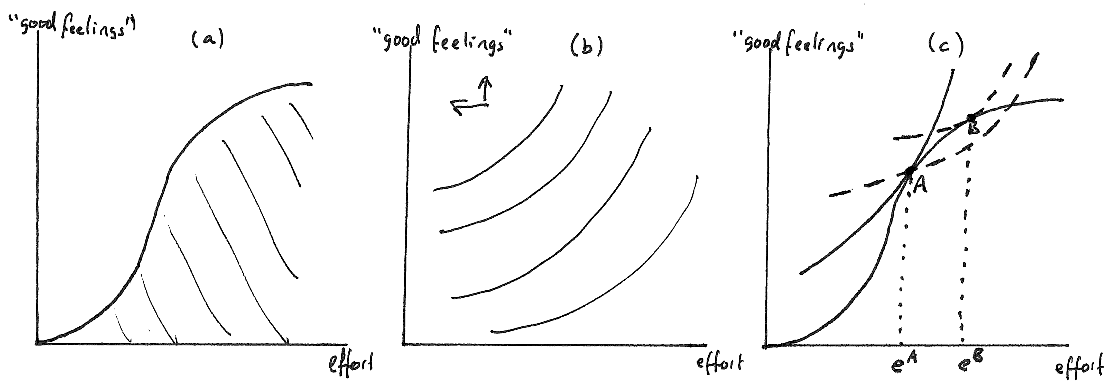
You, on the other hand, face a production process with $A = 1.1$. Following the same procedure, you can then calculate that your optimal production plan is $(\ell, x) \approx (17.73, 479.93)$ which gives profit of $\pi \approx 67.8$. Since profit is positive, you will indeed produce this level of output.

11.12 Policy Application: *Politicians as Producers of "Good Feelings": Consider a politician who has to determine how much effort he will exert in his re-election campaign.*

A: *We can model such a politician as a "producer of good feelings among voters".*

(a) *Begin with a graph that puts "effort" on the horizontal axis and the "good voter feelings" on the vertical axis. Assume that the marginal payoff from exerting effort initially increases with additional effort but eventually declines. Illustrate this politicians feasible "production plans".*

Answer: This is illustrated in panel (a) of Graph 11.13.



Graph 11.13: Politicians as Producers

(b) *Suppose that the politician dislikes expending effort but likes the higher probability of winning re-election that results from good voter feelings. Assume that tastes are rational, continuous and convex. Illustrate what indifference curves for this politician will look like.*

Answer: This is illustrated in panel (b) of Graph 11.13.

(c) *Combining your two graphs, illustrate the optimal level of effort expended by a politician during his re-election campaign.*

Answer: This is done in panel (c) where the optimal level of effort for the politician whose indifference curves are solid (as opposed to those that are dashed in the graph) is indicated by e^A at the optimal "production plan" A.

(d) *Now suppose that the politician's opponent in the campaign has the same "production technology". Suppose further that, at any "production plan" in the model, the opponent's indifference curve has a shallower slope than the incumbent. Assuming the candidate who has produced more good voter feelings will win, will the incumbent or the challenger win this election?*

Answer: This second politician's tastes therefore have indifference curves that look like the dashed ones in panel (c) of the graph. The dashed indifference curve that goes through A but has a shallower slope necessarily gives rise to production plans that are feasible and that this politician would prefer. The highest indifference curve he can get to is tangent at B, giving rise to effort level e^B. Since $e^B > e^A$, this challenger defeats the incumbent.

B: *Let effort be denoted by ℓ and "good voter feelings" by x. Suppose that a politician's tastes are defined by $u(x,\ell) = x - \alpha\ell$, and suppose that the production frontier for producing "good feelings" among voters is given by $x = \ell^2 - 0.25\ell^3$.*

(a) *When effort ℓ is on the horizontal and x is on the vertical, what is the marginal rate of substitution for this politician?*

Answer: The MRS is the negative of the partial derivative with respect to the "good" on the horizontal axis divided by the partial derivative with respect to the good on the vertical; i.e.

$$MRS = -\left(\frac{\partial u/\partial \ell}{\partial u/\partial x}\right) = -\left(\frac{-\alpha}{1}\right) = \alpha. \tag{11.105}$$

(b) *What does your answer imply for the shape of indifference curves?*

Answer: The indifference curves are straight lines with slope α (just as isoprofits are straight lines with slope w/p).

(c) *Setting this up similar to a profit maximization problem, solve for the politician's optimal level of effort.*

Answer: We can write this problem as the constrained optimization problem

$$\max_{\ell,x} x - \alpha\ell \text{ subject to } x = \ell^2 - 0.25\ell^3, \tag{11.106}$$

or, substituting the constraint into the objective, we can write it as an unconstrained problem (as we have done for single-input producers):

$$\max_{\ell} \ell^2 - 0.25\ell^3 - \alpha\ell. \tag{11.107}$$

Taking the first derivative with respect to ℓ and setting to zero, we get $2\ell - 0.75\ell^2 - \alpha = 0$ or, rearranging terms slightly,

$$-0.75\ell^2 + 2\ell - \alpha = 0. \tag{11.108}$$

For an equation of the form $ax^2 + bx + c = 0$, the quadratic formula gives the solutions

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{11.109}$$

Using this to solve equation (11.108) (where $a = -0.75$, $b = 2$ and $c = -\alpha$), we get

$$\ell = \frac{-2 + \sqrt{4 - 3\alpha}}{-1.5} \text{ and } \ell = \frac{-2 - \sqrt{4 - 3\alpha}}{-1.5} \tag{11.110}$$

We know from our intuitive graphs that only one of these two solutions is a true optimum — the other is a tangency that occurs on the non-convave portion of the production function to the left of the true optimum. Thus, the higher of these two will be the true optimum. It is easy to see that the second of the solutions is unambiguously larger than the first — so the true solution is

$$\ell = \frac{-2 - \sqrt{4 - 3\alpha}}{-1.5}. \tag{11.111}$$

(d) *Compare the optimal effort level for the politician for whom $\alpha = 1$ and the politician for whom $\alpha = 0.77$.*

Answer: Plugging the two values for α into equation (11.111), we get $\ell = 2$ when $\alpha = 1$ and $\ell = 2.2$ when $\alpha = 0.77$. Thus, as α decreases, effort level ℓ increases.

(e) *Which one will win the election? Explain how this makes sense intuitively.*

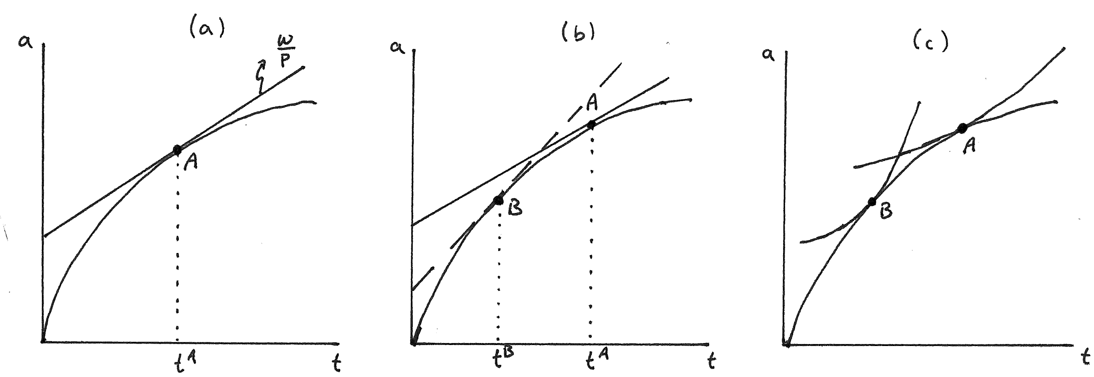
Answer: The politician with $\alpha = 0.77$ will win the election because he will put in more effort. This makes intuitive sense in light of our conclusions in part A and in light of the fact that we determined earlier that α is the MRS. As α falls, the slope of the indifference curves become shallower — and with shallower indifference curves, we get higher levels of effort as the tangency with the production set necessarily occurs farther to the left.

11.13 Policy Application: *Determining Optimal Class Size: Public policy makers are often pressured to reduce class size in public schools in order to raise student achievement.*

A: *One way to model the production process for student achievement is to view the "teacher/student" ratio as the input. For purposes of this problem, let t be defined as the number of teachers per 1000 students; i.e.* $t = 20$ *means there are 20 teachers per 1,000 students. Class size in a school of 1000 students is then equal to* $1000/t$.

(a) *Most education scholars believe that the increase in student achievement from reducing class size is high when class size is high but diminishes as class size falls. Illustrate how this translates into a production frontier with t on the horizontal axis and average student achievement a on the vertical.*

Answer: This production frontier is pictured in panel (a) of Graph 11.14 — with large slope (or marginal product) initially that falls as t increases. Note that t is not class size — when t is small, there are few teachers per 1000 students, which implies a large class size. In our graph, class size therefore falls as t increases.



Graph 11.14: Setting Class Size

(b) *Consider a school with 1,000 students. If the annual salary of a teacher is given by w, what is the cost of raising the input t by 1 — i.e. what is the cost per unit of the input t?*

Answer: Since t is the number of teachers per 1000 students, we need to hire one more teacher to increase t by 1. Thus, the per unit cost of t is w.

(c) *Suppose a is the average score on a standardized test by students in the school, and suppose that the voting public is willing to pay p for each unit increase in a. Illustrate the "production plan" that the local school board will choose if it behaves analogously to a profit maximizing firm.*

Answer: This is also illustrated in panel (a) of Graph 11.14 where A is the optimal "production plan" given the willingness p of voters to get a one unit increase in achievement. The input t^A then results in class size of $1000/t^A$.

(d) *What happens to class size if teacher salaries increase?*

Answer: If teacher salaries increase, w goes up — which implies w/p increases and the isoprofits become steeper. As a result, the optimal t falls — which implies class size increases. This is illustrated in panel (b) of Graph 11.14.

(e) *How would your graph change if the voting public's willingness to pay per unit of a decreases as a increases?*

Answer: This would in effect imply that p is high when a is low but decreases as a increases. A high p implies a relatively flat slope — thus, we would get isoprofit curves that, rather than being straight lines as in panels (a) and (b) of the graph, would be shaped as depicted in panel (c). (Note: The isoprofit tangent at B represents the curve when teacher salaries are higher.)

(f) *Now suppose that you are analyzing two separate communities that fund their equally sized schools from tax contributions by voters in each school district. They face the same production technology, but the willingness to pay for marginal improvements in a is lower in community 1 than in community 2 at every production plan. How do the isoprofit maps differ for the two communities?*

Answer: At every "production plan" the isoprofits of the community with lower willingness to pay would be steeper — indicating that a greater increase in achievement would be required for every increase in t in order for the community to remain equally well off.

(g) *Illustrate how this will result in different choices of class size in the two communities.*

Answer: This can be illustrated in a graph identical to the one pictured in panel (c) of Graph 11.14 where the community with the lower willingness to pay optimizes at B while the other community optimizes at A.

(h) *Suppose that the citizens in each of the two communities described above were identical in every way except that those in community 1 have a different average income level than those in community 2. Can you hypothesize which of the two communities has greater average income?*

Answer: Community 1 is poorer. Even though they care as much about education, their willingness to pay is lower simply because they have fewer resources. (This should be familiar from consumer theory — people with identical tastes can have very different demands for goods if they have different incomes.)

(i) *Higher level governments often subsidize local government contributions to public education, particularly for poorer communities. What changes in your picture of a community's optimal class size setting when such subsidies are introduced?*

Answer: You can think of these subsidies as either reducing the teacher salaries w (since the higher level government now shares in the expense) or as raising the willingness to pay p since the voters know that they get more resources if they spend more on education. Whether viewed as a decrease in w or and increase in p, the impact on w/p is the same: w/p falls — which implies that isoprofit curves become shallower. This in turn has the impact of increasing the optimal choice of t (as illustrated in panels (b) and (c) of Graph 11.14) — which is the same as saying that class size will decrease.

B: *Suppose the production technology for average student achievement is given by $a = 100t^{0.75}$, and suppose again that we are dealing with a school that has 1000 students.*

(a) *Let w denote the annual teacher salary in thousands of dollars and let p denote the community's marginal willingness to pay for an increase in student achievement. Calculate the "profit maximizing" class size.*

Answer: We need to solve the maximization problem

$$\max_{a,t} \; pa - wt \;\text{ subject to }\; a = 100t^{0.75} \tag{11.112}$$

which can be written as the unconstrained optimization problem

$$\max_{t} \; a = 100t^{0.75}p - wt. \tag{11.113}$$

The first order condition for this problem is $75t^{-0.25}p - w = 0$ and can be solved to get

$$t = \left(\frac{75p}{w}\right)^4. \tag{11.114}$$

Since class size is $1000/t$, the optimal class size is

$$\text{Class size} = \frac{1000w^4}{(75p)^4}. \tag{11.115}$$

(b) *What is the optimal class size when $w = 60$ and $p = 2$?*

Answer: Plugging these into the expression we just derived, we get an optimal class size of 25.6 students per class.

(c) *What happens to class size as teacher salaries change?*

Answer: We can simply take the derivative of our optimal class size expression with respect to w to see if it is positive or negative. Since it is clearly positive, we know that class size increases as teacher salaries increase (and decreases as teacher salaries decrease).

(d) *What happens to class size as the community's marginal willingness to pay for student achievement changes?*

Answer: This time we would take the derivative of our optimal class size expression with respect to p. Since this derivative is clearly negative, we know that class size will increase as p decreases (and decrease as p increases).

(e) *What would change if the state government subsidizes the local contribution to school spending?*

Answer: This would effectively lower w — and thus decrease class size. Alternatively you could think of it as raising the local willingness to spend money on schools — which would also have the effect of decreasing class size.

(f) *Now suppose that the community's marginal willingness to pay for additional student achievement is a function of the achievement level. In particular, suppose that $p(a) = Ba^{\beta-1}$ where $\beta \le 1$. For what values of β and B is the problem identical to the one you just solved?*

Answer: If $B = p$ and $\beta = 1$, the problem is identical to what we solved before.

(g) *Solve for the optimal t given the marginal willingness to pay of $p(a)$. What is the optimal class size when $B = 3$ and $\beta = 0.95$ (assuming again that $w = 60$.*

Answer: We would now solve the problem

$$\max_{a,t}\ p(a)a - wt = Ba^{\beta-1}a - wt = Ba^{\beta} - wt \text{ subject to } a = 100t^{0.75} \tag{11.116}$$

which can be written as the unconstrained maximization problem

$$\max_{t}\ B\left(100t^{0.75}\right)^{\beta} - wt. \tag{11.117}$$

The first order condition for this problem is

$$0.75\beta B(100^{\beta})t^{0.75\beta-1} - w = 0, \tag{11.118}$$

which solves to

$$t = \left(\frac{w}{0.75\beta B\left(100^{\beta}\right)}\right)^{1/(0.75\beta-1)} \tag{11.119}$$

Substituting in $w = 60$, $\beta = 0.95$ and $B = 3$, we then get $t \approx 37.27$. Given that class size is $1000/t$, we get that the optimal class size under these conditions is approximately 26.83 students per class.

(h) *Under the parameter values just specified, does class size respond to changes in teacher salaries as it did before?*

Answer: Under the parameter values specified above, the optimal input level t becomes

$$t = \left(\frac{w}{0.75(0.95)(3)\left(100^{0.95}\right)}\right)^{1/(0.75(0.95)-1)} \approx 169.79w^{-3.48}. \tag{11.120}$$

This implies an optimal class size of

$$\text{Class Size} = \frac{1000}{t} = \frac{1000}{169.79w^{-3.48}} \approx 5.89w^{3.48}. \tag{11.121}$$

The derivative of class size with respect to w is then clearly positive — which implies that class size increases as teacher salaries increase.