SOLUTIONS

12

Production with Multiple Inputs

Solutions for *Microeconomics: An Intuitive Approach with Calculus*

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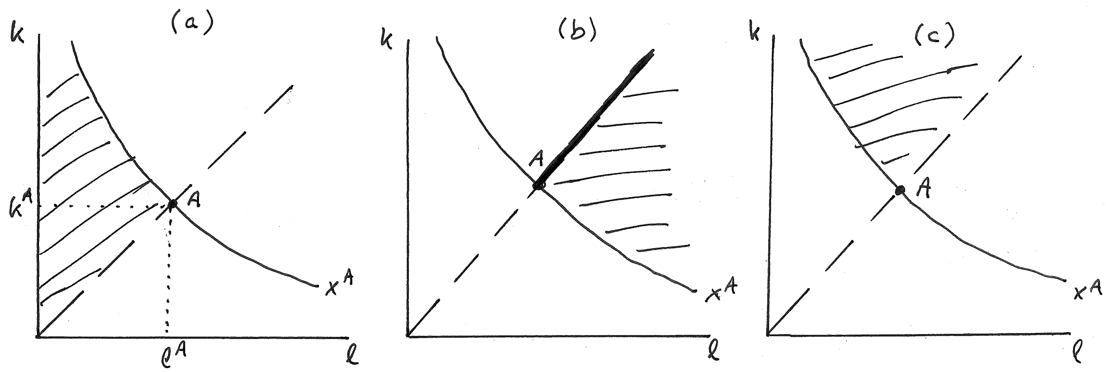
- Each end-of-chapter exercise begins on a new page. This is to facilitate maximum flexibility for instructors who may wish to share answers to some but not all exercises with their students.
- If you are assigning only the A-parts of exercises in *Microeconomics: An Intuitive Approach with Calculus*, you may wish to instead use the solution set created for the companion book *Microeconomics: An Intuitive Approach*.
- *Solutions to Within-Chapter Exercises are provided in the student* Study Guide.

12.1 *In our development of producer theory, we have found it convenient to assume that the production technology is homothetic.*

A: *In each of the following, assume that the production technology you face is indeed homothetic. Suppose further that you currently face input prices (w^A, r^A) and output price p^A — and that, at these prices, your profit maximizing production plan is $A = (\ell^A, k^A, x^A)$.*

(a) *On a graph with ℓ on the horizontal and k on the vertical, illustrate an isoquant through the input bundle (ℓ^A, k^A). Indicate where all cost minimizing input bundles lie given the input prices (w^A, r^A).*

Answer: This is depicted in panel (a) of Graph 12.1. Since the isocosts must be tangent at the profit maximizing input bundle A, homotheticity implies that all tangencies of isocosts with isoquants lie on the ray from the origin that passes through A.



Graph 12.1: Changing Prices and Profit Maximization

(b) *Can you tell from what you know whether the shape of the production frontier exhibits increasing or decreasing returns to scale along the ray you indicated in (a)?*

Answer: You cannot tell whether the production frontier has increasing or decreasing returns to scale along the entire ray from the origin.

(c) *Can you tell whether the production frontier has increasing or decreasing returns to scale around the production plan $A = (\ell^A, k^A, x^A)$?*

Answer: Yes, you can tell that it must have decreasing returns to scale at A — because the isoprofit must be tangent at that point in order for A to be the profit maximizing production plan.

(d) *Now suppose that wage increases to w'. Where will your new profit maximizing production plan lie relative to the ray you identified in (a)?*

Answer: When w increases, the isocosts become steeper — which implies that they are tangent to the isoquants to the left of the ray that goes through A. Thus, the new ray on which all cost minimizing production plans lie is steeper than the ray drawn in panel (a) of Graph 12.1. Since the new profit maximizing production plan must lie on that ray (because profit maximization implies cost minimization), the new profit maximizing production plan must lie to the left of the ray that passes through A.

(e) *In light of the fact that supply curves shift to the left as input prices increase, where will your new profit maximizing input bundle lie relative to the isoquant for x^A?*

Answer: The leftward shift of supply curves as w increases implies that the profit maximizing output level falls. Thus, the new profit maximizing input bundle must lie *below* the x^A isoquant.

(f) *Combining your insights from (d) and (e), can you identify the region in which your new profit maximizing bundle will lie when wage increases to w'?*

Answer: This is illustrated as the shaded area in panel (a) of Graph 12.1. The shaded area emerges from the insight in (d) that the new profit maximizing bundle lies to the *left* of the ray through A and from the insight in (e) that it must lie *below* the isoquant for x^A.

(g) *How would your answer to (f) change if wage fell instead?*

Answer: If wage falls instead, then the isocosts become shallower — which implies that all cost minimizing bundles will now lie to the *right* of the ray through A. A drop in w will furthermore shift the output supply curve to the right — which implies that the profit maximizing production plan will involve an increase in the production of x. Thus, the new profit maximizing plan must lie to the *right* of the ray through A (because profit maximization implies cost minimization) and it must lie *above* the isoquant for x^A (because output increases). This is indicated as the shaded area in panel (b) of Graph 12.1.

(h) *Next, suppose that, instead of wage changing, the output price increases to p'. Where in your graph might your new profit maximizing production plan lie? What if p decreases?*

Answer: When output price p changes, the slopes of the isocosts (which are equal to $-w/r$) remain unchanged. Thus, all cost minimizing production plans remain on the ray through A. Since supply curves slope up, an increase in p will cause an increase in output — implying that the new profit maximizing production plan lies *above* the isoquant for x^A. Thus, when p increases, the new profit maximizing production plan lies on the bold portion of the ray through A as indicated in panel (b) of Graph 12.1. When p decreases, on the other hand, output falls — which implies that the new profit maximizing production plan lies on the dashed portion of the ray through A in panel (b) of the graph.

(i) *Can you identify the region in your graph where the new profit maximizing plan would lie if instead the rental rate r fell?*

Answer: If r falls, the isocosts become steeper — implying the ray containing all cost minimizing production plans will be steeper than the ray through A. Thus, cost minimization implies that the new profit maximizing input bundle will lie to the *left* of the ray through A. A decrease in r further implies a shift in the supply curve to the right — which implies that output will increase. Thus, the profit maximizing input bundle must lie *above* the isoquant for x^A. This gives us the region to the *left* of the ray through A and *above* the isoquant x^A — which is equal to the shaded region in panel (c) of Graph 12.1.

B: *Consider the Cobb-Douglas production function $f(\ell,k)=A\ell^{\alpha}k^{\beta}$ with $\alpha,\beta>0$ and $\alpha+\beta<1$.*

(a) *Derive the demand functions $\ell(w,r,p)$ and $k(w,r,p)$ as well as the output supply function $x(w,r,p)$.*

Answer: These result from the profit maximization problem

$$\max_{\ell,k,x}\; px-w\ell-rk \;\text{ subject to }\; x=A\ell^{\alpha}k^{\beta} \tag{12.1}$$

which can also be written as

$$\max_{\ell,k}\; pA\ell^{\alpha}k^{\beta}-w\ell-rk. \tag{12.2}$$

Taking first order conditions and solving these, we then get input demand functions

$$\ell(w,r,p)=\left(\frac{pA\alpha^{(1-\beta)}\beta^{\beta}}{w^{(1-\beta)}r^{\beta}}\right)^{1/(1-\alpha-\beta)} \quad\text{and}\quad k(w,r,p)=\left(\frac{pA\alpha^{\alpha}\beta^{(1-\alpha)}}{w^{\alpha}r^{(1-\alpha)}}\right)^{1/(1-\alpha-\beta)}. \tag{12.3}$$

Plugging these into the production function and simplifying, we also get the output supply function

$$x(w,r,p)=\left(\frac{Ap^{(\alpha+\beta)}\alpha^{\alpha}\beta^{\beta}}{w^{\alpha}r^{\beta}}\right)^{1/(1-\alpha-\beta)} \tag{12.4}$$

(b) *Derive the conditional demand functions $\ell(w,r,x)$ and $k(w,r,x)$.*

Answer: We need to solve the cost minimization problem

$$\min_{\ell,k} \ w\ell + rk \ \text{ subject to } \ x = A\ell^{\alpha}k^{\beta}. \tag{12.5}$$

Setting up the Lagrangian and solving the first order conditions, we then get the conditional input demand functions

$$\ell(w,r,x) = \left(\frac{\alpha r}{\beta w}\right)^{\beta/(\alpha+\beta)}\left(\frac{x}{A}\right)^{1/(\alpha+\beta)} \ \text{ and } \ k(w,r,x) = \left(\frac{\beta w}{\alpha r}\right)^{\alpha/(\alpha+\beta)}\left(\frac{x}{A}\right)^{1/(\alpha+\beta)} \tag{12.6}$$

(c) *Given some initial prices (w^A, r^A, p^A), verify that all cost minimizing bundles lie on the same ray from the origin in the isoquant graph.*

Answer: Dividing the conditional input demands by one another, we get

$$\frac{k(w^A,r^A,x)}{\ell(w^A,r^A,x)} = \frac{\beta w^A}{\alpha r^A}. \tag{12.7}$$

Thus, regardless of what isoquant x we try to reach, the ratio of capital to labor that minimizes the cost of reaching that isoquant is independent of x — implying that all cost minimizing input bundles lie on a ray from the origin.

(d) *If w increases, what happens to the ray on which all cost minimizing bundles lie?*

Answer: If w increases to w', the ratio of capital to labor becomes

$$\frac{\beta w'}{\alpha r^A} > \frac{\beta w^A}{\alpha r^A}; \tag{12.8}$$

i.e. the ray becomes steeper as firms substitute away from labor and toward capital.

(e) *What happens to the profit maximizing input bundles?*

Answer: We see from the input demand equations in (12.3) that both labor and capital demand fall as w increases. (Similarly, we see from equation (12.4) that output supply falls.)

(f) *How do your answers change if w instead decreases?*

Answer: When wage falls to w'', we get that the ray on which cost minimizing bundles occur is

$$\frac{\beta w''}{\alpha r^A} < \frac{\beta w^A}{\alpha r^A}; \tag{12.9}$$

i.e. the ray becomes shallower. From the input demand functions, we also see that demand for labor and capital increase — as does output (as seen in the output supply function).

(g) *If instead p increases, does the ray along which all cost minimizing bundles lie change?*

Answer: The ray along which cost minimizing bundles lie is defined by the ratio of conditional capital to conditional labor demand — which is

$$\frac{k(w,r,x)}{\ell(w,r,x)} = \frac{\beta w}{\alpha r}. \tag{12.10}$$

Since this does not depend on p, we can see that the ray does not depend on output price. This should make sense: Cost minimization does not take output price into account since all it asks is: "what is the least cost way of producing x?"

(h) *Where on that ray will the profit maximizing production plan lie?*

Answer: Since the ray of cost minimizing input bundles remains unchanged, we know that the new profit maximizing plan lies somewhere on that ray. From the output supply equation (12.4), we see that output increases with p. Thus, the new profit maximizing production plan lies above the initial isoquant and on the same ray as the initial profit maximizing production plan.

(i) *What happens to the ray on which all cost minimizing input bundles lie if r falls? What happens to the profit maximizing input bundle?*

Answer: If r falls to r', we get

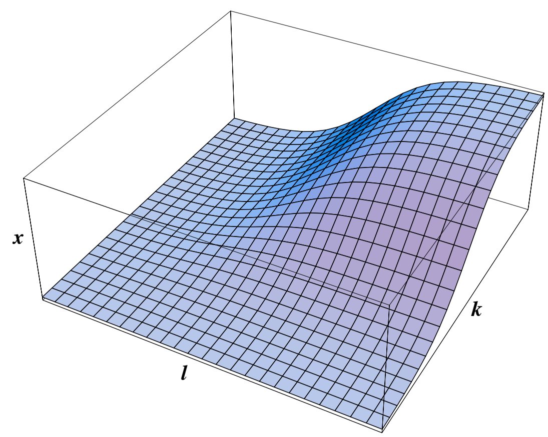
$$\frac{\beta w^A}{\alpha r'} > \frac{\beta w^A}{\alpha r^A}; \tag{12.11}$$

i.e. the ray on which cost minimizing input bundles lie will be steeper as firms substitute toward capital and away from labor. From the output supply equation (12.4), we can also see that a decrease in r results in an increase in output — thus, the new profit maximizing input bundle lies above the initial isoquant and to the left of the initial ray along which cost minimizing input bundles occurred.

12.2 *We have said that economic profit is equal to economic revenue minus economic cost — where cash inflows or outflows are not "real" economic revenues or costs unless they are in fact impacted by the economic decisions of the firm. Suppose that a firm uses both labor ℓ and capital k in its production of x — and that no output can be produced without at least some of each input.*

A: *In the short run, however, it can only change the level of labor input because it has already committed to a particular capital input level for the coming months. Assume that the firm's homothetic production process is one that has initially increasing but eventually decreasing returns to scale — and that the marginal product of each input is initially increasing but eventually decreasing. (The full production frontier would then look something like what we have plotted in Graph 12.2.)*

(a) *Suppose the firm is currently implementing the profit maximizing production plan denoted $A = (\ell^A, k^A, x^A)$. Given input prices w and r and output price p, what is the expression for the profit this firm earns.*



Graph 12.2: Production Frontier with two Inputs

Answer: Profit= $\pi = px^A - w\ell^A - rk^A$.

(b) *Now consider the short run where capital is fixed at k^A. Graph the short run production function for this firm.*

Answer: This is done in Graph 12.3 (next page) where the typical shape arises from the fact that marginal product of labor is initially increasing but eventually decreasing.

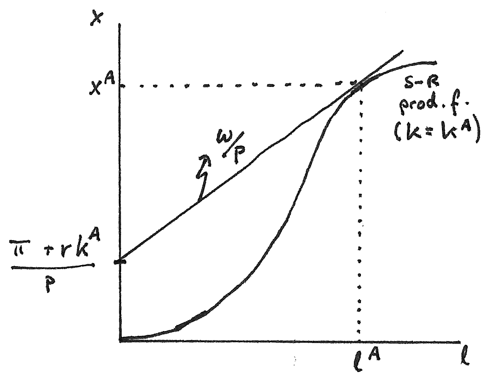
(c) *Add to this graph the slice of the isoprofit plane that is tangent to the production frontier at A. Indicate its slope and vertical intercept.*

Answer: In part (a), we indicated that profit at A is just revenue minus all the costs; i.e. Profit= $\pi = px^A - w\ell^A - rk^A$. When k cannot vary but ℓ and x can, we can simply write the equation for profit as $\pi = px - w\ell - rk^A$ — and this can be solved for x to give us

$$x = \left[\frac{\pi + rk^A}{p}\right] + \left(\frac{w}{p}\right)\ell. \qquad (12.12)$$

This is the equation for the isoprofit slice that goes through A but holds capital fixed at k^A. The bracketed term is then the intercept of the isoprofit, and the term in parenthesis is the slope.

(d) *Given that we learned in Chapter 11 that the vertical intercept of the isoprofit is equal to profit (along that isoprofit) divided by output price, what does the vertical intercept in your graph suggest is the profit for this firm when viewed from the short run perspective.*



Graph 12.3: Short Run Production

Answer: We know that in our short run picture, the vertical intercept is Profit/p. Since our intercept is $(\pi + rk^A)/p$, this would imply that profit when viewed from the short run perspective is $(\pi + rk^A)$ — i.e. it is what we call "profit" in the long run (π) *plus* the expense of the capital that is fixed in the short run (rk^A).

(e) *Explain why the short run perspective of economic profit differs in this case from the long run perspective.*

Answer: It differs because capital is not a real economic cost when viewed from the short run perspective. The firm already committed to rent the capital k^A *regardless of how much it produces.* Thus, the expense on capital is unaffected by any economic decision the firm makes in the short run — and so it is not a real economic cost of producing. Put differently, it is not something the firm gives up by producing — it is simply something that it pays regardless of what it chooses to do. It is what we will call a "sunk cost". Thus, the short run profit is simply revenue minus the real economic cost of hiring labor — i.e.

$$\text{Short Run Profit} = \pi)SR = px - w\ell = \text{Long Run Profit} + rk^A = \pi)LR + rk^A. \tag{12.13}$$

(f) True or False*: It is possible for a firm to be earning zero profit in the long run but positive profit when viewed from a short-run perspective.*

Answer: This is true — long run profit includes expenses that are economic costs in the long run but not in the short run.

B: *Suppose that, instead of the production process described in part A, the production frontier is characterized by the Cobb-Douglas production function $x = f(\ell, k) = A\ell^\alpha k^\beta$ with $\alpha + \beta < 1$ and $A, \alpha,$ and β all greater than zero.*

(a) *Does this production process have increasing, decreasing or constant returns to scale?*

Answer: It exhibits decreasing returns to scale since the exponents on the Cobb-Douglas function sum to less than 1.

(b) *Set up the profit maximization problem.*

Answer: The profit maximization problem is

$$\max_{\ell,k,x} px - w\ell - rk \quad \text{subject to} \quad x = A\ell^\alpha k^\beta \tag{12.14}$$

which can be written as the unconstrained maximization problem

$$\max_{\ell,k} pA\ell^{\alpha}k^{\beta} - w\ell - rk. \tag{12.15}$$

(c) *Solve this for the optimal production plan.*

Answer: The first order conditions are

$$\alpha pA\ell^{\alpha-1}k^{\beta} = w \quad \text{and} \quad \beta pA\ell^{\alpha}k^{\beta-1} = r. \tag{12.16}$$

Solving the second for k, we get

$$k = \left(\frac{r}{\beta pA}\right)^{1/(\beta-1)} \ell^{-\alpha/(\beta-1)}, \tag{12.17}$$

and substituting this into the first first order condition, we can (after some tedious work with exponents) solve for the labor demand function

$$\ell(w,r,p) = \left(\frac{pA\alpha^{(1-\beta)}\beta^{\beta}}{w^{(1-\beta)}r^{\beta}}\right)^{1/(1-\alpha-\beta)}. \tag{12.18}$$

Substituting this into our equation (12.17) (and again slogging through some work with exponents), we get the capital demand function

$$k(w,r,p) = \left(\frac{pA\alpha^{\alpha}\beta^{(1-\alpha)}}{w^{\alpha}r^{(1-\alpha)}}\right)^{1/(1-\alpha-\beta)} \tag{12.19}$$

Finally, we can plug these input demand functions back into the production function to find the supply function

$$x(w,r,p) = A\left(\frac{(pA)^{(\alpha+\beta)}\alpha^{\alpha}\beta^{\beta}}{w^{\alpha}r^{\beta}}\right)^{1/(1-\alpha-\beta)} = \left(\frac{Ap^{(\alpha+\beta)}\alpha^{\alpha}\beta^{\beta}}{w^{\alpha}r^{\beta}}\right)^{1/(1-\alpha-\beta)}. \tag{12.20}$$

(d) *Now consider the short run profit maximization problem for the firm that is currently employing $\overline{k}$ of capital. Write down the firm's short run production function and its short run profit maximization problem.*

Answer: The short run production function $g(\ell)$ is then simply

$$x = g(\ell) = (A\overline{k}^{\beta})\ell^{\alpha} = B\ell^{\alpha} \quad \text{where} \quad B = A\overline{k}^{\beta} \tag{12.21}$$

Since capital is not a choice variable in the short run, the short run profit maximization problem is

$$\max_{\ell,x} px - w\ell \quad \text{subject to} \quad x = B\ell^{\alpha}, \tag{12.22}$$

which can be written as the unconstrained maximization problem

$$\max_{\ell} pB\ell^{\alpha} - w\ell. \tag{12.23}$$

(e) *Solve for the short run labor demand and output supply functions.*

Answer: The first order condition for the unconstrained maximization problem is $\alpha pB\ell^{\alpha-1} = w$ which solves for the labor demand function

$$\ell(w,p) = \left(\frac{\alpha pB}{w}\right)^{1/(1-\alpha)} = \left(\frac{\alpha p\left(A\overline{k}^{\beta}\right)}{w}\right)^{1/(1-\alpha)}. \tag{12.24}$$

Substituting this back into the production function, we get the supply function

$$x(w,p) = B\left[\left(\frac{\alpha pB}{w}\right)^{1/(1-\alpha)}\right]^{\alpha} = \left[B\left(\frac{\alpha p}{w}\right)^{\alpha}\right]^{1/(1-\alpha)} = \left[A\overline{k}^{\beta}\left(\frac{\alpha p}{w}\right)^{\alpha}\right]^{1/(1-\alpha)} \tag{12.25}$$

(f) *Suppose that the short run fixed capital is equal to the long run optimal quantity you calculated in part (c). Demonstrate that the firm would then choose the same amount of labor in the short run as it does in the long run.*

Answer: Substituting equation (12.19) for $\overline{k}$ into equation (12.24), a little work with exponents demonstrates this equivalence.

(g) *Finally, illustrate that profit is larger from the short run perspective than the long run perspective.*

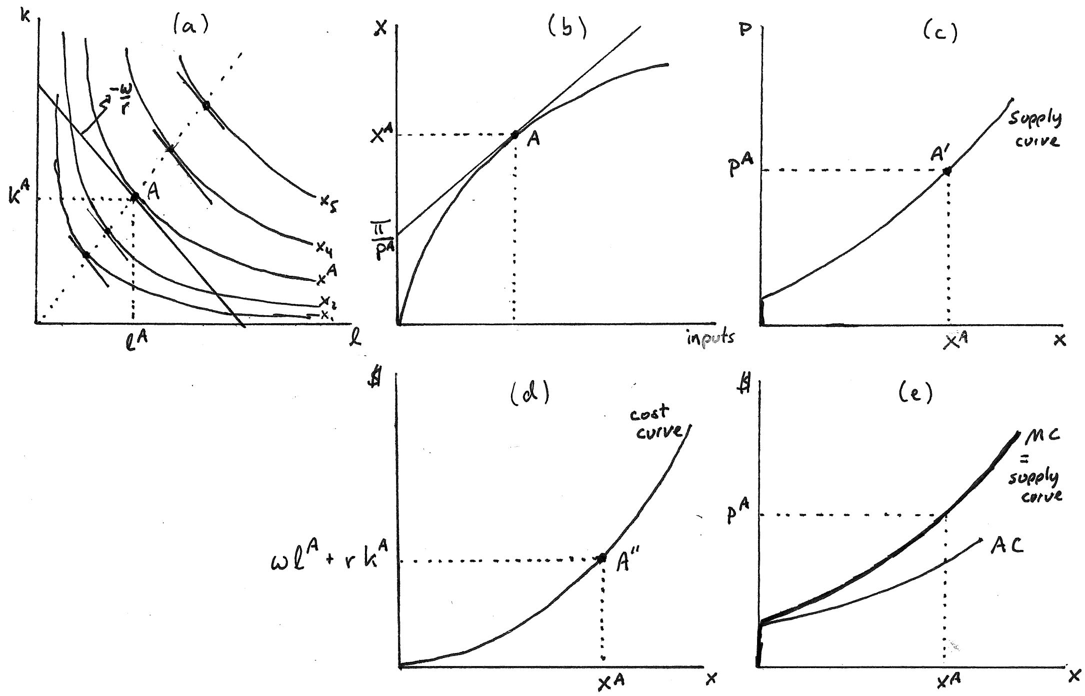
Answer: The short run profit is $px - w\ell$ since k is not a choice variable in the short run. The long run profit is $px - w\ell - rk$. We just demonstrated above that a firm's choice of labor input is the same in the short and long run assuming that capital is fixed at the long run optimal quantity — thus output is the same in the short run decision as in the long run decision. The only difference between short and long run profit is therefore the inclusion of the cost of capital in the long run — and profit from the short run perspective therefore exceeds long run profit.

12.3 *Consider again the two ways in which we can view the producer's profit maximization problem.*

A: *Suppose a homoethetic production technology involves two inputs, labor and capital, and that its producer choice set is fully convex.*

(a) *Illustrate the production frontier in an isoquant graph with labor on the horizontal axis and capital on the vertical.*

Answer: This is done in panel (a) of Graph 12.4. Since the producer choice set is convex, the horizontal slices represented by the isoquants must have the usual convex shape. In addition, the homotheticity property implies that the slopes (or TRS) of the isoquants are the same along any ray from the origin.



Graph 12.4: 2 Ways to Derive Output Supply

(b) *Does this production process have increasing or decreasing returns to scale? How would you be able to see this on an isoquant graph like the one you have drawn?*

Answer: It has decreasing returns to scale — because the entire producer choice set is convex. You would only see this in an isoquant map if the isoquants are accompanied by output numbers that increase at a decreasing rate along any ray from the origin.

(c) *For a given wage w and rental rate r, show in your graph where the cost minimizing input bundles lie. What is true at each such input bundle?*

Answer: The input prices give us the slope of the isocost lines — which is $(-w/r)$. The isocost drawn in panel (a) is tangent at A — implying that (ℓ^A, k^A) is the cheapest input bundle that can produce the output level x^A. Since the production process is homothetic, it implies that all isoquants have the same slope along the ray from the origin through A.

This further implies that all cost minimizing input bundles for the various output levels (represented by the isoquants) lie on this ray. Put differently, along this ray it is always true that $TRS = -w/r$ — the condition for cost minimization.

(d) *On a separate graph, illustrate the vertical slice (of the production frontier) that contains all these cost minimizing input bundles.*

Answer: Panel (b) of Graph 12.4 illustrates this vertical slice whose shape emerges from the decreasing returns to scale of the production process.

(e) *Assuming output can be sold at p^A, use a slice of the isoprofit plane to show the profit maximizing production plan. What, in addition to what is true at all the cost-minimizing input bundles, is true at this profit maximizing plan?*

Answer: This is also illustrated in panel (b) where the slice of the isoprofit plane is tangent at A. Since this is the profit maximizing plan, it must also be true that $p^A MP_\ell^A = w$ and $p^A MP_k^A = r$ — i.e. the marginal revenue product of each input is equal to that input's price. (Of course this automatically implies that $TRS^A = -w/r$ — which can be shown by simply dividing the two previous profit maximizing conditions by each other.)

(f) *If output price changes, would you still profit maximize on this vertical slice of the production frontier? What does the supply curve (which plots output on the horizontal and price on the vertical) look like?*

Answer: Yes, you would still produce on the same slice. This can be seen in panel (a) — a change in p changes nothing in panel (a). Thus, the cost minimizing input bundles remain unchanged, and — since profit maximization implies cost minimization — the profit maximizing plan must therefore lie on this slice. As p changes, the slope of the isoprofit line in (b) changes, becoming steeper when p falls and shallower when it rises. Thus, as p increases, output increases — and as p decreases, output decreases. This results in a shape for the supply curve as drawn in panel (c) of Graph 12.4.

(g) *Now illustrate the (total) cost curve (with output on the horizontal and dollars on the vertical axis). How is this derived from the vertical slice of the production frontier that you have drawn before?*

Answer: The vertical slice of the production frontier in panel (b) illustrates that it gets increasingly difficult to produce additional units of output as the inputs are increased in proportion to one another. This implies that the cost of increasing output will rise faster and faster as output increases — giving us the shape for the cost curve in panel (d) of Graph 12.4. This shape is essentially the inverse of the shape of the production frontier slice in (a). For the output quantity x^A, for instance, this cost is simply calculated by going back to panel (a) and checking how much of each input is required to produce x^A. We then multiply each input quantity by how much that input costs per unit to determine the total cost of producing x^A.

(h) *Derive the marginal and average cost curves and indicate where in your picture the supply curve lies.*

Answer: This is done in panel (e) of Graph 12.4 where the MC is simply the slope of the (total) cost curve from (d) — a slope that starts small (i.e. shallow) and becomes increasingly large (i.e. steeper). As always, the AC begins where the MC. Since MC is increasing throughout, this implies that AC always lies below MC. The supply curve is then, as always, the part of the MC curve that lies above the AC.

(i) *Does the supply curve you drew in part (f) look similar to the one you drew in part (h)?*

Answer: Yes — because the two methods of deriving the supply curve are equivalent.

B: *Suppose that the production technology is fully characterized by the Cobb-Douglas production function $x = f(\ell, k) = A\ell^\alpha k^\beta$ with $\alpha + \beta < 1$ and $A, \alpha,$ and β all greater than zero.*

(a) *Set up the profit maximization problem (assuming input prices w and r and output price p). Then solve for the input demand and output supply functions. (Note: This is identical to parts B(b) and (c) of exercise 12.2 — so if you have solved it there, you can simply skip to part (b) here.)*

Answer: See answers to parts B(b) and (c) of exercise 12.2. The input demand functions we calculated there are

$$\ell(w,r,p) = \left(\frac{pA\alpha^{(1-\beta)}\beta^{\beta}}{w^{(1-\beta)}r^{\beta}}\right)^{1/(1-\alpha-\beta)} \text{ and } k(w,r,p) = \left(\frac{pA\alpha^{\alpha}\beta^{(1-\alpha)}}{w^{\alpha}r^{(1-\alpha)}}\right)^{1/(1-\alpha-\beta)} \quad (12.26)$$

and the output supply function was

$$x(w,r,p) = \left(\frac{Ap^{(\alpha+\beta)}\alpha^{\alpha}\beta^{\beta}}{w^{\alpha}r^{\beta}}\right)^{1/(1-\alpha-\beta)}. \quad (12.27)$$

(b) *Now set up the cost minimization problem and solve for the first order conditions.*

Answer: This problem is

$$\min_{\ell,k} \; w\ell + rk \text{ subject to } x = A\ell^{\alpha}k^{\beta}. \quad (12.28)$$

The Lagrange function for this problem is

$$\mathscr{L}(\ell,k,\lambda) = w\ell + rk + \lambda\left(x - A\ell^{\alpha}k^{\beta}\right) \quad (12.29)$$

giving rise to first order conditions

$$\begin{aligned} \frac{\partial\mathscr{L}}{\partial\ell} &= w - \lambda A\alpha\ell^{\alpha-1}k^{\beta} = 0 \\ \frac{\partial\mathscr{L}}{\partial k} &= r - \lambda A\beta\ell^{\alpha}k^{\beta-1} = 0 \\ \frac{\partial\mathscr{L}}{\partial\lambda} &= x - A\ell^{\alpha}k^{\beta} = 0 \end{aligned} \quad (12.30)$$

(c) *Solve for the conditional labor and capital demands.*

Answer: Moving the negative terms in each of the first two first order conditions to the other side, and dividing the two conditions by each other, we get

$$\frac{w}{r} = \frac{\alpha k}{\beta\ell} \text{ or } k = \frac{\beta w\ell}{\alpha r}. \quad (12.31)$$

Substituting the latter expression for k into the third first order condition and solving for ℓ, we then get the conditional labor demand function

$$\ell(w,r,x) = \left(\frac{\alpha r}{\beta w}\right)^{\beta/(\alpha+\beta)}\left(\frac{x}{A}\right)^{1/(\alpha+\beta)} \quad (12.32)$$

and substituting this back into the expression for k from equation (12.31), we get the conditional capital demand function

$$k(w,r,x) = \left(\frac{\beta w}{\alpha r}\right)^{\alpha/(\alpha+\beta)}\left(\frac{x}{A}\right)^{1/(\alpha+\beta)}. \quad (12.33)$$

(d) *Derive the cost function and simplify the function as much as you can. (*Hint*: You can check your answer with the cost function given for the same production process in exercise 12.4.) Then derive from this the marginal and average cost functions.*

Answer: The cost function is simply the sum of the conditional input demands multiplied by the respective input prices; i.e.

$$\begin{aligned} C(w,r,x) &= w\ell(w,r,x) + rk(w,r,x) \\ &= w\left(\frac{\alpha r}{\beta w}\right)^{\beta/(\alpha+\beta)}\left(\frac{x}{A}\right)^{1/(\alpha+\beta)} + r\left(\frac{\beta w}{\alpha r}\right)^{\alpha/(\alpha+\beta)}\left(\frac{x}{A}\right)^{1/(\alpha+\beta)}. \end{aligned} \quad (12.34)$$

This can be written as

$$C(w,r,x) = \left[w\left(\frac{\alpha r}{\beta w}\right)^{\beta/(\alpha+\beta)} + r\left(\frac{\beta w}{\alpha r}\right)^{\alpha/(\alpha+\beta)} \right] \left(\frac{x}{A}\right)^{1/(\alpha+\beta)} \tag{12.35}$$

which, with a little algebra, simplifies to

$$C(w,r,x) = (\alpha+\beta)\left(\frac{x w^{\alpha} r^{\beta}}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)}. \tag{12.36}$$

The marginal cost function is then simply the derivative of the cost function with respect to x; i.e.

$$MC = \frac{\partial C(w,r,x)}{\partial x} = \left(\frac{w^{\alpha} r^{\beta}}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)} x^{(1-\alpha-\beta)/(\alpha+\beta)}. \tag{12.37}$$

Finally, the average cost function is simply

$$AC(w,r,x) = \frac{C(w,r,x)}{x} = (\alpha+\beta)\left(\frac{w^{\alpha} r^{\beta}}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)} x^{(1-\alpha-\beta)/(\alpha+\beta)}. \tag{12.38}$$

(e) *Use your answers to derive the supply function. Compare your answer to what you derived in (a).*

Answer: To derive the supply function, we set price equal to marginal cost and solve for x; i.e. we start with

$$MC = \left(\frac{w^{\alpha} r^{\beta}}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)} x^{(1-\alpha-\beta)/(\alpha+\beta)} = p. \tag{12.39}$$

Dividing through by the term in parentheses and taking both sides to the power $(\alpha+\beta)/(1-\alpha-\beta)$, we get

$$x(w,r,p) = \left[p\left(\frac{A\alpha^{\alpha}\beta^{\beta}}{w^{\alpha} r^{\beta}}\right)^{1/(\alpha+\beta)} \right]^{(\alpha+\beta)/(1-\alpha-\beta)} = \left(\frac{A p^{(\alpha+\beta)}\alpha^{\alpha}\beta^{\beta}}{w^{\alpha} r^{\beta}}\right)^{1/(1-\alpha-\beta)}, \tag{12.40}$$

exactly the same as what we derived in (a) through direct profit maximization. (This entire function is the supply function since MC lies above AC everywhere $x > 0$. You can see this by noticing from the AC and MC functions that $AC = (\alpha+\beta)MC$. Since $(\alpha+\beta) < 1$, this implies $AC < MC$ everywhere.)

(f) *Finally, derive the (unconditional) labor and capital demands. Compare your answers to those in (a).*

Answer: We now simply need to substitute $x(w,r,p)$ from above in for x in the conditional input demand equations (12.32) and (12.33) — and once we do that, we get back the unconditional labor and capital demands that are identical to those in the equations (12.26) in part (a).

12.4 *In upcoming chapters, we will often assume that the average cost curve is U-shaped.*

A: *Indicate for each statement below whether you believe that the description of the firm's situation would lead to a U-shaped average cost curve.*

(a) *The firm's production frontier initially exhibits increasing returns to scale but, beginning at some output quantity $\overline{x}$, it exhibits decreasing returns to scale.*

Answer: Yes, this results in the U-shaped average cost curve as described in the text. The initial increasing returns to scale imply that it is initially getting easier and easier to produce more — implying that cost will rise by less and less as more is produced. But eventually, decreasing returns implies it is getting harder and harder to produce more — implying the cost of producing additional units goes up faster and faster. Thus, average costs will initially fall but will eventually rise.

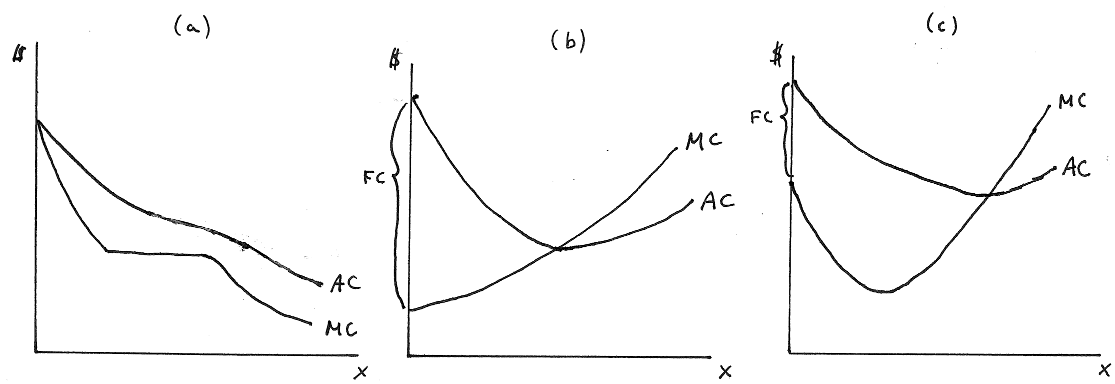
(b) *The firm's production frontier initially exhibits decreasing returns to scale but, beginning at some output quantity $\overline{x}$, it exhibits increasing returns to scale.*

Answer: No — this would result in an inverse U-shape. Initially it is getting harder and harder to produce — meaning average cost is increasing; but eventually it gets easier and easier to produce — meaning average costs will fall.

(c) *The firm's production process initially has increasing returns to scale, then — in some interval from $\underline{x}$ to $\overline{x}$ — it has constant returns to scale, followed by decreasing returns to scale.*

Answer: Yes, the *AC* curve would begin downward sloping (as it becomes easier and easier to produce more along the increasing returns to scale portion of the production process), would continue to slope down during the constant returns to scale portion and would increase at some point after production enters the decreasing returns to scale portion.

(d) *The firm's production process initially has increasing returns to scale, then — in some interval from $\underline{x}$ to $\overline{x}$ — it has constant returns to scale, followed by once again increasing returns to scale.*



Graph 12.5: Average Cost Curves

Answer: No — the average cost curve would slope down throughout. This is true even during the constant returns to scale portion of production where marginal costs are flat — because marginal cost is still below average cost. This is illustrated in panel (a) of Graph 12.5.

(e) *The production process for the firm has decreasing returns to scale throughout — but, before ever producing the first unit of output, the firm incurs annually a fixed cost FC (such as a large license fee) that must be paid if production is to occur.*

Answer: Yes, this results in a U-shaped average cost curve. Average costs start very high — because of the fixed cost — but falls so long as *MC* lies below *AC*. Since the production process has decreasing returns to scale, eventually MC and AC must cross. This is illustrated in panel (b) of Graph 12.5.

(f) *The firm incurs the same annual FC as in (e) — but it's production process initially has increasing returns to scale before eventually switching to decreasing returns to scale.*

Answer: Yes, this results in the U-shaped average cost curve for reasons similar to those in (e) — because of the eventual decreasing returns to scale that cause MC to slope up. It is illustrated in panel (c) of Graph 12.5.

B: *We will explore production processes with initially increasing and eventually decreasing returns to scale in exercises 12.5 and 12.6. Here we instead focus on exploring the impact of recurring fixed costs (like annual license fees) on the shape of cost curves. Consider, as we did in exercises 12.2 and 12.3, the Cobb-Douglas production function $x = f(\ell, k) = A\ell^{\alpha}k^{\beta}$. In exercise 12.3B(d), you should have concluded that the cost function for this production process is*

$$C(w,r,x) = (\alpha+\beta)\left(\frac{xw^{\alpha}r^{\beta}}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)}. \tag{12.41}$$

(a) *In problem 12.3, this cost function was derived for the case where $\alpha+\beta<1$ and A, α, and β are all greater than zero. Is the cost function still valid for the case where $\alpha+\beta \geq 1$?*

Answer: Yes, it is still valid. To see why, consider the graphical depiction of the cost minimization on an isoquant graph. Pick some isoquant x — and ask what the least cost way of producing that level of output is. Your answer has nothing to do with the vertical shape of the production frontier (that tells us about returns to scale) — it only has to do with the shape of the horizontal slice. Our Lagrange based cost minimization method is therefore valid as long as this cost minimization problem has an interior solution — which it has as long as the isoquants have the usual convex shape. (Our mathematical profit-maximization approach, of course, does not give us the right solution when the Cobb-Douglas function has increasing returns to scale (i.e. when $\alpha+\beta>1$) — because now the true profit maximum would be to produce an infinite amount.)

(b) *Are marginal and average cost curves for this production process upward or downward sloping? What does your answer depend on?*

Answer: The marginal and average cost functions are

$$MC = \left(\frac{w^{\alpha}r^{\beta}}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)} x^{(1-\alpha-\beta)/(\alpha+\beta)} \quad \text{and} \tag{12.42}$$

$$AC = (\alpha+\beta)\left(\frac{w^{\alpha}r^{\beta}}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)} x^{(1-\alpha-\beta)/(\alpha+\beta)}. \tag{12.43}$$

To tell whether the MC and AC curves are upward or downward sloping, we need to check the sign of the derivative of each of these functions with respect to x:

$$\frac{\partial MC}{\partial x} = \left(\frac{1-\alpha-\beta}{\alpha+\beta}\right)\left(\frac{w^{\alpha}r^{\beta}}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)} x^{(1-2(\alpha+\beta))/(\alpha+\beta)} \quad \text{and} \tag{12.44}$$

$$\frac{\partial AC}{\partial x} = (1-\alpha-\beta)\left(\frac{w^{\alpha}r^{\beta}}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)} x^{(1-2(\alpha+\beta))/(\alpha+\beta)}. \tag{12.45}$$

The only ambiguity in terms of whether these derivatives are positive or negative arises from the term $(1-\alpha-\beta)$ — if it is positive, both derivatives are positive; if it is negative, both derivatives are negative; and if it is zero, both derivatives are zero. Put differently, MC and AC slope up as long as $\alpha+\beta<1$, slope down as long as $\alpha+\beta>1$ and are flat if $\alpha+\beta=1$. This makes intuitive sense since we know that $\alpha+\beta<1$ implies decreasing returns to scale, $\alpha+\beta>1$ implies increasing returns to scale, and $\alpha+\beta=1$ implies constant returns to scale.

(c) *Suppose that the firm incurs a fixed cost FC that has to be paid each period before production starts. How does this change the (total) cost function, the marginal cost function and the average cost function?*

Answer: The total cost function now becomes

$$C(w,r,p) = FC + (\alpha+\beta)\left(\frac{xw^{\alpha}r^{\beta}}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)}. \tag{12.46}$$

The new FC term drops out, however, as we take the derivative with respect to x to get the marginal cost function; i.e. marginal cost is unchanged as

$$MC = \left(\frac{w^{\alpha}r^{\beta}}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)} x^{(1-\alpha-\beta)/(\alpha+\beta)}. \tag{12.47}$$

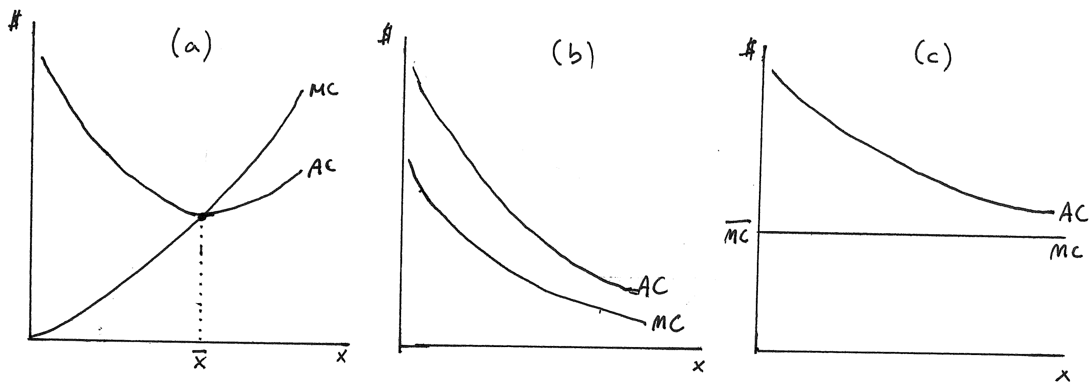
The AC function changes to

$$AC = \frac{FC}{x} + (\alpha+\beta)\left(\frac{w^{\alpha}r^{\beta}}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)} x^{(1-\alpha-\beta)/(\alpha+\beta)}. \tag{12.48}$$

(d) *Suppose that $\alpha+\beta<1$. What is the relationship between MC and AC now?*

Answer: This relationship is depicted in panel (a) of Graph 12.6. The AC begins high because of the FC term but declines rapidly as x (in the denominator) increases. The derivative of MC with respect to x is positive (since $\alpha+\beta<1$) — so the MC curve slopes up. They cross in the graph at $\overline{x}$ which we can calculate by setting AC equal to MC and solving for x. This gives us

$$\overline{x} = \left(\frac{FC}{1-\alpha-\beta}\right)^{(\alpha+\beta)}\left(\frac{A\alpha^{\alpha}\beta^{\beta}}{w^{\alpha}r^{\beta}}\right). \tag{12.49}$$



Graph 12.6: MC and AC Curves for $\alpha+\beta<1$, $\alpha+\beta>1$, and $\alpha+\beta=1$

(e) *How does your answer differ if $\alpha+\beta>1$? What if $\alpha+\beta=1$?*

Answer: These two cases are depicted in panels (b) and (c) of Graph 12.6. Note that when $\alpha+\beta>1$, the x term in the MC equation is taken to a negative power. Thus, the MC curve starts high — and it has downward slope because the derivative of MC with respect to x

is negative when $\alpha + \beta > 1$. The AC starts above MC because of the FC term — and it continues to slope down since MC never crosses it. (You can see from the expression for $\overline{x}$ in equation (12.49) that AC and MC actually cross at a negative (and thus economically irrelevant) output level because $(1-\alpha-\beta) < 0$ when $\alpha+\beta > 1$.)

When $\alpha + \beta = 1$ and we have constant returns to scale, the expressions for MC and AC without the FC become

$$MC = \frac{w^{\alpha} r^{\beta}}{A\alpha^{\alpha}\beta^{\beta}} = AC. \tag{12.50}$$

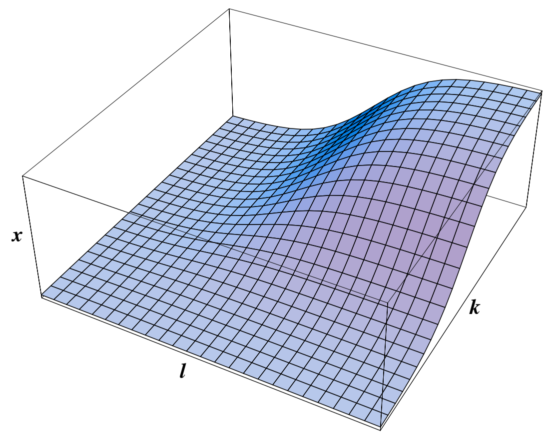
In panel (c) of Graph 12.6, this is equal to $\overline{MC}$. However, once FC is included, the AC function becomes

$$AC = \frac{FC}{x} + \frac{w^{\alpha} r^{\beta}}{A\alpha^{\alpha}\beta^{\beta}} = \frac{FC}{x} + \overline{MC}. \tag{12.51}$$

Thus, AC starts above MC and converges to MC as x increases (and thus FC/x falls).

12.5 *In the absence of recurring fixed costs (such as those in exercise 12.4), the U-shaped cost curves we will often graph in upcoming chapters presume some particular features of the underlying production technology when we have more than 1 input.*

A: *Consider the following production technology, where output is on the vertical axis (that ranges from 0 to 100) and the inputs capital and labor are on the two horizontal axes. (The origin on the graph is the left-most corner).*



Graph 12.7: Production Frontier with two Inputs

(a) *Suppose that output and input prices result in some optimal production plan A (that is not a corner solution). Describe in words what would be true at A relative to what we described as an isoprofit plane at the beginning of this chapter.*

Answer: The isoprofit plane $\pi = px - w\ell - rk$ would have to be tangent to the production frontier — with no other portion of the isoprofit plane intersecting the frontier. It is like a sheet of paper tangent to a "mountain" that is initially getting steeper but eventually becomes shallower. This implies that the isoprofit plane that is tangent at A has a positive vertical intercept.

(b) *Can you tell whether this production frontier has increasing, constant or decreasing returns to scale?*

Answer: The production frontier has initially increasing but eventually decreasing returns to scale — i.e. along every horizontal ray from the origin, the slice of the production frontier has the "sigmoid" shape that we used throughout Chapter 11.

(c) *Illustrate what the slice of this graphical profit maximization problem would look like if you held capital fixed at its optimal level k^A.*

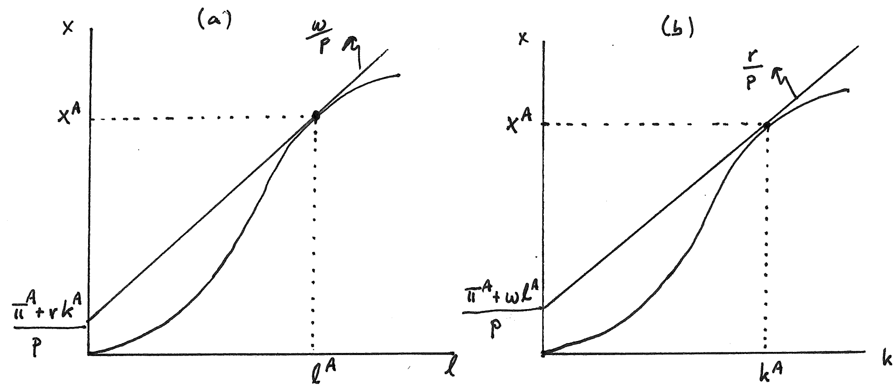
Answer: This is illustrated in panel (a) of Graph 12.8 (next page). The tangency of the isoprofit plane shows up as a tangency of the line $x = [(\pi^A + rk^A)/p] + (w/p)\ell$, where the bracketed term is the vertical intercept and the (w/p) term is the slope. (This is just derived from solving the expression $\pi^A = px - w\ell - rk^A$ for x.)

(d) *How would the slice holding labor fixed at its optimal level ℓ^A differ?*

Answer: It would look similar except for re-labeling as in panel (b) of the graph.

(e) *What two conditions that have to hold at the profit maximizing production plan emerge from these pictures?*

Answer: In panels (a) and (b) of Graph 12.8, the slopes of the isoprofit lines are tangent to the slopes of the production frontier with one of the inputs held fixed. The slope of the production frontier at (ℓ^A, x^A) in panel (a) is the marginal product of labor at that production plan;



Graph 12.8: Holding k^A and ℓ^A fixed

i.e. MP_ℓ^A. And the slope of the production frontier at (k^A, x^A) in panel (b) is the marginal product of capital at that production plan; i.e. MP_k^A. Thus, the conditions that emerge are

$$MP_\ell^A = \frac{w}{p} \text{ and } MP_k^A = \frac{r}{p}. \tag{12.52}$$

(f) *Do you think there is another production plan on this frontier at which these conditions hold?*

Answer: Yes — this would occur on the increasing returns to scale portion of the production frontier where an isoprofit "sheet" is tangent to the lower side of the frontier. This "sheet" will, however, have a negative intercept — implying negative profit.

(g) *If output price falls, the profit maximizing production plan changes to once again meet the conditions you derived above. Might the price fall so far that no production plan satisfying these conditions is truly profit maximizing?*

Answer: A decrease in p will cause the isoprofit planes to become steeper — causing the profit maximizing production plan to slide down the production frontier as the tangent isoprofit now happens at a steeper slope. This implies that the vertical intercept also slides down — with profit falling. If the price falls too much, this intercept will become negative — implying that the true profit maximizing production plan becomes (0,0,0). Put differently, if the price falls too much, the firm is better off not producing at all rather than producing at the tangency of an isoprofit with the production frontier.

(h) *Can you tell in which direction the optimal production plan changes as output price increases?*

Answer: As output price increases, the isoprofit plane becomes shallower — which implies that the tangency with the production frontier slides up in the direction of the shallower portion of the frontier. Thus, the production plan will involve more of each input and more output.

B: *Suppose your production technology is characterized by the production function*

$$x = f(\ell, k) = \frac{\alpha}{1 + e^{-(\ell-\beta)} + e^{-(k-\gamma)}} \tag{12.53}$$

where e is the base of the natural logarithm. Given what you might have learned in one of the end-of-chapter exercises in Chapter 11 about the function $x = f(\ell) = \alpha/(1 + e^{-(\ell-\beta)})$, can you see how the shape in Graph 12.16 emerges from this extension of this function?

Answer: Even though this question was not meant to be answered directly, the graph given in part A of the question depicts this function for the case where $\alpha = 100$ and $\beta = \gamma = 5$. The graph was generated using the software package Mathematica (as are the other machine generated graphs in some of the answers in this Chapter). As you can see, the function takes on the shape that has initially increasing and eventually diminishing slope along slices holding each input fixed (as well as along rays from the origin.) Note that ℓ and k enter symmetrically given that $\beta = \gamma$ — and the two inputs appear on the axes in the plane from which the surface emanates. The vertical axis in the graph is output x.

(a) *Set up the profit maximization problem.*

Answer: The problem is

$$\max_{x,\ell,k} px - w\ell - rk \text{ subject to } x = \frac{\alpha}{1+e^{-(\ell-\beta)}+e^{-(k-\gamma)}} \tag{12.54}$$

which can also be written as the unconstrained maximization problem

$$\max_{\ell,k} \frac{\alpha p}{1+e^{-(\ell-\beta)}+e^{-(k-\gamma)}} - w\ell - rk. \tag{12.55}$$

(b) *Derive the first order conditions for this optimization problem.*

Answer: We simply take derivatives with respect to w and r and set them to zero. Thus, we get

$$\frac{\alpha p e^{-(\ell-\beta)}}{\left(1+e^{-(\ell-\beta)}+e^{-(k-\gamma)}\right)^2} = w \text{ and } \frac{\alpha p e^{-(k-\gamma)}}{\left(1+e^{-(\ell-\beta)}+e^{-(k-\gamma)}\right)^2} = r. \tag{12.56}$$

(c) *Substitute $y = e^{-(\ell-\beta)}$ and $z = e^{-(k-\gamma)}$ into the first order conditions. Then, with the first order conditions written with w and r on the right hand sides, divide them by each other and derive from this an expression $y(z,w,r)$ and the inverse expression $z(y,w,r)$.*

Answer: These substitutions lead to the first order conditions becoming

$$\frac{\alpha p y}{(1+y+z)^2} = w \text{ and } \frac{\alpha p z}{(1+y+z)^2} = r. \tag{12.57}$$

Dividing the two equations by each other, we can then derive

$$y(z,w,r) = \frac{wz}{r} \text{ and } z(y,w,r) = \frac{ry}{w}. \tag{12.58}$$

(d) *Substitute $y(z,w,r)$ into the first order condition that contains r. Then manipulate the resulting equation until you have it in the form $az^2 + bz + c$ (where the terms a, b and c may be functions of w, r, α and p). (*Hint*: It is helpful to multiply both sides of the equation by r.) The quadratic formula then allows you to derive two "solutions" for z. Choose the one that uses the negative rather than the positive sign in the quadratic formula as your "true" solution $z^*(\alpha,p,w,r)$.*

Answer: Substituting $y(z,w,r)$ into the second expression in equation (12.57) and multiplying both sides by the denominator, we get

$$\alpha p z = r\left(1+\frac{wz}{r}+z\right)^2. \tag{12.59}$$

Multiplying the right hand side by r lets us reduce it to

$$r^2\left(1+\frac{wz}{r}+z\right)^2 = (r+wz+rz)^2 = (r+(w+r)z)^2. \tag{12.60}$$

Thus, when we multiply both sides of equation (12.59) by r, we get

$$\alpha r p z = (r+(w+r)z)^2. \tag{12.61}$$

Expanding the left hand side and grouping terms, we then get

$$(w+r)^2 z^2 + [2r(w+r) - \alpha r p]z + r^2 = 0. \tag{12.62}$$

This is now in the form we need to apply the quadratic formula to solve for z. The problem tells us to use the version of the formula that has a negative rather than positive sign in front of the square root — thus

$$z^*(\alpha,p,w,r) = \frac{-[2r(w+r)-\alpha r p] - \sqrt{[2r(w+r)-\alpha r p]^2 - 4(w+r)^2 r^2}}{2(w+r)^2}. \tag{12.63}$$

(e) *Substitute $z(y,w,r)$ into the first order condition that contains w and solve for $y^*(\alpha,p,w,r)$ in the same way you solved for $z^*(\alpha,p,w,r)$ in the previous part.*

Answer: Substituting $z(y,w,r)$ into the first expression in equation (12.57) and multiplying both sides by the denominator, we get

$$\alpha p y = w\left(1+y+\frac{ry}{w}\right)^2. \tag{12.64}$$

Multiplying the right hand side by w lets us reduce it to

$$w^2\left(1+y+\frac{ry}{w}\right)^2 = (w+wy+ry)^2 = (w+(w+r)y)^2. \tag{12.65}$$

Thus, when we multiply both sides of equation (12.64) by w, we get

$$\alpha w p y = (w+(w+r)y)^2. \tag{12.66}$$

Expanding the left hand side and grouping terms, we then get

$$(w+r)^2 y^2 + [2w(w+r) - \alpha w p]z + w^2 = 0. \tag{12.67}$$

This is now in the form we need to apply the quadratic formula to solve for y. The problem tells us to use the version of the formula that has a negative rather than positive sign in front of the square root — thus

$$y^*(\alpha,p,w,r) = \frac{-[2w(w+r)-\alpha w p] - \sqrt{[2w(w+r)-\alpha w p]^2 - 4(w+r)^2 w^2}}{2(w+r)^2}. \tag{12.68}$$

(f) *Given the substitutions you did in part (c), you can now write $e^{-(\ell-\beta)} = y^*(\alpha,p,w,r)$ and $e^{-(k-\gamma)} = z^*(\alpha,p,w,r)$. Take natural logs of both sides to solve for labor demand $\ell(w,r,p)$ and capital demand $k(w,r,p)$ (which will be functions of the parameters α, β and γ.)*

Answer: Taking natural logs of $e^{-(\ell-\beta)} = y^*(\alpha,p,w,r)$ and $e^{-(k-\gamma)} = z^*(\alpha,p,w,r)$ gives us

$$-(\ell-\beta) = \ln y^*(\alpha,p,w,r) \text{ and } -(k-\gamma) = \ln z^*(\alpha,p,w,r) \tag{12.69}$$

which can be solved for ℓ and k to get the input demand functions:

$$\ell(w,r,p) = \beta - \ln y^*(\alpha,p,w,r) \text{ and } k(w,r,p) = \gamma - \ln z^*(\alpha,p,w,r). \tag{12.70}$$

(g) *How much labor and capital will this firm demand if $\alpha = 100$, $\beta = \gamma = 5 = p$, $w = 20 = r$? (It might be easiest to type the solutions you have derived into an Excel spreadsheet in which you can set the parameters of the problem.) How much output will the firm produce? How does your answer change if r falls to $r = 10$? How much profit does the firm make in the two cases.*

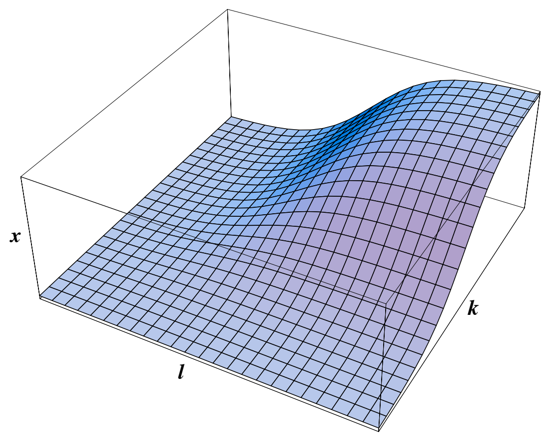
Answer: The firm would initially hire approximately 8.035 units of labor and capital to produce 91.23 units of output. When $r = 10$, the optimal production plan would change to $(\ell,k,y) = (8.086, 8.780, 93.59)$ — i.e. the firm would increase production primarily by hiring more capital but also by hiring slightly more labor. Profit is 134.74 in the first case and 218.42 in the second.

(h) *Suppose you had used the other "solutions" in parts (d) and (e) — the ones that emerge from using the quadratic formula in which the square root term is added rather than subtracted. How would your answers to (g) be different — and why did we choose to ignore this "solution"?*

Answer: The solution for the initial values given in part (g) would then have been $(\ell, k, y) \approx (3.35, 3.35, 8.77)$ and this would change to $(\ell, k, y) \approx (2.72, 3.42, 6.41)$ when r falls to 10. This would be an odd outcome — with a drop in the input price r, the problem suggests that output will fall. It is wrong because profit in both cases is negative — meaning these are not profit maximizing production plans. (Profit in the first case is -90.19 and in the second -56.61.)

12.6 *We will now re-consider the problem from exercise 12.5 but will focus on the two-step optimization method that starts with cost minimization.*

A: *Suppose again that you face a production process such as the one depicted in Graph 12.9.*



Graph 12.9: Production Frontier with two Inputs

(a) *What do the horizontal slices — the map of isoquants — of this production process look like? Does this map satisfy our usual notion of convexity as "averages better than extremes"?*

Answer: The isoquants have the usual shape — and therefore satisfy the usual "average are better than extremes" notion of convexity.

(b) *From this map of isoquants, how would you be able to infer the vertical shape of the production frontier? Do you think the producter choice set is convex?*

Answer: You would be able to infer it from the rate at which the numbering of the isoquants changes along rays from the origin. For the production frontier in this example, you would see that numbering initially increasing at an increasing rate (on the increasing returns to scale portion) but eventually increasing at a decreasing rate (on the decreasing returns to scale portion). Since the producer choice set has a frontier with increasing returns to scale along some portion, it is not a convex set (even though the horizontal slices are always convex.)

(c) *Suppose this production frontier is homothetic. For a given set of input prices* (w, r)*, what can you conclude about how the cost minimizing input bundles in your isoquant map will be related to one another.*

Answer: You can conclude that the cost minimizing input bundles will all lie on the same ray from the origin in the isoquant graph. This is because if $-w/r = TRS$ at some input bundle, then the same holds for all input bundles on that ray (due to the homotheticity property).

(d) *What can you conclude about the shape of the cost curve for a given set of input prices?*

Answer: Any vertical slice along a ray from the origin will have the sigmoid shape — initially increasing at an increasing rate, then increasing at a decreasing rate. The cost curve will have the inverse shape — initially increasing at a decreasing rate and eventually increasing at an increasing rate. This is because the initial part of the production frontier has increasing returns to scale — meaning it becomes easier and easier to produce additional output. Along that part of the production process, it therefore becomes cheaper and cheaper to produce additional units — meaning that the (total) cost of production will increase at a slower and slower rate. Once decreasing returns to scale set in, however, it becomes increasingly difficult to produce additional output — which implies that each additional output will add more and more to the (total) cost of production. Thus, the (total) cost eventually increases at a faster and faster rate with additional production.

(e) *What will the average and marginal cost curves look like?*

Answer: They will have the usual U-shapes precisely for the reasons that the (total) cost curve initially increases at a decreasing rate but eventually increases at an increasing rate. As always, the marginal and average cost curves will start at the same intercept; the marginal cost curve will decrease faster and reach its lowest point earlier than the average cost curve; and the marginal cost curve will cross the average cost curve at its lowest point.

(f) *Suppose again that $A = (\ell^A, k^A, x^A)$ is a profit maximizing production plan at the current prices (and suppose that A is not a corner solution). Illustrate the isoquant that represents the profit maximizing output quantity x^A. Using the conditions that have to hold for this to be a profit maximum, can you demonstrate that these imply the producer is cost minimizing at A?*

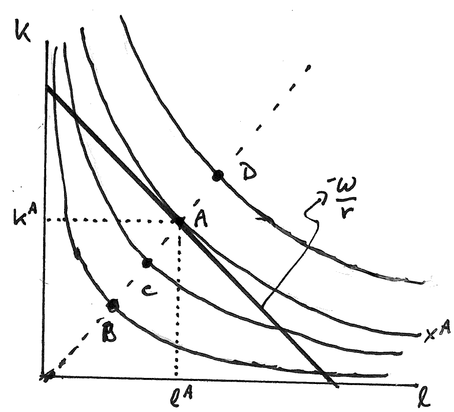
Answer: In Graph 12.10, point A falls on the isoquant labeled x^A and contains input levels (ℓ^A, k^A). If this is a profit maximizing production plan, then it must be the case that the marginal revenue product of each input is equal to that input's price; i.e.

$$pMP_\ell^A = w \text{ and } pMP_k^A = r. \quad (12.71)$$

If we divide the two equations by each other, we get

$$\frac{MP_\ell^A}{MP_k^A} = \frac{w}{r} \text{ or, put differently, } TRS^A = -\frac{w}{r}. \quad (12.72)$$

We therefore know that at A, there is a tangency between an isocost curve with slope $-w/r$ and the isoquant x^A which has a slope equal to the technical rate of substitution. Since this is the condition for cost minimization, we have demonstrated that the producer is cost minimizing at A.



Graph 12.10: Cost Minimizing and Profit Maximizing

(g) *Where else does the cost minimizing condition hold? Do the profit maximizing conditions hold there as well?*

Answer: Given the homotheticity of the production frontier, this cost minimization condition holds on *every* isoquant in Graph 12.10 along the ray that emanates from the origin and passes through the profit maximizing point A. At each of these input bundles, the technical rate of substitution is equal to $-w/r$ — which is the condition that has to be satisfied for cost minimization. However, the profit maximizing conditions are not satisfied anywhere on this ray other than at A.

(h) *What happens to output as p falls? What happens to the ratio of capital to labor in the production process (assuming the production process is indeed homothetic)?*

Answer: As p falls, the intersection of price with the MC curve occurs at a lower level of output — just as the tangency of the isoprofit plane with the production frontier in exercise 12.5 slid up to a higher output level when p increased. Thus, production falls — which means we will end up producing on a lower isoquant in Graph 12.10. However, since w and r have not changed, the slopes of isocosts have not changed — which implies all cost minimizing bundles continue to lie on the same ray from the origin as before. This further implies that, if we indeed continue to produce, the ratio of capital to labor in our production remains unchanged (even as we hire less labor and less capital). Put differently, we will reduce our labor and capital inputs by the same proportion. All that happens is that a point lower on the ray than A becomes the new profit maximizing point at which the profit maximizing conditions hold. Of course, if price falls too far — i.e. if it falls below the lowest point on the average cost curve — the firm will stop producing altogether.

B: *Consider the same production function as the one introduced in part B of exercise 12.5.*

(a) *Write down the problem you would need to solve to determine the least cost input bundle to produce some output level x.*

Answer: The optimization problem would be

$$\min_{\ell,k} w\ell + rk \text{ subject to } x = \frac{\alpha}{1+e^{-(\ell-\beta)}+e^{-(k-\gamma)}}. \tag{12.73}$$

(b) *Set up the Lagrange function and derive the first order conditions for this problem.*

Answer: To solve this, we need to set up the Lagrange function

$$\mathcal{L}(\ell,k,\lambda) = w\ell + rk + \lambda\left(\frac{\alpha}{1+e^{-(\ell-\beta)}+e^{-(k-\gamma)}} - x\right). \tag{12.74}$$

The first two first order conditions are then

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\ell} &= w - \frac{\lambda\alpha e^{-(\ell-\beta)}}{\left(1+e^{-(\ell-\beta)}+e^{-(k-\gamma)}\right)^2} = 0\\ \frac{\partial\mathcal{L}}{\partial k} &= r - \frac{\lambda\alpha e^{-(k-\gamma)}}{\left(1+e^{-(\ell-\beta)}+e^{-(k-\gamma)}\right)^2} = 0\end{aligned} \tag{12.75}$$

and the final first order condition is simply the constraint in the optimization problem (which results from taking the partial of the Lagrangian with respect to λ).

(c) *To make the problem easier to solve, substitute $y = e^{-(\ell-\beta)}$ and $z = e^{-(k-\gamma)}$ into the first order conditions and solve for y and z as functions of w, r, x (and α).*

Answer: Substituting these into the first order conditions (and moving negative terms to the right hand sides), we get

$$\begin{aligned} w &= \frac{\lambda\alpha y}{(1+y+z)^2}\\ r &= \frac{\lambda\alpha z}{(1+y+z)^2}\\ x &= \frac{\alpha}{1+y+z}.\end{aligned} \tag{12.76}$$

Dividing the first by the second, we get $w/r = y/z$. Solving for y, we get $y = (wz)/r$. Substituting this into the final first order condition, we get

$$x = \frac{\alpha}{1+(wz/r)+z} = \frac{\alpha r}{r+(w+r)z}. \tag{12.77}$$

Solving for z, we then get

$$z(w,r,x) = \frac{r(\alpha - x)}{x(w+r)}, \tag{12.78}$$

and plugging this into our previous expression $y = (wz)/r$, we get

$$y(w,r,x) = \frac{w(\alpha - x)}{x(w+r)}. \tag{12.79}$$

(d) *Recognizing that y and z were placeholders for $e^{-(\ell-\beta)}$ and $e^{-(k-\gamma)}$, use your answers now to solve for the conditional input demands $\ell(w,r,x)$ and $k(w,r,x)$.*

Answer: We can therefore write equations (12.79) and (12.78) as

$$e^{-(\ell-\beta)} = \frac{w(\alpha - x)}{x(w+r)} \text{ and } e^{-(k-\gamma)} = \frac{r(\alpha - x)}{x(w+r)}. \tag{12.80}$$

Since $\ln e^x = x$, we can then solve these equations to get our conditional labor and capital demands:

$$\ell(w,r,x) = \beta - \ln\left(\frac{w(\alpha - x)}{x(w+r)}\right) \text{ and } k(w,r,x) = \gamma - \ln\left(\frac{r(\alpha - x)}{x(w+r)}\right). \tag{12.81}$$

(e) *Derive from your answer the cost function for this firm — i.e. derive the function that tells you the least it will cost to produce any output quantity x for any set of input prices. Can you guess the shape of this function when α, β, γ, w and r are held fixed?*

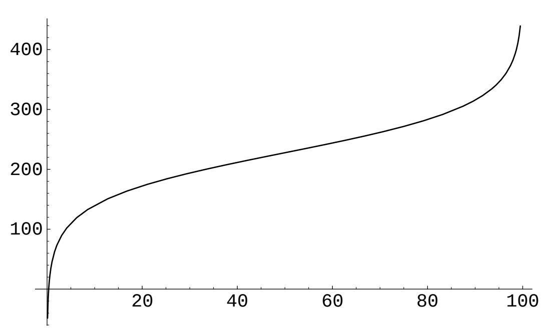
Answer: We simply need to multiply the conditional input demands by input prices and add them; i.e.

$$C(w,r,x) = w\left(\beta - \ln\left(\frac{w(\alpha - x)}{x(w+r)}\right)\right) + r\left(\gamma - \ln\left(\frac{r(\alpha - x)}{x(w+r)}\right)\right) \tag{12.82}$$

which can also be written as

$$C(w,r,x) = w\left[\beta - \ln\left(\frac{w}{w+r}\right)\right] + r\left[\gamma - \ln\left(\frac{r}{w+r}\right)\right] - (w+r)\ln\left(\frac{\alpha - x}{x}\right) \tag{12.83}$$

Graph 12.11 illustrates the shape of c when $\alpha = 100$, $\beta = \gamma = 5$ and $w = r = 20$, with x on the horizontal axis and the cost of production in \$'s on the vertical.



Graph 12.11: Cost function when $\alpha = 100$, $\beta = \gamma = 5$ and $w = r = 20$

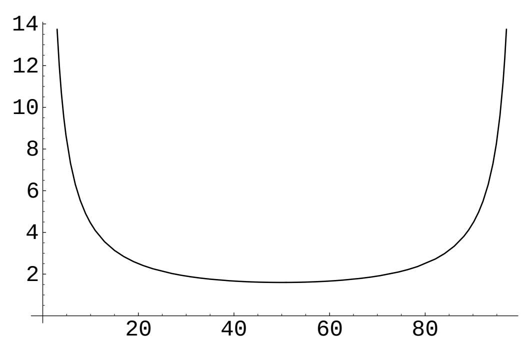
This shape arises from the fact that the underlying production function has initially increasing returns to scale and eventually decreasing returns to scale. Thus, the cost of producing additional units of output initially rises at a decreasing rate (giving rise to an increasingly shallow cost curve) but eventually rises at an increasing rate (giving rise to an increasingly steep cost curve.

(f) *Derive the marginal cost function. Can you guess its shape when α, w and r are held fixed?*

Answer: This is just

$$\frac{\partial C(w,r,x)}{\partial x} = \frac{w+r}{\alpha - x} + \frac{w+r}{x} = \frac{\alpha(w+r)}{x(\alpha - x)} \tag{12.84}$$

Graph 12.12 illustrates the shape of this marginal cost when $\alpha = 100$ and $w = r = 20$, with output x on the horizontal axis and the marginal cost in $'s on the vertical. This is the familiar U-shape that emerges with production processes that have initially increasing returns to scale but eventually decreasing returns.



Graph 12.12: Marginal cost function when $\alpha = 100$ and $w = r = 20$

(g) *Use your expression of the marginal cost curve to derive the supply function. Can you picture what this looks like when it is inverted to yield a supply curve (with input prices held fixed)?*

Answer: For the profit maximizing firm, the optimal output will occur where $p = MC$; i.e. where

$$p = \frac{\alpha(w+r)}{x(\alpha - x)}. \tag{12.85}$$

Writing this in the form $ax^2 + bx + c = 0$ so that we can apply the quadratic formula, we get

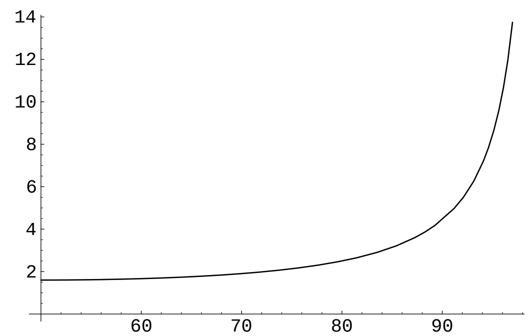
$$px^2 - \alpha p x + \alpha(w+r) = 0, \tag{12.86}$$

and applying the quadratic formula, we get the two solutions

$$x = \frac{\alpha p + \sqrt{\alpha^2 p^2 - 4p\alpha(w+r)}}{2p} \text{ and } x = \frac{\alpha p - \sqrt{\alpha^2 p^2 - 4p\alpha(w+r)}}{2p}. \tag{12.87}$$

The two solutions emerge from the fact that price intersects MC twice when the MC curve is U-shaped — with only the higher quantity representing the profit maximizing quantity. Clearly the first of the two solutions is greater than the second — thus, the supply function is given by

$$x(w,r,p) = \frac{\alpha p + \sqrt{\alpha^2 p^2 - 4p\alpha(w+r)}}{2p} \tag{12.88}$$



Graph 12.13: Supply curve when $\alpha = 100$ and $w = r = 20$

which is derived from the upward sloping portion of the MC curve. The supply curve we usually draw has x on he horizontal and p on the vertical axis — i.e. it is of the form in equation (12.85) which is the inverse of $x(w,r,p)$ with w and r held fixed. For the values $\alpha = 100$ and $w = 20 = r$, the graph of our derived supply curve is then given in Graph 12.13 where p is on the vertical and x on the horizontal axis.. Note that the graph's origin is (50,0) — i.e. production begins at 50 units when price hits the lowest point of the marginal cost curve.

(h) *In (g) of exercise 12.5, you were asked to calculate the profit maximizing output level when $\alpha = 100$, $\beta = \gamma = 5 = p$ and $w = r = 20$. You did so using the input demand functions calculated from the profit maximization problem. You can now use the supply function derived from the cost minimization problem to verify your answer (which should have been 91.23 units of output.) Then verify that your answer is also the same as it was before (93.59) when r falls to 10.*

<u>Answer</u>: Plugging these values into the function $x(w,r,p)$, we get

$$x = \frac{100(5) + \sqrt{100^2(5^2) - 4(5)(100)(20+20)}}{2(5)} = 91.23. \tag{12.89}$$

Doing the same when r falls to 10, we get

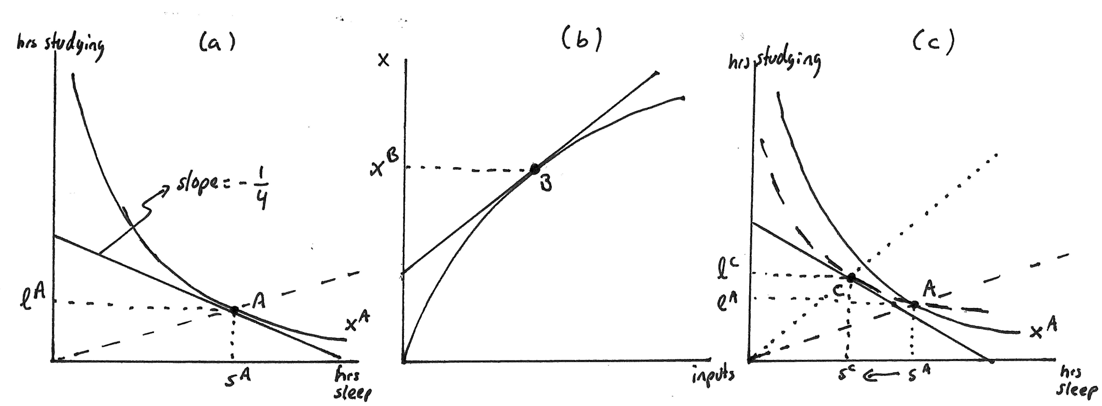
$$x = \frac{100(5) + \sqrt{100^2(5^2) - 4(5)(100)(20+10)}}{2(5)} = 93.59. \tag{12.90}$$

12.7 Everyday Application: *To Study or to Sleep?: Research suggests that successful performance on exams requires preparation (i.e. studying) and rest (i.e. sleep). Neither by itself produces good exam grades — but in the right combination they maximize exam performance.*

A: *We can then model exam grades as emerging from a production process that takes hours of studying and hours of sleep as inputs. Suppose this production process is homothetic and has decreasing returns to scale.*

(a) *On a graph with hours of sleep on the horizontal axis and hours of studying on the vertical, illustrate an isoquant that represents a particular exam performance level x^A.*

Answer: This is illustrated in panel (a) of Graph 12.14 where x^A represents the isoquant with all the input bundles that can produce exam grade x^A.



Graph 12.14: Studying and Sleeping

(b) *Suppose you are always willing to pay \$5 to get back an hour of sleep and \$20 to get back an hour of studying. Illustrate on your graph the least cost way to get to the exam grade x^A.*

Answer: This is illustrated in panel (a) of the graph with the addition of the isocost line that is tangent at A — which implies the least cost way to get exam grade x^A is to sleep s^A hours and study ℓ^A hours.

(c) *Since the production process is homothetic, where in your graph are the cost minimizing ways to get to the other exam grade isoquants?*

Answer: The cost minimizing input bundles will all lie on a ray from the origin through A — because the slopes of the isoquants are the same along any such ray, and at A the slope of the isocost is equal to the slope of the isoquant.

(d) *Using your answer to (c), can you graph a vertical slice of the production frontier that contains all the cost minimizing sleep/study input bundles?*

Answer: This is illustrated in panel (b) of Graph 12.14 where the vertical slice along the ray from the origin in panel (a) is graphed. It has a concave shape because the production process is assumed to have decreasing returns to scale.

(e) *Suppose you are willing to pay \$p for every additional point on your exam. Can you illustrate on your graph from (d) the slice of the "isoprofit" that gives you your optimal exam grade? Is this necessarily the same as the exam grade x^A from your previous graph?*

Answer: This is simply a slice of an isoprofit plane described by $\pi = px - 20\ell - 5s$, where π stands for the highest possible "profit", x is the exam grade, ℓ is the hours spent studying and s is the hours spent sleeping. It is tangent at B — with x^B being the optimal exam grade. This is not necessarily the same as x^A. We had chosen x^A arbitrarily and used it to show on what ray all cost minimizing input bundles lie. x^B lies on that ray — but does not necessarily overlap with x^A.

(f) *What would change if you placed a higher value on each exam point?*

Answer: If you place a higher value on exam grades, nothing in panel (a) will change since none of the items in that graph were derived from the knowledge of p. Neither will the production frontier slice in (b) change since the technology for producing exam grades has not changed — just the value you place on them. The only thing that changes is that the slice of the isoprofit that is tangent to the production frontier in panel (b) becomes shallower — implying that the optimal exam grade is higher.

(g) *Suppose a new caffeine/Gingseng drink comes on the market — and you find it makes you twice as productive when you study. What in your graphs will change?*

Answer: This drink has changed the production technology — so any object in your graphs that comes from the production technology will change. In particular, panel (c) illustrates the original x^A isoquant with the original cost minimizing input bundle A. If the drink makes studying twice as productive, the slope of the new isoquant must be shallower at A than it was before — resulting in the new dashed isoquant. The new cost minimizing input bundle for exam grade x^A is then C — with less sleep and more studying. Since the production technology is homothetic, this implies that all the new cost minimizing ways of getting to different exam grades will lie on the (dotted) ray from the origin through C. The vertical slice of the new production technology will then also differ.

B: *Suppose that the production technology described in part A can be captured by the production function $x = 40\ell^{0.25}s^{0.25}$ — where x is your exam grade, ℓ is the number of hours spent studying and s is the number of hours spent sleeping.*

(a) *Assume again that you'd be willing to pay \$5 to get back an hour of sleep and \$20 to get back an hour of studying. If you value each exam point at p, what is your optimal "production plan"?*

Answer: We need to solve something quite analogous to a profit maximization problem

$$\max_{\ell,s,x} px - 20\ell - 5s \quad \text{subject to} \quad x = 40\ell^{0.25}s^{0.25} \tag{12.91}$$

which can also be written as the unconstrained optimization problem

$$\max_{\ell,x} 40p\ell^{0.25}s^{0.25} - 20\ell - 5s. \tag{12.92}$$

The two first order conditions are

$$10p\ell^{-0.75}s^{0.25} = 20 \quad \text{and} \quad 10p\ell^{0.25}s^{-0.75} = 5. \tag{12.93}$$

Solving these, we get the "input demand" equations

$$\ell(p) = 0.50p^2 \quad \text{and} \quad s(p) = 2p^2. \tag{12.94}$$

And plugging these into the production function, we get the exam grade "supply" function

$$x(p) = 40(0.50p^2)^{0.25}(2p^2)^{0.25} = 40p. \tag{12.95}$$

The optimal "production plan" therefore entails getting a grade of $40p$ by studying for $0.5p^2$ hours and sleeping $2p^2$ hours.

(b) *Can you arrive at the same answer using the Cobb-Douglas cost function (given in problem 12.4)?*

Answer: Using this cost function and substituting $A = 40$, $\alpha = \beta = 0.25$, $w = 20$ and $r = 5$, we get

$$C(x) = 0.5\left(\frac{20^{0.25}(5^{0.25})x}{40(0.25^{0.25})(0.25^{0.25})}\right)^2 = 0.0125x^2. \tag{12.96}$$

The marginal cost is then

$$MC(x) = \frac{\partial C(x)}{\partial x} = 0.025x. \tag{12.97}$$

Setting this equal to p and solving for x, we get $x(p) = 40p$, exactly as we did before.

(c) *What is your optimal production plan when you value each exam point at $2?*

Answer: You would study for 2 hours, sleep for 8 hours and earn an 80 on the exam.

(d) *How much would you have to value each exam point in order for you to put in the effort and sleep to get a 100 on the exam.*

Answer: You would have to value each exam point at $2.50. You would then study for 3.125 hours, sleep for 12.5 hours and earn a 100.

(e) *What happens to your optimal production plan as the value you place on each exam point increases?*

Answer: It is easy to see from the equations (12.94) and (12.95) that p always enters positively. As the value you place on your exam increases, you will therefore study and sleep more — and earn a higher grade.

(f) *What changes if the caffeine/Gingseng drink described in A(g) is factored into the problem?*

Answer: The underlying technology changes — which means the production function would have to change in a way that reflects this. For every 1 hour of studying, you would now get the benefit that you previously received from 2 hours of studying. Thus, the new production function would be

$$x = 40(2\ell)^{0.25}s^{0.25} \approx 47.57\ell^{0.25}s^{0.25}. \tag{12.98}$$

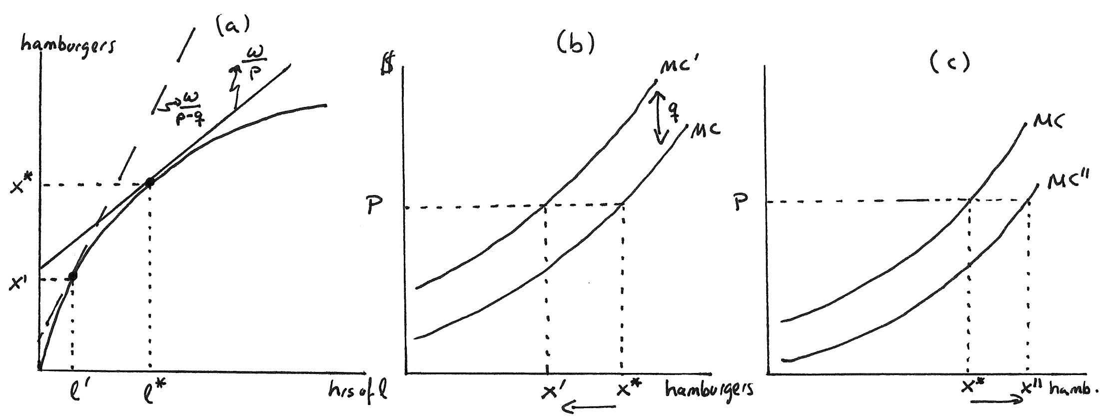
For the previous values of sleep and study time, you can check that you would have to value an exam point by only about $1.77 in order to make a 100 on the exam — and you would put in 2.22 hours of study time with 8.86 hours of sleep.

12.8 Everyday and Business Application: *Fast Food Restaurants and Grease: Suppose you run a fast food restaurant that produces only greasy hamburgers using labor that you hire at wage w. There is, however, no way to produce the hamburgers without also producing lots of grease that has to be hauled away. In fact, the only way for you to produce a hamburger is to also produce 1 ounce of grease. You therefore also have to hire a service that comes around and picks up the grease at a cost of q per ounce.*

A: *Since we are assuming that each hamburger comes with 1 ounce of grease that has to be picked up, we can think of this as a single input production process (using only labor) that produces 2 outputs — hamburgers and grease — in equal quantities.*

(a) *On a graph with hours of labor on the horizontal axis and hamburgers on the vertical, illustrate your production frontier assuming decreasing returns to scale. Then illustrate the profit maximizing plan assuming for now that it does not cost anything to have grease picked up (i.e. assume $q = 0$.)*

Answer: This is illustrated in panel (a) of Graph 12.15 where the initial (solid) isoprofit is tangent to the production frontier at production plan (ℓ^*, x^*).



Graph 12.15: Hamburgers and Grease

(b) *Now suppose $q > 0$. Can you think of a way of incorporating this into your graph and demonstrating how an increase in q changes the profit maximizing production plan?*

Answer: This is also illustrated in panel (a) of Graph 12.15. Now, the slope of the isoprofit is $w/(p-q)$ because the price per hamburger net of the cost of hauling the associated grease is $(p-q)$ rather than p. As a result, the new isoprofit is tangent at production plan (ℓ', x') — with less output and less labor input than before.

(c) *Illustrate the marginal cost curves with and without q — and then illustrate again how the cost of having grease picked up (i.e. $q > 0$) alters the profit maximizing production choice.*

Answer: The marginal cost curve is upward sloping because of the decreasing returns to scale of the production process. This is illustrated in panel (b) of Graph 12.15 as MC when $q = 0$ and as MC' when $q > 0$. Notice that the MC curve shifts up in a parallel way — because each hamburger now costs q more to produce than before. At a hamburger price of p, this implies that the profit maximizing quantity of hamburgers falls from x^* to x'.

(d) *With increasing fuel prices, the demand for hybrid cars that run partially on gasoline and partially on used cooking grease has increased. As a result, fast food chains report that they no longer have to pay to have grease picked up — in fact, they are increasingly being paid for their grease. (In essence, one of the goods you produce used to have a negative price but now has a positive price.) How does this change how many hamburgers are being produced at your fast food restaurant?*

Answer: This is illustrated in panel (c) of Graph 12.15 where the marginal cost curve MC'' is now below the original MC because q is now a "negative" cost. As a result, output increases from x^* to x''.

(e) *We have done all our analysis under the assumption that labor is the only input into hamburger production. Now suppose that labor and capital were both needed in a homothetic, decreasing returns to scale production process. Would any of your conclusions change?*

Answer: Given some input prices w and r, the homotheticity of the production process implies that we will operate on a vertical slice along a ray that emanates from the origin. This slice will look exactly like the production frontier graphed for the single input case in panel (a) of Graph 12.15 and a similar change in the slice of the isoprofits will result in the same conclusion. Similarly, the marginal cost curve will again shift by exactly q — leading to the same pictures as in panels (b) and (c).

(f) *We have also assumed throughout that producing one hamburger necessarily entails producing exactly one ounce of grease. Suppose instead that more or less grease per hamburger could be achieved through the purchase of fattier or less fatty hamburger meat. Would you predict that the increased demand for cooking grease in hybrid vehicles will cause hamburgers at fast food places to increase in cholesterol as higher gasoline prices increase the use of hybrid cars?*

Answer: Yes, as grease turns from being a cost to the firm to being a product that raises revenues, the firm will substitute toward fattier beef — thus increasing the amount of cholesterol in hamburgers.

B: *Suppose that the production function for producing hamburgers x is $x = f(\ell) = A\ell^{\alpha}$ where $\alpha < 1$. Suppose further that, for each hamburger that is produced, 1 ounce of grease is also produced.*

(a) *Set up the profit maximization problem assuming that hamburgers sell for price p and grease costs q (per ounce) to be hauled away.*

Answer: The profit maximization problem is

$$\max_{\ell,x} px - w\ell - qx \text{ subject to } x = A\ell^{\alpha} \tag{12.99}$$

which can also be written as

$$\max_{\ell} (p-q)A\ell^{\alpha} - w\ell. \tag{12.100}$$

(b) *Derive the number of hours of labor you will hire as well as the number of hamburgers you will produce.*

Answer: Differentiating the objective function in equation (12.100) with respect to ℓ and solving for ℓ, we get

$$\ell = \left(\frac{\alpha(p-q)A}{w}\right)^{1/(1-\alpha)}. \tag{12.101}$$

Substituting this into the production function, we get output level

$$x = A\left[\left(\frac{\alpha(p-q)A}{w}\right)^{1/(1-\alpha)}\right]^{\alpha} = A^{1/(1-\alpha)}\left(\frac{\alpha(p-q)}{w}\right)^{\alpha/(1-\alpha)}. \tag{12.102}$$

(c) *Determine the cost function (as a function of w, q and x).*

Answer: Inverting the production function, we get the conditional labor demand function

$$\ell(w,x) = \left(\frac{x}{A}\right)^{1/\alpha}. \tag{12.103}$$

The cost function is then simply the conditional labor demand function multiplied by the cost of labor w plus the cost of hauling away the grease; i.e.

$$C(w,x,q) = w\left(\frac{x}{A}\right)^{1/\alpha} + qx. \tag{12.104}$$

(d) *Derive from this the marginal cost function.*

Answer: Taking the derivative of the cost function with respect to x, we get

$$MC(w,x,q) = \left(\frac{w}{\alpha A^{1/\alpha}}\right) x^{(1-\alpha)/\alpha} + q. \tag{12.105}$$

(e) *Use the marginal cost function to determine the profit maximizing number of hamburgers and compare your answer to what you got in (b).*

Answer: Setting the marginal cost function equal to price p and solving for x, we get

$$x = A^{1/(1-\alpha)} \left(\frac{\alpha(p-q)}{w}\right)^{\alpha/(1-\alpha)} \tag{12.106}$$

which is identical to what we derived in (b).

(f) *How many hours of labor will you hire?*

Answer: Plugging our result for x back into the conditional labor demand function in equation (12.103), we get

$$\ell = \left(\frac{\alpha(p-q)A}{w}\right)^{1/(1-\alpha)} \tag{12.107}$$

which is again identical to what we derived in part (b).

(g) *How does your production of hamburgers change as grease becomes a commodity that people will pay for (rather than one you have to pay to have hauled away)?*

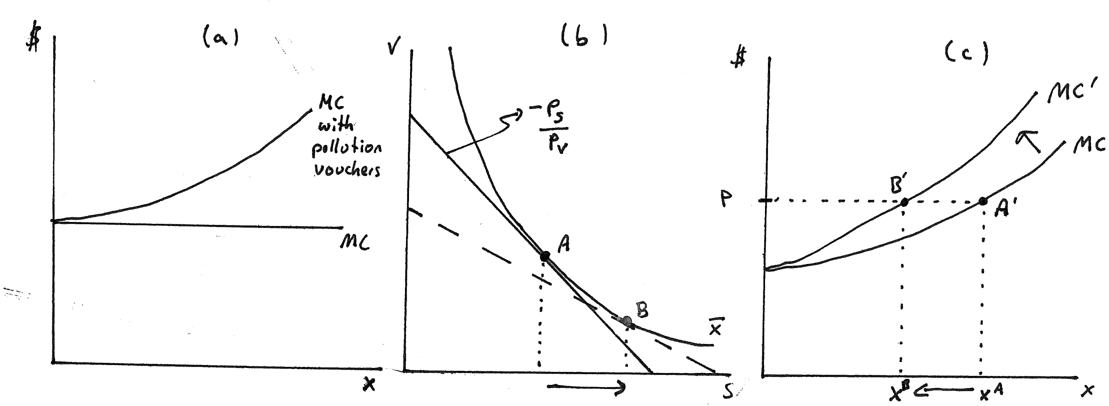
Answer: It is easy to see from our output equation that, as q becomes smaller and turns negative, output increases.

12.9 Business and Policy Application: *Investing in Smokestack Filters under Cap-and-Trade: On their own, firms have little incentive to invest in pollution abating technologies such as smokestack filters. As a result, governments have increasingly turned to "cap-and-trade" programs. Under these programs, discussed in more detail in Chapter 21, the government puts an overall "cap" on the amount of permissible pollution and firms are permitted to pollute only to the extent to which they own sufficient numbers of pollution permits or "vouchers". If a firm does not need all of its vouchers, it can sell them at a market price p_v to firms that require more.*

A: *Suppose a firm produces x using a technology that emits pollution through smokestacks. The firm must ensure that it has sufficient pollution vouchers v to emit the level of pollution that escapes the smokestacks, but it can reduce the pollution by installing increasingly sophisticated smokestack filters s.*

(a) *Suppose that the technology for producing x requires capital and labor and, without considering pollution, has constant returns to scale. For a given set of input prices (w, r), what does the marginal cost curve look like?*

Answer: The *MC* curve is flat when the production technology has constant returns to scale. This is depicted in panel (a) of Graph 12.16.



Graph 12.16: Cap-and-Trade and Smokestack Filters

(b) *Now suppose that relatively little pollution is emitted initially in the production process, but as the factory is used more intensively, pollution per unit of output increases — and thus more pollution vouchers have to be purchased per unit absent any pollution abating smokestack filters. What does this do to the marginal cost curve assuming some price p_v per pollution voucher and assuming the firm does not install smokestack filters?*

Answer: It causes the *MC* curve to be upward sloping as depicted in panel (a) of Graph 12.16.

(c) *Considering carefully the meaning of "economic cost", does your answer to (b) depend on whether the government gives the firm a certain amount of vouchers or whether the firm starts out with no vouchers and has to purchase whatever quantity is necessary for its production plan?*

Answer: It does not depend on whether the vouchers are owned by the firm or the firm has to purchase them. In both cases, the opportunity cost of using a pollution voucher to emit pollution in production is p_v. If the firm owns the voucher, it foregoes the opportunity to sell it at p_v to another firm that wishes to buy more vouchers. If the firm does not own vouchers, it must directly pay p_v per voucher.

(d) *Suppose that smokestack filters are such that initial investments in filters yield high reductions in pollution, but as additional filters are added, the marginal reduction in pollution declines. You can now think of the firm as using two additional inputs — pollution vouchers and smokestack filters — to produce output x legally. Does the overall production technology now have increasing, constant or decreasing returns to scale?*

Answer: The overall technology now has decreasing returns to scale. This is because, whether the firm uses pollution vouchers or smokestack filters or some combination of the two, it has to expend increasing resources to deal with its pollution output for any marginal increase in production.

(e) *Next, consider a graph with "smokestack filters" s on the horizontal and "pollution vouchers" v on the vertical axis. Illustrate an isoquant that shows different ways of reaching a particular output level $\overline{x}$ legally — i.e. without polluting illegally. Then illustrate the least cost way of reaching this output level (not counting the cost of labor and capital) given p_v and p_s.*

Answer: This is illustrated in panel (b) of Graph 12.16 where A is the cost minimizing bundle of smokestack filters and pollution vouchers to produce $\overline{x}$ when the prices of filters and vouchers are p_s and p_v.

(f) *If the government imposes additional limits on pollution by removing some of the pollution vouchers from the market, p_v will increase. How much will this affect the number of smokestack filters used in any given firm assuming output does not change? What does your answer depend on?*

Answer: The increase in p_v will cause isocosts to become shallower. If output does not change from $\overline{x}$, this will lead to a change in the cost minimizing bundle to B — causing the firm to use fewer vouchers and more smokestack filters. The size of the adjustment depends on the degree of substitutability between vouchers and smokestack filters in production. In other words, if it is relatively easy for the firm to install additional smokestack filters, the effect will be bigger than if it is not.

(g) *What happens to the overall marginal cost curve for the firm (including all costs of production) as p_v increases? Will output increase or decrease?*

Answer: This is illustrated in panel (c) of Graph 12.16. The marginal cost of production increases as p_v increases, rotating the MC curve from MC to MC'. For a given output price p, this implies that the profit maximizing output falls from x^A to x^B.

(h) *Can you tell whether the firm will buy more or fewer smokestack filters as p_v increases? Do you think it will produce more or less pollution?*

Answer: It is not clear whether the firm will buy more or fewer smokestack filters — because it is not clear by how much the firm will reduce its output. We know from panel (c) that the firm will produce less, and we know from panel (b) that, for the same level of output, it will buy more filters. But if the firm decreases production sufficiently much, it may end up buying fewer filters. No matter what, however, it will produce less pollution — because it produces less output with more filters for that level of output than it would have used before.

(i) True or False*: The Cap-and-Trade system reduces overall pollution by getting firms to use smokestack filters more intensively and by causing firms to reduce how much output they produce.*

Answer: This is true. As we have shown, the firm uses more smokestack filters for any given output level (panel (b) of the graph) but also produces less output (panel (c)).

B: *Suppose the cost function (not considering pollution) is given by $C(w,r,x) = 0.5w^{0.5}r^{0.5}x$, and suppose that the tradeoff between using smokestack filters s and pollution vouchers v to achieve legal production is given by the Cobb-Douglas production technology $x = f(s,v) = 50s^{0.25}v^{0.25}$.*

(a) *In the absence of cap-and-trade policies, does the production process have increasing, decreasing or constant returns to scale?*

Answer: The marginal cost function derived from $C(w,r,x)$ is

$$MC(w,r,x) = \frac{\partial C(w,r,x)}{\partial x} = 0.5w^{0.5}r^{0.5}. \qquad (12.108)$$

This function is independent of x — i.e. the marginal cost is constant, which implies constant returns to scale.

(b) *Ignoring for now the cost of capital and labor, derive the cost function for producing different output levels as a function of p_s and p_v — the price of a smokestack filter and a pollution voucher.(You can derive this directly or use the fact that we know the general form of cost functions for Cobb-Douglas production functions from what is given in problem 12.4).*

Answer: Plugging in $A = 50$ and $\alpha = \beta = 0.25$ into the cost function given in problem 12.4, we get

$$C(p_s, p_v, x) = 0.5\left(\frac{x p_s^{0.25} p_v^{0.25}}{50(0.25^{0.25})(0.25^{0.25})}\right)^2 = 0.0008 p_s^{0.5} p_v^{0.5} x^2. \qquad (12.109)$$

(c) *What is the full cost function $C(w, r, p_s, p_v)$? What is the marginal cost function?*

Answer: The cost of producing output level x is then simply the cost of labor and capital plus the cost of complying with the requirement that pollution is produced legally; i.e.

$$C(w, r, p_s, p_v) = 0.5w^{0.5}r^{0.5}x + 0.0008 p_s^{0.5} p_v^{0.5} x^2. \qquad (12.110)$$

The marginal cost function is then

$$MC(w, r, p_s, p_v) = 0.5w^{0.5}r^{0.5} + 0.0016 p_s^{0.5} p_v^{0.5} x. \qquad (12.111)$$

(d) *For a given output price p, derive the supply function.*

Answer: We set p equal to MC and solve for x to get

$$x(w, r, p_s, p_v, p) = \frac{p - 0.5w^{0.5}r^{0.5}}{0.0016 p_s^{0.5} p_v^{0.5}}. \qquad (12.112)$$

(e) *Using Shephard's lemma, can you derive the conditional smokestack filter demand function?*

Answer: Shephard's lemma tells us that the partial derivative of the cost function with respect to an input price is equal to the conditional input demand function for that input; i.e.

$$s(w, r, p_s, p_v, x) = \frac{\partial C(w, r, p_s, p_v)}{\partial p_s} = 0.0004 \frac{p_v^{0.5} x^2}{p_s^{0.5}}. \qquad (12.113)$$

(f) *Using your answers, can you derive the (unconditional) smokestack filter demand function?*

Answer: If we plug the supply function $x(w, r, p_s, p_v, p)$ into the conditional smokestack filter demand function $s(w, r, p_s, p_v, x)$, we will get the unconditional smokestack filter demand function. We then get

$$s(w, r, p_v, p_s, p) = \frac{625(p - 0.5w^{0.5}r^{0.5})^2}{4 p_v^{0.5} p_s^{1.5}}. \qquad (12.114)$$

(g) *Use your answers to illustrate the effect of an increase in p_v on the demand for smokestack filters holding output fixed as well as the effect of an increase in p_v on the profit maximizing demand for smokestack filters.*

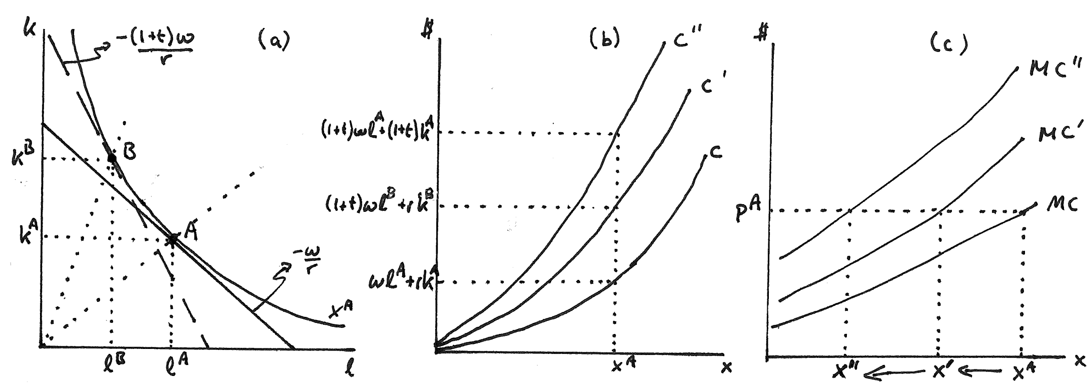
Answer: The derivative of the conditional demand function $s(w, r, p_s, p_v, x)$ with respect to p_v is positive — indicating that we will buy more smokestack filters conditional on producing the same quantity of output as before. The derivative of the unconditional filter demand $s(w, r, p_v, p_s, p)$ with respect to p_v, however, is negative — indicating that we will buy fewer pollution filters when we arrive at our new profit maximizing production plan. This is not because we pollute more — but rather because our supply function $x(w, r, p_s, p_v, p)$ tells us that we will produce sufficiently less such that we will need fewer overall filters even through we use more filters for the quantity that we do produce than we would have before.

12.10 Policy Application: *Taxes on Firms: There are several ways in which governments tax firms — including taxes on labor, capital or profits. As we will see in Chapter 19, it is not at all immediately clear whether taxes on labor or capital are paid by firms even when tax laws specify that firms will pay them. For now, we will simply assume that we know that some share of taxes on inputs are real costs to firms. It is also not clear that governments can easily identify economic profit of firms — or that price-taking firms usually make such profits (as we will see in Chapter 14). Again, we will simply assume these issues away for now.*

A: *Suppose a firm employs labor ℓ and capital k to produce output x using a homothetic, decreasing returns to scale technology.*

(a) *Suppose that, at the current wage w, rental rate r and output price p, the firm has identified $A = (x^A, \ell^A, k^A)$ as its profit maximizing production plan. Illustrate an isoquant corresponding to x^A and show how (ℓ^A, k^A) must satisfy the conditions of cost minimization.*

Answer: The isoquant corresponding to x^A is depicted in panel (a) of Graph 12.17. At the input bundle (ℓ^A, k^A), the isocost with slope $(-w/r)$ is tangent — implying that x^A is being produced at the least cost possible (where $TRS = -w/r$).



Graph 12.17: Taxes on Firms

(b) *Translate this to a graph of the cost curve that holds w and r fixed — indicating where in your isoquant graph the underlying input bundles lie for this cost curve.*

Answer: Given that the technology is homothetic, all cost minimizing input bundles lie on the same ray from the origin — i.e. the ray from the origin through A in panel (a) of the graph so long as w and r do not change. The cost curve C in panel (b) is derived from this vertical slice of the production frontier — and takes the shape depicted from the fact that the production frontier has decreasing returns to scale. At production level x^A, the cost is indicated as $w\ell^A + rk^A$ because the input bundle (ℓ^A, k^A) is the least cost bundle that will achieve output level x^A.

(c) *Show how x^A emerges as the profit maximizing production level on the marginal cost curve that is derived from the cost curve you illustrated in (b).*

Answer: In panel (c) of Graph 12.17, the curve MC depicts the marginal cost curve that is simply the slope of the cost curve C from panel (b) of the graph. The profit maximizing output level occurs where $p = MC$ — or where $p^A = MC$ when the output price is p^A.

(d) *Now suppose that the government taxes labor — causing the cost of labor for the firm to increase to $(1+t)w$. What changes in your pictures — and how will this effect the profit maximizing production plan?*

Answer: In panel (a), the slope of isocosts changes from $(-w/r)$ to $(-(1+t)w/r)$ — which implies the cost minimizing input bundles now lie on the ray that passes through B. In other words, for any given output level, the firm substitutes away from labor (which has become more expensive) and toward capital. As a result of the higher input costs, the total cost curve shifts up in panel (b) to C' — with x^A now costing $(1+t)w\ell^B + rk^B$. The MC curve also increases as a result — to MC' in panel (c) of the graph. If output price remains at p^A, this implies that the profit maximizing output level falls to x'.

(e) *What happens if the government instead imposes a tax on capital that raises the real cost of capital to $(1+t)r$?*

Answer: In panel (a), the new slope of the isocosts would be shallower — with slope $-w/((1+t)r)$ rather than $-w/r$. (This is not depicted in panel (a) of the graph). As a result, the new cost minimizing input bundles would lie on a ray that is shallower than that which passes through A — with firms substituting away from capital and toward labor. Panels (b) and (c) would change as in the case of a tax on labor — with new cost curve C' and new marginal cost curve MC'.

(f) *What happens if instead the government imposes a tax on both capital and labor — causing the cost of capital and labor to increase by the same proportion (i.e. to $(1+t)w$ and $(1+t)r$.)*

Answer: In this case, the new slope of isocosts would be

$$\frac{-(1+t)w}{(1+t)r} = \frac{-w}{r}. \tag{12.115}$$

Thus, the slope would be unchanged by the taxes — causing the cost minimizing input bundles to remain on the ray through A as if there were no taxes on capital and labor. This is because the relative prices of capital and labor have not changed. Although the firms will use the same input bundles as before for any output quantity — those input bundles now cost more. The cost curve would therefore shift up even more than it does when only one of the inputs is taxed — to C'' in panel (b) of Graph 12.17. The output level x^A, for instance, would now be $(1+t)w\ell^A + (1+t)rk^A$. This implies a higher marginal cost curve — MC'' in panel (c) of the graph. And this in turn implies that, so long as output price remains at p^A, the profit maximizing output quantity falls to x''.

(g) *Now suppose the government instead taxes economic profit at some rate $t<1$. Thus, if the firm makes pre-tax profit π, the firm gets to keep only $(1-t)\pi$.*

Answer: This would not affect any of the graphs we have drawn in panels (a) through (c) of Graph 12.17 at the beginning of the problem. No input prices change — so the isocost lines have the same slope as originally. Thus, all cost minimizing input bundles lie on the ray that passes through A — and the cost function is C with marginal cost MC. The profit maximizing output level is then x^A. After some thought, this should make intuitive sense: If (ℓ^A, k^A, x^A) is the profit maximizing production plan when prices are (w, r, p), this means that no other production plan can get the firm a higher profit. If the government says it will take half of the firm's profit, then implementing the same production plan as before still results in exactly the same profit — which means the highest possible half profit that the firm can possibly attain. Maximizing before tax profit π and after tax profit $(1-t)\pi$ is thus exactly the same problem — resulting in the same solution.

B: *Suppose your firm has a decreasing returns to scale, Cobb-Douglas production function of the form $x = A\ell^\alpha k^\beta$ for which you may have previously calculated input and output demands as well as the cost function. (The latter is also given in problem 12.4).*

(a) *If you have not already done so, calculate input demand and output supply functions. (You can do so directly using the profit maximization problem, or you can use the cost function given in problem 12.4 to derive these.)*

Answer: We illustrated the way to solve this through profit maximization in problem 12.2. In problem 12.3 we showed how to calculate the input demand and output supply functions using cost minimization. So now we just illustrate how we would calculate these if all we knew was the cost function given in problem 12.4 which was

$$C(w,r,x) = (\alpha+\beta)\left(\frac{xw^\alpha r^\beta}{A\alpha^\alpha\beta^\beta}\right)^{1/(\alpha+\beta)}. \tag{12.116}$$

We would get the output supply function by setting marginal cost equal to price and solving for x. This was already done in part B(e) of problem 12.3 where we got

$$x(w,r,p) = \left(\frac{Ap^{(\alpha+\beta)}\alpha^{\alpha}\beta^{\beta}}{w^{\alpha}r^{\beta}}\right)^{1/(1-\alpha-\beta)}. \tag{12.117}$$

To get the input demand functions, we can substitute this into the conditional input demand functions (as we did in B(f) of problem 12.3). However, if we only know the cost function to start with, we would first have to derive the conditional input demand functions from the cost function — which we can do using Shephard's lemma that tells us the conditional input demands are simply given by the derivative of the cost function with respect to the input prices. Thus

$$\begin{aligned}\ell(w,r,x) = \frac{\partial C(w,r,x)}{\partial w} &= \frac{\alpha}{\alpha+\beta}(\alpha+\beta)\left(\frac{r^{\beta}x}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)} w^{-\beta/(\alpha+\beta)} \\ &= \left(\frac{\alpha r}{w\beta}\right)^{\beta/(\alpha+\beta)} + \left(\frac{x}{A}\right)^{1/(\alpha+\beta)}.\end{aligned} \tag{12.118}$$

We can similarly derive the conditional capital demand function as

$$k(w,r,x) = \frac{\partial C(w,r,x)}{\partial r} = \left(\frac{\beta w}{\alpha r}\right)^{\alpha/(\alpha+\beta)}\left(\frac{x}{A}\right)^{1/(\alpha+\beta)}. \tag{12.119}$$

Plugging the supply function $x(w,r,p)$ in for x into these conditional input demand equations, we then get the unconditional labor and capital demand functions

$$\ell(w,r,p) = \left(\frac{pA\alpha^{(1-\beta)}\beta^{\beta}}{w^{(1-\beta)}r^{\beta}}\right)^{1/(1-\alpha-\beta)} \text{ and } k(w,r,p) = \left(\frac{pA\alpha^{\alpha}\beta^{(1-\alpha)}}{w^{\alpha}r^{(1-\alpha)}}\right)^{1/(1-\alpha-\beta)} \tag{12.120}$$

(b) *Derive the profit function and check that it is correct by checking whether Hotelling's Lemma works.*

Answer: The profit function is $\pi(w,r,p) = px(w,r,p) - w\ell(w,r,p) - rk(w,r,p)$. When we plug the output supply and input demand equations from above into this expression, we get

$$\pi(w,r,p) = (1-\alpha-\beta)\left(\frac{Ap\alpha^{\alpha}\beta^{\beta}}{w^{\alpha}r^{\beta}}\right)^{1/(1-\alpha-\beta)}. \tag{12.121}$$

Hotelling's Lemma implies that the partial derivative of the profit function with respect to p yields the output supply function $x(w,r,p)$ and the partial derivatives of the profit function with respect to the input prices yield the negative of the input demand functions $\ell(w,r,p)$ and $k(w,r,p)$. These indeed hold.

(c) *If you have not already done so, derive the conditional input demand functions. (You can do so directly by setting up the cost minimization problem, or you can employ Shephard's Lemma and use the cost function given in problem 12.4.)*

Answer: Employing Shephard's Lemma, we get

$$\ell(w,r,x) = \frac{\partial C(w,r,x)}{\partial w} = \left(\frac{\alpha r}{\beta w}\right)^{\beta/(\alpha+\beta)}\left(\frac{x}{A}\right)^{1/(\alpha+\beta)} \tag{12.122}$$

and

$$k(w,r,x) = \frac{\partial C(w,r,x)}{\partial r} = \left(\frac{\beta w}{\alpha r}\right)^{\alpha/(\alpha+\beta)}\left(\frac{x}{A}\right)^{1/(\alpha+\beta)}. \tag{12.123}$$

(d) *Consider a tax on labor that raises the labor costs for firms to* $(1+t)w$. *How does this affect the various functions in the duality picture for the firm?*

Answer: We would replace w with $(1+t)w$ in each of these equations. Consider first the output supply function $x(w,r,p)$ which now becomes

$$x(w,r,p) = \left(\frac{Ap^{(\alpha+\beta)}\alpha^{\alpha}\beta^{\beta}}{((1+t)w)^{\alpha}r^{\beta}}\right)^{1/(1-\alpha-\beta)}. \tag{12.124}$$

This implies that, as t increases, output falls. Next, consider the input demand functions that now become

$$\ell(w,r,p) = \left(\frac{pA\alpha^{(1-\beta)}\beta^{\beta}}{((1+t)w)^{(1-\beta)}r^{\beta}}\right)^{1/(1-\alpha-\beta)} \text{ and}$$
$$k(w,r,p) = \left(\frac{pA\alpha^{\alpha}\beta^{(1-\alpha)}}{((1+t)w)^{\alpha}r^{(1-\alpha)}}\right)^{1/(1-\alpha-\beta)} \tag{12.125}$$

which implies that, as t increases, the demand for both labor and capital falls. Plugging $(1+t)w$ in for w in the profit function also indicates that profit falls with increases in t.

Now consider the functions from the cost minimization problem. The conditional input demand functions become

$$\ell(w,r,x) = \left(\frac{\alpha r}{\beta((1+t)w)}\right)^{\beta/(\alpha+\beta)}\left(\frac{x}{A}\right)^{1/(\alpha+\beta)} \tag{12.126}$$

and

$$k(w,r,x) = \left(\frac{\beta((1+t)w)}{\alpha r}\right)^{\alpha/(\alpha+\beta)}\left(\frac{x}{A}\right)^{1/(\alpha+\beta)}. \tag{12.127}$$

Thus, conditional on producing x, labor input falls and capital input increases — i.e. the firm substitutes away from labor (which has become more expensive) and toward capital (which is now relatively cheaper). The overall cost of producing x increases with t, as indicated by the new cost function

$$C(w,r,x) = (\alpha+\beta)\left(\frac{x((1+t)w)^{\alpha}r^{\beta}}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)} \tag{12.128}$$

(e) *Repeat for a tax on capital that raises the capital cost for the firm to* $(1+t)r$.

Answer: Going through steps analogous to those in part (d), we get that output, input demands and profit falls as t increases. On the cost minimization side of the duality picture, conditional labor demand increases and conditional capital demand decreases as firms substitute away from capital and toward labor for any output quantity x. Finally, costs increase.

(f) *Repeat for simultaneous taxes on labor and capital that raise the cost of labor and capital to* $(1+t)w$ *and* $(1+t)r$.

Answer: This time, we replace w with $(1+t)w$ and r with $(1+t)r$ in our various functions. Our output supply function then becomes

$$x(w,r,p) = \left(\frac{Ap^{(\alpha+\beta)}\alpha^{\alpha}\beta^{\beta}}{((1+t)w)^{\alpha}((1+t)r)^{\beta}}\right)^{1/(1-\alpha-\beta)}. \tag{12.129}$$

This implies that, as t increases, output falls — by more than if either of the two taxes were implemented individually. Next, consider the input demand functions that now become

$$\ell(w,r,p) = \left(\frac{pA\alpha^{(1-\beta)}\beta^{\beta}}{((1+t)w)^{(1-\beta)}((1+t)r)^{\beta}}\right)^{1/(1-\alpha-\beta)} \text{ and}$$
$$k(w,r,p) = \left(\frac{pA\alpha^{\alpha}\beta^{(1-\alpha)}}{((1+t)w)^{\alpha}((1+t)r)^{(1-\alpha)}}\right)^{1/(1-\alpha-\beta)} \quad (12.130)$$

which implies that, as t increases, the demand for both labor and capital falls — again by more than if either tax were implemented individually. Similarly, we can see in the profit function that profit falls by more than if either of the taxes were implemented by itself.

Now consider the functions from the cost minimization problem. The conditional input demand functions become

$$\ell(w,r,x) = \left(\frac{\alpha((1+t)r)}{\beta((1+t)w)}\right)^{\beta/(\alpha+\beta)}\left(\frac{x}{A}\right)^{1/(\alpha+\beta)} = \left(\frac{\alpha r}{\beta w}\right)^{\beta/(\alpha+\beta)}\left(\frac{x}{A}\right)^{1/(\alpha+\beta)} \quad (12.131)$$

and

$$k(w,r,x) = \left(\frac{\beta((1+t)w)}{\alpha((1+r)}\right)^{\alpha/(\alpha+\beta)}\left(\frac{x}{A}\right)^{1/(\alpha+\beta)} = \left(\frac{\beta w}{\alpha r}\right)^{\alpha/(\alpha+\beta)}\left(\frac{x}{A}\right)^{1/(\alpha+\beta)}. \quad (12.132)$$

Thus, conditional on producing x, labor and capital input do not change as a result of the simultaneous introduction of the two taxes — because the *relative* prices of labor and capital have not changed. (In our graphical analysis, this is equivalent to saying that the slope of the isocosts have not changed because the $(1+t)$ in the numerator and denominator of the slope term cancel.) At the same time, while we continue to use the same input bundles to produce any given level of output, those input bundles now cost more — thus the cost of producing any quantity x increases as shown in the cost function

$$C(w,r,x) = (\alpha+\beta)\left(\frac{x((1+t)w)^{\alpha}((1+t)r)^{\beta}}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)} \quad (12.133)$$

(g) *Repeat for a tax on profits as described in part A(g).*

Answer: The profit tax does not show up in any of our functions — because no prices are changed by the tax. Thus, no choice of the firm is changed — not the output supply and input demand choices, nor the conditional input demands. All that happens is that profit falls by the percentage of the tax. But if the firm was initially making positive profit, it will still make positive profit under the tax — because the tax only takes part of the profit. It should make sense that none of the firm's choices change: If the firm was maximizing profit before the tax, then it should not do anything differently if the government takes some fraction of the profit. Maximizing a fraction of profit is the same as maximizing profit. You can see this by simply setting up the profit maximization problem when the government collects some fraction t of the firm's overall profit, leaving the firm with $(1-t)\pi$. This problem would be

$$\max_{\ell,k,x}\,(1-t)(px - w\ell - rk) \quad \text{subject to} \quad x = f(\ell,k) \quad (12.134)$$

which can be written as the unconstrained optimization problem

$$\max_{\ell,k}\,(1-t)(pf(\ell,k) - w\ell - rk). \quad (12.135)$$

The first order conditions are then

$$(1-t)p\frac{\partial f(\ell,k)}{\partial \ell} = (1-t)w \quad \text{and} \quad (1-t)p\frac{\partial f(\ell,k)}{\partial k} = (1-t)r. \quad (12.136)$$

Since $(1-t)$ appears on both sides of each equation, we can cancel it and are left with the usual first order conditions in the absence of taxes; i.e.

$$p\frac{\partial f(\ell,k)}{\partial \ell} = w \quad \text{and} \quad p\frac{\partial f(\ell,k)}{\partial k} = r \tag{12.137}$$

which then can be solved for input demand functions and output supply — all of which is unaffected by t.