SOLUTIONS

13

Production Decisions in the Short and Long Run

Solutions for *Microeconomics: An Intuitive Approach with Calculus*

Apart from end-of-chapter exercises provided in the student *Study Guide*, these solutions are provided for use by instructors. (End-of-Chapter exercises with solutions in the student *Study Guide* are so marked in the textbook.)

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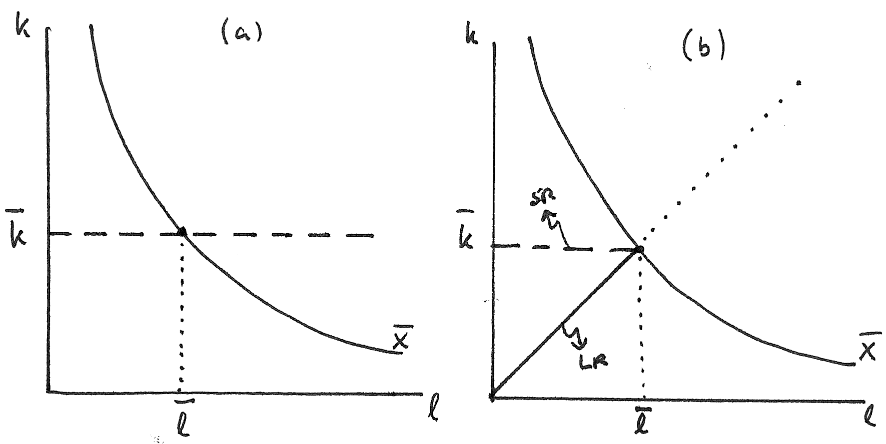
- Each end-of-chapter exercise begins on a new page. This is to facilitate maximum flexibility for instructors who may wish to share answers to some but not all exercises with their students.
- If you are assigning only the A-parts of exercises in *Microeconomics: An Intuitive Approach with Calculus*, you may wish to instead use the solution set created for the companion book *Microeconomics: An Intuitive Approach*.
- *Solutions to Within-Chapter Exercises are provided in the student* Study Guide.

13.1 *The following problem explores the relationship between maximizing profit in the short and long run when capital is fixed in the short run.*

A: *Suppose you have a homothetic production technology and you face output price p and input prices (w, r).*

(a) *On a graph with labor ℓ on the horizontal and capital k on the vertical axis, draw an isoquant and label a point on that isoquant as $(\overline{\ell}, \overline{k})$.*

Answer: This is done in both panels of Graph 13.1.



Graph 13.1: Short Run and Long Run Profit Maximization

(b) *Suppose that the point in your graph represents a profit maximizing production plan. What has to be true at this point?*

Answer: It must be true that $pMP_k = r$ and $pMP_\ell = w$ — i.e. the marginal revenue products of the inputs must be equal to their input prices.

(c) *In your graph, illustrate the slice along which the firm must operate in the short run.*

Answer: This slice is indicated by the horizontal (dashed) line in panel (a) of Graph 13.1. Capital is fixed at $\overline{k}$ along this line.

(d) *Suppose that the production technology has decreasing returns to scale throughout. If p falls, can you illustrate all the possible points in your graph where the new profit maximizing production plan will lie in the long run? What about the short run?*

Answer: This is illustrated in panel (b) of Graph 13.1. The horizontal dashed line is a portion of the short run production slice — that portion that yields less output than the output along the isoquant $\overline{x}$. Since we know from Chapter 11 that short run production falls as price falls, any point along this dashed line can result from a decrease in p. The solid portion of the ray emanating from the origin represents the production plans that could result in the long run when firms can adjust capital. Again, we know that output will fall with a decrease in price, but in the long run capital will adjust so that the ratio of capital to labor remains constant. This is because input prices have not changed. Thus the ratio of input prices remains unchanged, which means all cost minimizing input bundles lie on this ray from the origin through the original profit maximizing bundle $(\overline{\ell}, \overline{k})$. (And, of course, it must be the case that any new long run profit maximizing production plan — while involving less output — still produces output in the least costly way.)

(e) *What condition that is satisfied in the long run will typically not be satisfied in the short run?*

Answer: Since the firm can only adjust labor in the short run, the firm will let go of labor until $pMP_\ell = w$ — i.e. until the marginal revenue product of labor is equal to the wage. However, the firm cannot adjust capital in the short run — which implies that the marginal revenue product condition will not hold for capital in the short run. In fact, since the firm will release capital in the long run, we know that it must be the case that $pMP_k < r$ at the short run profit maximum.

(f) *What qualification would you have to make to your answer in (d) if the production process had initially increasing but eventually decreasing returns to scale?*

Answer: In that case, there is a portion of the solid part of the ray (in panel (b) of the Graph) along which the firm will not profit maximize no matter how much price falls — because the firm would make a negative profit along that portion. Without knowing more, we cannot tell exactly where the solid part of the ray in the graph should end in this case — but we do know it should not extend all the way down to the origin in the graph.

B: *Consider the Cobb-Douglas production function $x = f(\ell, k) = A\ell^\alpha k^\beta$.*

(a) *For input prices (w, r) and output price p, calculate the long run input demand and output supply functions assuming $0 < \alpha, \beta \leq 1$ and $\alpha + \beta < 1$.*

Answer: Solving the usual profit maximization problem

$$\max_{\ell,k} \; pA\ell^\alpha k^\beta - w\ell - rk, \tag{13.1}$$

we get the input demand functions

$$\ell(w,r,p) = \left(\frac{pA\alpha^{(1-\beta)}\beta^\beta}{w^{(1-\beta)}r^\beta}\right)^{1/(1-\alpha-\beta)} \text{ and } k(w,r,p) = \left(\frac{pA\alpha^\alpha\beta^{(1-\alpha)}}{w^\alpha r^{(1-\alpha)}}\right)^{1/(1-\alpha-\beta)}. \tag{13.2}$$

Plugging these into the production function and simplifying, we then get the output supply function

$$x(w,r,p) = \left(\frac{Ap^{(\alpha+\beta)}\alpha^\alpha\beta^\beta}{w^\alpha r^\beta}\right)^{1/(1-\alpha-\beta)}. \tag{13.3}$$

(b) *How would your answer change if $\alpha + \beta \geq 1$?*

Answer: If $\alpha + \beta = 1$, then the production function has constant returns to scale — which implies that either all output levels are profit maximizing or only one of the corners (i.e. zero or infinity) is optimal. If $\alpha + \beta > 1$, the production function has increasing returns to scale — which implies that the firm should produce an infinite amount of output. (Of course this does not make sense in light of the fact that we are assuming price-taking behavior — i.e. we are assuming firms small enough relative to the market such that they cannot influence price.)

(c) *Suppose that capital is fixed at $\overline{k}$ in the short run. Calculate the short run input demand and output supply functions.*

Answer: We need to use the short run slice of the production function that holds $\overline{k}$ fixed — i.e. $x = \left[A\overline{k}^\beta\right]\ell^\alpha$. The short run profit maximization problem is then

$$\max_{\ell} \; p\left[A\overline{k}^\beta\right]\ell^\alpha - w\ell, \tag{13.4}$$

where we do not take into account the *expense* of capital that is fixed and we treat the bracketed term as a constant. (Even if we did include it, it would drop out since only ℓ is a choice variable and the capital expense term would simply drop out as we take first order conditions). Solving this in the usual way, we get the short run labor demand function

$$\ell_{\overline{k}}(p,w) = \left(\frac{\alpha pA\overline{k}^\beta}{w}\right)^{1/(1-\alpha)}. \tag{13.5}$$

Substituting this back into the production function and simplifying, we get the short run output supply function

$$x_{\overline{k}}(p,w) = \left(A\overline{k}^{\beta}\right)^{1/(1-\alpha)} \left(\frac{\alpha p}{w}\right)^{\alpha/(1-\alpha)}. \tag{13.6}$$

(d) *What has to be true about α and β for these short run functions to be correct?*

Answer: It has to be the case that the production process has decreasing returns to scale — i.e. $\alpha + \beta < 1$. Otherwise, the true solution is a corner solution that will not be picked up by our usual optimization method.

(e) *Suppose $\overline{k} = k(w,r,p)$ (where $k(w,r,p)$ is the long run capital demand function you calculated in part (a).) What is your optimal short run labor demand and output supply in that case?*

Answer: If we plug our long run capital demand function in for $\overline{k}$ in the short run labor demand function, we get, after simplifying the expression,

$$\ell_{k(w,r,p)}(p,w) = \left(\frac{pA\alpha^{(1-\beta)}\beta^{\beta}}{w^{(1-\beta)}r^{\beta}}\right)^{1/(1-\alpha-\beta)}. \tag{13.7}$$

Similarly, if we plug the long run capital demand function in for $\overline{k}$ in short run supply function, we get

$$x_{k(w,r,p)}(p,w) = \left(\frac{Ap^{(\alpha+\beta)}\alpha^{\alpha}\beta^{\beta}}{w^{\alpha}r^{\beta}}.\right)^{1/(1-\alpha-\beta)}. \tag{13.8}$$

(f) *How do your answers compare to the long run labor demand function $\ell(w,r,p)$ and the long run supply function $x(w,r,p)$ you calculated in part (a)? Can you make intuitive sense of this?*

Answer: The short run labor demand and output supply functions calculated above are exactly equal to the long run labor demand and output supply functions calculated earlier; i.e.

$$\ell_{k(w,r,p)}(p,w) = \ell(w,r,p) \quad \text{and} \quad x_{k(w,r,p)}(p,w) = x(w,r,p). \tag{13.9}$$

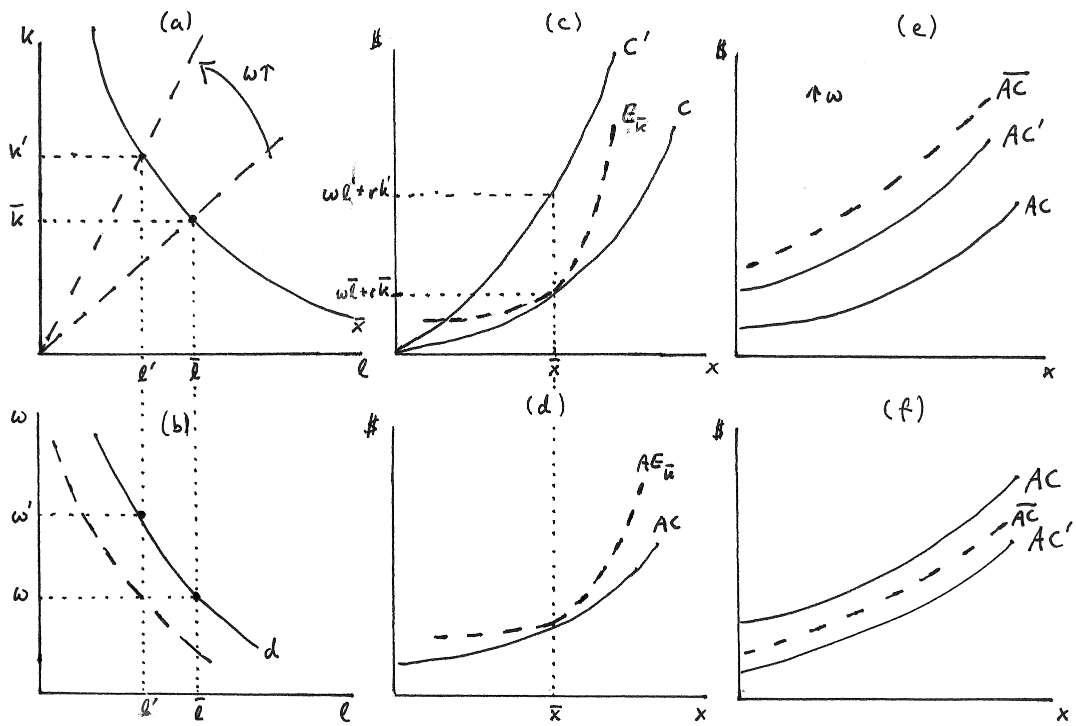
This should make intuitive sense: If I provide you in the short run with the optimal level of capital for you to reach the long run profit maximum at current input and output prices, then you'll choose your labor input exactly as you would in the long run. Put differently, we are "fixing" capital at exactly the long run quantity — which means there is nothing to keep you from implementing the long run profit maximizing production plan immediately.

13.2 *The following problem explores issues similar to those in exercise 13.1 but instead of thinking directly about profit maximization, we will think about cost minimization on the way to profit maximization.*

A: *Suppose you have a homothetic production technology and you face input prices (w, r).*

(a) *On a graph with labor ℓ on the horizontal and capital k on the vertical axis, illustrate a ray along which all cost minimizing production plans might lie for a given set of input prices. Does your answer depend on whether the production technology has increasing or decreasing returns to scale (or some combination of these)?*

Answer: This is illustrated in panel (a) of Graph 13.2 where the lower of the two dashed rays emerging from the origin represents the ray along which all cost minimizing bundles lie. The answer does not depend on the returns to scale of the production function — because regardless of what these returns to scale are, the cost minimizing way to produce a given level of output simply identifies the tangency between the isoquant for that output level and the isocost that has slope $-w/r$. These all lie on the same ray because of the assumption of homotheticity of the production process.



Graph 13.2: Costs and Expenditures

(b) *Illustrate in your graph an isoquant corresponding to some output level $\bar{x}$. What has to be true at the intersection of the ray and the isoquant?*

Answer: This is also illustrated in panel (a) where the isoquant corresponding to $\bar{x}$ intersects the ray at the input level $(\bar{\ell}, \bar{k})$. At that point, the technical rate of substitution TRS is equal to $-w/r$ or, put differently, the ratio of the marginal product of labor to the marginal product of capital is equal to the w/r.

(c) *Show what happens to the ray of cost minimizing input bundles if w increases to w'. Then illustrate how you would derive the conditional labor demand curve for producing $\overline{x}$.*

Answer: The isocosts would become steeper when w increases, implying that the tangencies to isoquants will fall to the left of where they fell originally. This implies that the new cost minimizing input bundles will lie on a steeper ray — such as the second ray illustrated in panel (a) of Graph 13.2. At the original w, we therefore would use $\overline{\ell}$ labor input, while at the new w' we would use ℓ'. This gives us two points on the conditional labor demand curve that is graphed in panel (b).

(d) *From this point forward, suppose that the production technology has decreasing returns to scale. Illustrate how you would derive the firm's long run cost curve for the original input prices.*

Answer: This is illustrated in panel (c) of Graph 13.2. For output level $\overline{x}$, the cost minimizing input bundle $(\overline{\ell},\overline{k})$ costs $w\overline{\ell}+r\overline{k}$ — which is one point on the cost curve C. The shape for the remainder of the cost curve emerges from the fact that the production technology as decreasing returns to scale. This implies that the slice of the production function along the lower ray in panel (a) has diminishing slope (as it gets more and more difficult to produce more output), which in turn implies that the cost curve C has a increasing slope that increases at an increasing rate.

(e) *What happens to the cost curve when w increases to w'?*

Answer: When w increases, the output level $\overline{x}$ costs more produce with the new cost minimizing input bundle (ℓ',k'). All other cost minimizing input bundles (for different output levels) lie on the same higher ray in panel (a) — with the shape of the new cost curve C' in panel (c) again emerging from the decreasing returns to scale of the production process.

(f) *Suppose that you are initially producing at the intersection of your original isoquant (corresponding to $\overline{x}$) and the original ray. If w remained unchanged, where would your (short run)* expenditure *curve fall on your graph with the long run cost curve?*

Answer: The short run expenditure curve includes the short run cost of labor as well as the expense on the fixed level of capital $\overline{k}$. If we are producing $\overline{x}$, this expense is then exactly the same as the long run cost — because in the long run we would continue to use exactly the same input bundle. But if we are producing either more or less than $\overline{x}$, then the short run expenditure will be higher than the long run cost because capital cannot be adjusted to its long run optimal level in the short run. The dashed curve in panel (c) of Graph 13.2 is then the short run expenditure curve given a fixed level of capital $\overline{k}$.

(g) *Translate your cost/expenditure curve graph to a graph with the average (long run) cost and average (short run) expenditure curves.*

Answer: This is done in panel (d) of Graph 13.2 where the short run average expenditure $AE_{\overline{k}}$ is equal to the long run average cost AC at the output level $\overline{x}$ but higher everywhere else. The average cost curve for the higher wage would lie above AC in the graph (as is shown in panel (e) for the higher average cost AC').

(h) *How does the average (long run) cost curve change when w increases to w'? If you also graphed a cost curve that removed the substitution effect, where would it generally lie relative to the original and final cost curve? What would its precise location depend on?*

Answer: This is illustrated in panel (e) of Graph 13.2. The new average cost curve AC' lies above AC (because the (total) cost C' lies above C in panel (b).) If the firm did not substitute away from labor and toward capital, the total cost would be even higher — above C' in panel (b) — which implies that the average cost that only incorporates this direct effect of the wage increase (but not the substitution effect) would lie above AC' in panel (e). This is illustrated as $\overline{AC}$ in the graph. The precise location of $\overline{AC}$ will depend on how substitutable capital and labor are in production. If they are relatively complementary, then there won't be much of a substitution effect and AC' and $\overline{AC}$ will lie close to one another. If, on the other hand, the inputs are relatively substitutable, then the firm is able to reduce costs a lot by substituting away from labor and toward capital — creating a larger gap between AC' and $\overline{AC}$.

(i) *Now suppose that instead of wage increasing, the rental rate on capital r fell to r'. What happens to the conditional labor demand curve that you graphed in part (c)?*

Answer: This would cause a change in the cost minimizing ray in panel (a) similar to what we illustrated for an increase in the wage. If the decrease in r caused us to end up on the higher of the rays in panel (a) of Graph 13.2, then the firm would use (ℓ', k') to produce $\overline{x}$. Since the wage is still w (because it was a change in r that caused the change in behavior), this implies that we would use less labor at w than we did originally — causing the conditional labor demand for producing $\overline{x}$ to shift in as illustrated by the dashed curve in panel (b).

(j) *Repeat (h) for the change in the rental rate.*

Answer: This is illustrated in panel (f) of Graph 13.2 where we begin at the original average cost curve AC and end at the new lower average cost curve AC' when r falls to r'. If the firm did not substitute and still produced at the original (shallower) ray in panel (a) of the graph, it would not experience as much of a drop in its average costs — and therefore would end up somewhere between the two average cost curves. This is illustrated as $\overline{AC}$ in the graph. Again, the precise location of $\overline{AC}$ will depend on the degree of substitutability of capital and labor in production. If they are relatively complementary, then most of the cost reduction comes from the direct rather than the substitution effect — implying that $\overline{AC}$ lies close to AC'. If, on the other hand, the inputs are relatively substitutable, then more of the cost reduction comes from the firm's ability to substitute inputs — implying that $\overline{AC}$ lies closer to AC.

B: *Suppose again (as in exercise 13.1) that the production process is defined by the Cobb-Douglas production function* $x = f(\ell, k) = A\ell^{\alpha}k^{\beta}$.

(a) *For input prices* (w, r), *calculate the long run conditional input demand functions.*

Answer: To solve for the conditional input demand functions, we need to solve the problem

$$\min_{\ell,k} \; w\ell + rk \text{ subject to } x = A\ell^{\alpha}k^{\beta}. \tag{13.10}$$

Setting up the Lagrangian, taking first order conditions, and solving in the usual way, we get

$$\ell(w,r,x) = \left(\frac{\alpha r}{\beta w}\right)^{\beta/(\alpha+\beta)}\left(\frac{x}{A}\right)^{1/(\alpha+\beta)} \text{ and } k(w,r,x) = \left(\frac{\beta w}{\alpha r}\right)^{\alpha/(\alpha+\beta)}\left(\frac{x}{A}\right)^{1/(\alpha+\beta)}. \tag{13.11}$$

(b) *Do you need to assume* $0 < \alpha, \beta \leq 1$ *and* $\alpha + \beta < 1$ *in order for these to be valid?*

Answer: No, we do not need to assume this — i.e. we do not need to assume that the production function has decreasing returns to scale (as long as we assume $\alpha > 0$ and $\beta > 0$). Unlike for the profit maximization problem where profit maximizing production plans will lie at corners if the production function has increasing returns to scale, here we are simply asking what the least cost way of producing different levels of output is. And to produce some output level x, we need to reach the isoquant for that level of x in the least costly way if we produce x in a cost minimizing way — i.e. we need to find the tangency of the isocost with that isoquant.

(c) *Derive the long run total, marginal and average cost functions.*

Answer: Substituting $\ell(w,r,x)$ and $k(w,r,x)$ into the objective function of the minimization problem (and combining terms), we get the long run (total) cost function

$$C(w,r,x) = w\ell(w,r,x) + r(w,r,x) = (\alpha+\beta)\left(\frac{w^{\alpha}r^{\beta}x}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)}. \tag{13.12}$$

From this , we can derive the marginal cost function

$$MC(w,r,x) = \frac{\partial C(w,r,x)}{\partial x} = \left(\frac{w^{\alpha}r^{\beta}x^{(1-\alpha-\beta)}}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)} \tag{13.13}$$

and the average cost function

$$AC(w,r,x) = \frac{C(w,r,x)}{x} = (\alpha+\beta)\left(\frac{w^{\alpha}r^{\beta}x^{(1-\alpha-\beta)}}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)}. \tag{13.14}$$

(d) *Suppose output price is p. Use your answer to derive the firm's (long run) profit maximizing output supply function. Do you need to assume $0<\alpha,\beta\le 1$ and $\alpha+\beta<1$ for this to be valid? (If you have done exercise 13.1, check to make sure your answer agrees with what you concluded in part (a) of that exercise.)*

Answer: Setting MC equal to price and solving for x, we get

$$x(w,r,p)=\left(\frac{Ap^{(\alpha+\beta)}\alpha^{\alpha}\beta^{\beta}}{w^{\alpha}r^{\beta}}\right)^{1/(1-\alpha-\beta)} \tag{13.15}$$

which is of course the same function we derived directly through the profit maximization problem in the previous exercise. And now it is indeed necessary to assume $0<\alpha,\beta\le 1$ and $\alpha+\beta<1$ in order for this to be valid — because, unless the production function has decreasing returns to scale, the true profit maximizing production plan is a corner solution (with, for instance, optimal production of infinity when we have increasing returns to scale). The Lagrange method does not pick out such corner solutions — and, in the case of increasing returns, will actually suggest a production plan that results in negative profit.

(e) *From your answer, derive the firm's profit maximizing long run labor and capital demand functions. (You can again check your answers with those you derived through direct profit maximization in exercise 13.1.)*

Answer: To derive these, we can now simply plug the long run output supply function $x(w,r,p)$ into the conditional input demand functions $\ell(w,r,x)$ and $k(w,r,x)$ to get

$$\ell(w,r,p)=\ell(w,r,x(w,r,p))=\left(\frac{pA\alpha^{(1-\beta)}\beta^{\beta}}{w^{(1-\beta)}r^{\beta}}\right)^{1/(1-\alpha-\beta)} \tag{13.16}$$

and

$$k(w,r,p)=k(w,r,x(w,r,p))=\left(\frac{pA\alpha^{\alpha}\beta^{(1-\alpha)}}{w^{\alpha}r^{(1-\alpha)}}\right)^{1/(1-\alpha-\beta)}, \tag{13.17}$$

the same functions we derived directly through the profit maximization problem in the previous exercise.

(f) *Now suppose capital is fixed in the short run at $\overline{k}$. Derive the short run conditional input demand for labor.*

Answer: If capital is fixed at $\overline{k}$, the short run production function becomes $x=\left[A\overline{k}^{\beta}\right]\ell^{\alpha}$ where the bracketed term enters as a constant. Thus, there is only one possible technologically efficient production plan for each output level — the production plan that hires just enough labor to produce that level of output. This one technologically efficient production plan is then also the economically efficient production plan. While we could set up a cost minimization problem to solve for the short run conditional labor demand function, we can also simply invert the production function by solving it for ℓ. When we do that, we get the short run conditional labor demand function

$$\ell_{\overline{k}}(w,x)=\left(\frac{x}{A\overline{k}^{\beta}}\right)^{1/\alpha}. \tag{13.18}$$

Note that this is not actually a function of the input price w — because no matter what w is, we will always have to use this much labor to reach output level x.

(g) *Derive the short run (total) cost function as well as the short run marginal and average cost functions.*

Answer: The short run (total) cost function is then

$$C_{\overline{k}}(w,x)=w\ell_{\overline{k}}(w,x)=w\left(\frac{x}{A\overline{k}^{\beta}}\right)^{1/\alpha}. \tag{13.19}$$

From this, we can derive the short run marginal cost function

$$MC_{\overline{k}}(w,x) = \frac{\partial C_{\overline{k}}(w,x)}{\partial x} = \frac{w}{\alpha}\left(\frac{x^{(1-\alpha)}}{A\overline{k}^{\beta}}\right)^{1/\alpha} \tag{13.20}$$

and the short run average cost function

$$AC_{\overline{k}}(w,x) = w\left(\frac{x^{(1-\alpha)}}{A\overline{k}^{\beta}}\right)^{1/\alpha}. \tag{13.21}$$

(h) *Derive the short run supply curve.*

Answer: Setting price equal to short run marginal cost and solving for x, we get

$$x_{\overline{k}}(w,p) = \left(A\overline{k}^{\beta}\right)^{1/(1-\alpha)}\left(\frac{\alpha p}{w}\right)^{\alpha/(1-\alpha)}. \tag{13.22}$$

(i) True or False*: As long as the production function has decreasing returns the scale, the (short run) average* expenditure *curve will be U-shaped even though the short run average cost curve is not.*

Answer: This is true. It is easy to see that the short run AC curve is not U-shaped by taking the derivative of $AC_{\overline{k}}(w,x)$ with respect to x:

$$\frac{\partial AC_{\overline{k}}(w,x)}{\partial x} = \frac{\alpha w}{(1-\alpha)}\left(\frac{1}{A\overline{k}^{\beta}}\right)^{1/\alpha} x^{(1-2\alpha)/\alpha}. \tag{13.23}$$

This expression is strictly positive so long as $\alpha < 1$ — i.e. so long as the short run production function has decreasing returns to scale. Thus, the short run average cost curve is upward sloping throughout.

The short run average *expenditure* function is the short run average cost function plus the average expenditure on the fixed capital — i.e.

$$AE_{\overline{k}}(w,x) = AC_{\overline{k}}(w,x) + \frac{r\overline{k}}{x} = w\left(\frac{x^{(1-\alpha)}}{A\overline{k}^{\beta}}\right)^{1/\alpha} + \frac{r\overline{k}}{x}. \tag{13.24}$$

The derivative of this with respect to x is

$$\frac{\partial AE_{\overline{k}}(w,x)}{\partial x} = \frac{\alpha w}{(1-\alpha)}\left(\frac{1}{A\overline{k}^{\beta}}\right)^{1/\alpha} x^{(1-2\alpha)/\alpha} - \frac{r\overline{k}}{x^2}. \tag{13.25}$$

The sign of this expression is not unambiguously positive or negative. For small values of x, the negative term at the end will be large (in absolute value) — causing the derivative to be negative; but for large values of x, the reverse is true and the derivative becomes positive. Thus, the short run average cost curve initially slopes down but eventually slopes up — giving us the U-shape. You can in fact solve for the lowest point of the U by setting the derivative of AE equal to zero and solving for x.

(j) *What is the shape of the long run average cost curve? Can the Cobb-Douglas production function yield U-shaped long run average cost curves?*

Answer: Taking the derivative of the long run average cost function with respect to x, we get

$$\frac{\partial AC(w,r,x)}{\partial x} = (1-\alpha-\beta)\left(\frac{w^{\alpha}r^{\beta}}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)} x^{(1-2(\alpha-\beta))} \tag{13.26}$$

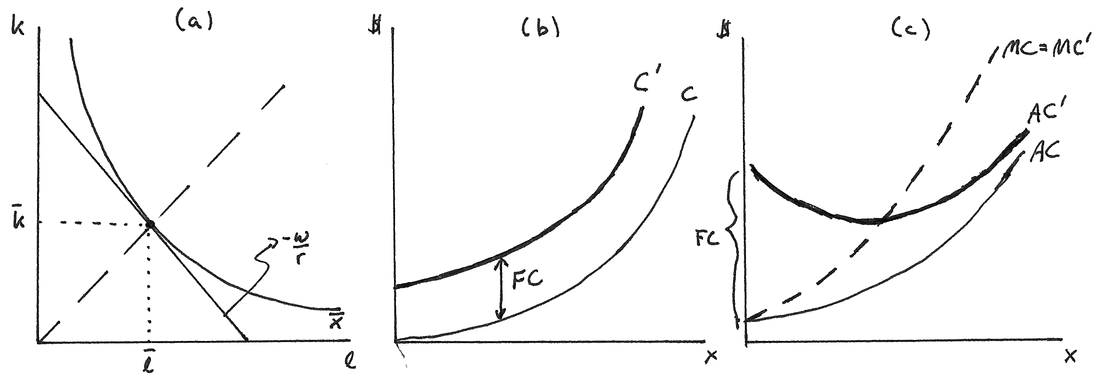
which is positive if $\alpha+\beta<1$, zero if $\alpha+\beta=1$ and negative if $\alpha+\beta>1$. Thus, the long run average cost curve slopes up for decreasing returns to scale, is constant for constant returns to scale and slopes down for increasing returns to scale (which should make sense). It is therefore not possible for the Cobb-Douglas production function to yield a U-shaped long run average cost curve — at least not in the absence of long run fixed costs that we will discuss in exercise 13.3.

13.3 *In this exercise we add a (long run) fixed cost to the analysis.*

A: *Suppose the production process for a firm is homothetic and has decreasing returns to scale.*

(a) *On a graph with labor ℓ on the horizontal and capital k on the vertical axis, draw an isoquant corresponding to output level $\overline{x}$. For some wage rate w and rental rate r, indicate the cost minimizing input bundle for producing $\overline{x}$.*

Answer: This is illustrated in panel (a) of Graph 13.3 where the input bundle $(\overline{\ell},\overline{k})$ is the cost minimizing input bundle to produce $\overline{x}$.



Graph 13.3: Fixed Costs

(b) *Indicate in your graph the slice of the production frontier along which all cost minimizing input bundles lie for this wage and rental rate.*

Answer: Since the production process is homothetic, all tangencies of isocosts with slope $-w/r$ with isoquants for different output levels will lie on the ray emanating from the origin and passing through $(\overline{\ell},\overline{k})$. This is also illustrated in panel (a) of Graph 13.3.

(c) *In two separate graphs, draw the (total) cost curve and the average cost curve with the marginal cost curve.*

Answer: This is illustrated in panels (b) and (c) of Graph 13.3 as C, AC and MC. Since the slice of the production frontier indicated by the dashed ray in panel (a) has decreasing returns, the shape of the cost function must be such that cost increases at an increasing rate as x goes up. The same then holds for AC, with the MC beginning at the same point as AC but lying above AC throughout.

(d) *Suppose that, in addition to paying for labor and capital, the firm has to pay a recurring fixed cost (such as a license fee). What changes in your graphs?*

Answer: Nothing changes in panel (a) — because the fact that the firm has to pay some cost to begin producing does not change how much labor and capital will be needed to reach different isoquants. The cost curve, however, shifts up to C', with C and C' parallel to each other and the difference being the FC. The marginal cost curve, however, remains unchanged since fixed costs do not enter the *additional* cost of producing output. Finally, the average cost curve moves up but, unlike the cost curve, not in a parallel fashion. It increases by the FC when $x = 1$ (because the average fixed cost is FC/x). As x increases, however, FC/x falls — which causes the new average cost curve AC' to converge to the original AC as x gets large.

(e) *What is the firm's exit price in the absence of fixed costs? What happens to that exit price when a fixed cost is added?*

Answer: In the absence of fixed costs, the firm's exit price is equal to the marginal cost of producing the first unit of output — because that is where the marginal cost curve crosses the AC curve. When fixed costs are introduced, however, the exit price rises to the lowest point of the new U-shaped average cost curve AC' where the unchanged MC curve crosses it. Thus, the exit price increases.

(f) *Does the firm's supply curve shift as we add a fixed cost?*

Answer: No, the supply curve does not shift, but it does become "shorter". It does not shift because the MC curve does not shift. It becomes "shorter" because the exit price increases. Thus, the supply curve before the introduction of the fixed cost is the entire MC curve in panel (c) of Graph 13.3, but after the FC is introduced, it shrinks to only the portion of the MC curve that lies above AC'.

(g) *Suppose that the cost minimizing input bundle for producing $\overline{x}$ that you graphed in part (a) is also the profit maximizing production plan before a fixed cost is considered. Will it still be the profit maximizing production plan after we include the fixed cost in our analysis?*

Answer: This will still be the profit maximizing production plan *if* it is optimal for the firm not to exit. In that case, price is sufficiently high relative to w and r such that it crosses MC above AC' in panel (c) of Graph 13.3. However, it may be the case that the introduction of FC implies that it is no longer profit maximizing to produce — and that a corner solution of producing nothing is optimal. This occurs if the price falls below AC'.

B: *As in exercises 13.1 and 13.2, suppose the production process is again characterized by the production function $x = f(\ell,k) = A\ell^{\alpha}k^{\beta}$ with $0<\alpha,\beta\le 1$ and $\alpha+\beta<1$.*

(a) *If you have not already done so in a previous exercise, derive the (long run) cost function for this firm.*

Answer: The long run cost function (solved from the cost minimization problem) is

$$C(w,r,x) = w\ell(w,r,x) + r(w,r,x) = (\alpha+\beta)\left(\frac{w^{\alpha}r^{\beta}x}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)}. \quad (13.27)$$

(b) *Now suppose that, in addition to the cost associated with inputs, the firm has to pay a recurring fixed cost of FC. Write down the cost minimization problem that includes this FC. Will the conditional input demand functions change as a result of the FC being included?*

Answer: The cost minimization problem would now be

$$\min_{\ell,k}\ w\ell + rk + FC \ \text{ subject to } \ x = A\ell^{\alpha}k^{\beta}. \quad (13.28)$$

When we now write down the Lagrange function and take derivatives to get the first order conditions from which we derive the conditional input demand functions, the FC term will disappear since it enters the objective function as a constant. Thus, the conditional input demand functions will be unchanged by the addition of a FC term. This should make intuitive sense: Just because the firm has to pay something like a license fee to start producing does not mean it doesn't need exactly as much capital and labor to produce any given level of output as it did before.

(c) *Write down the new cost function and derive the marginal and average cost functions from it.*

Answer: Since the conditional input demands are no different than they were before the FC, the new cost function is the same as the one derived in (a) except that we also have to include the fixed cost; i.e. it now becomes

$$C(w,r,x) = w\ell(w,r,x) + r(w,r,x) = (\alpha+\beta)\left(\frac{w^{\alpha}r^{\beta}x}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)} + FC. \quad (13.29)$$

From this, we can derive the marginal cost function

$$MC(w,r,x) = \frac{\partial C(w,r,x)}{\partial x} = \left(\frac{w^{\alpha}r^{\beta}x^{(1-\alpha-\beta)}}{A\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)} \quad (13.30)$$

and the average cost function

$$AC(w,r,x) = \frac{C(w,r,x)}{x} = (\alpha+\beta)\left(\frac{w^\alpha r^\beta x^{(1-\alpha-\beta)}}{A\alpha^\alpha \beta^\beta}\right)^{1/(\alpha+\beta)} + \frac{FC}{x}. \tag{13.31}$$

Note that the marginal cost function is the same as it would be without the FC but the AC function is not.

(d) *What is the shape of the average cost curve? How does its lowest point change with changes in the FC?*

Answer: We can infer the shape of the AC curve by checking whether its derivative with respect to x is positive or negative. The derivative is

$$\frac{\partial AC(w,r,x)}{\partial x} = \left[(1-\alpha-\beta)\left(\frac{w^\alpha r^\beta}{A\alpha^\alpha \beta^\beta}\right)^{1/(\alpha+\beta)} x^{(1-2(\alpha+\beta))/(\alpha+\beta)}\right] - \frac{FC}{x^2}. \tag{13.32}$$

The bracketed term is positive (since $\alpha+\beta<1$) but the FC term is negative. Furthermore, as x approaches zero, the first term is smaller than the absolute value of the second term — implying an initially negative sign and thus an initially downward sloping AC curve. As x gets larger, however, the absolute value of the FC term gets smaller, with the positive bracketed term eventually outweighing the negative FC term. Thus, at some point, the slope of the AC curve becomes positive. This implies a U-shape to the AC curve. And, as FC increases, only the second term changes while the bracketed first term remains the same. Thus, the AC curve will have negative slope for a larger range of x — implying that the bottom of the U moves to the right as FC increases. We can of course also infer this from the fact that the upward sloping MC curve is unchanged as FC increases — because the MC curve must cross AC no matter how high FC gets. Thus, as FC pushes up the AC curve, it must be that its lowest point slides up along the unchanged MC curve. And, to calculate the level of output at which the AC reaches its lowest point, we can set equation (13.32) to zero and solve for x to get

$$x = \left(\frac{A\alpha^\alpha \beta^\beta}{w^\alpha r^\beta}\right)\left(\frac{FC}{(1-\alpha-\beta)}\right)^{(\alpha+\beta)} \tag{13.33}$$

and note again that x increases as FC increases.

(e) *Does the addition of a FC term change the (long run) marginal cost curve? Does it change the long run supply curve?*

Answer: No, the FC term does not appear in the MC equation and thus has no impact on the marginal cost curve. This, of course, makes intuitive sense — a fixed cost is paid before production even begins and therefore does not impact the cost of producing additional units of output. Since the long run supply curve is a portion of the long run MC curve, we know that the supply curve is therefore not shifted by changes in the fixed cost. However, since the supply curve is that portion of MC that lies above the AC curve, and since AC shifts up with an increase in FC, the supply curve "shrinks" as FC increases.

(f) *How would you write out the profit maximization problem for this firm including fixed costs? If you were to solve this problem, what role would the FC term play?*

Answer: The profit maximization problem would then be

$$\max_{\ell,k}\; pA\ell^\alpha k^\beta - w\ell - rk - FC. \tag{13.34}$$

When we take first order conditions, we are taking derivatives of the objective function above — and thus FC disappears from the first order conditions that are used to solve the maximization problem. As a result, none of the input demand or output supply functions will change when FC is included in the profit maximization problem.

(g) *Considering not just the math but also the underlying economics, does the addition of the FC have any implications for the input demand and output supply functions?*

Answer: We see from the math that the functions produced by our optimization problem are unchanged. However, we also know that the math does not identify corner solutions.

In the absence of a fixed cost, we do not have to worry about such corner solutions so long as the production function has decreasing returns to scale — but when we add a fixed cost, we know from what we did in the earlier parts of the exercise that the *AC* curve takes on a U-shape as a result of the addition of the *FC*. Thus, while the exit price before a *FC* term is added is zero, it is now at the lowest point of the *AC* curve. This implies that the functions produced by the math are correct only for sufficiently high prices and/or sufficiently low wages and rental rates. If p gets too high relative to w and r, the firm should exit and produce nothing rather than the positive amount indicated by the math. This is analogous to our conclusion above that only a portion of the *MC* curve will be the supply curve when there is a *FC* while the entire *MC* curve is the supply curve (under decreasing returns to scale) when there is no fixed cost.

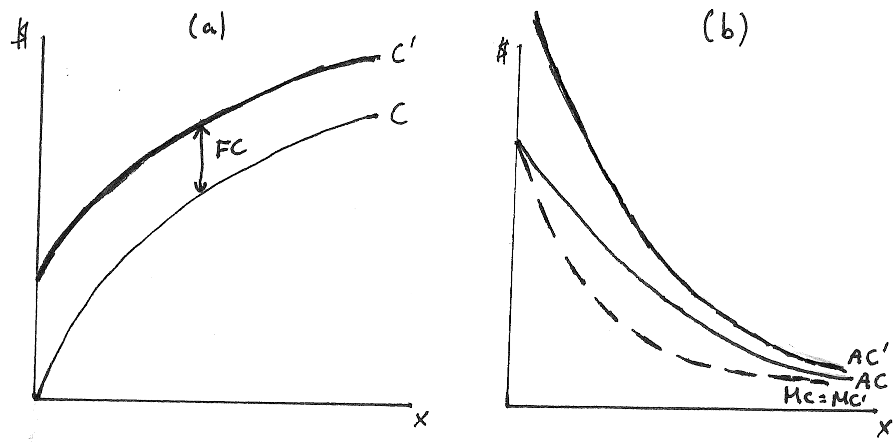
13.4 *Repeat exercise 13.3 assuming increasing rather than decreasing returns to scale. What changes in the analysis, and what does not change?*

A: Answer: Below is the answer to each of the parts of exercise 13.3 under the assumption of increasing returns to scale.

(a) The answer to (a) is identical to that in exercise 13.3.

(b) The answer to (b) is again identical to that in exercise 13.3.

(c) These are illustrated as *C*, *AC* and *MC* in panels (a) and (b) of Graph 13.4. The cost curve now has a concave shape because increasing returns to scale implies it is becoming increasingly cheap to produce additional units of output. This implies a downward sloping *MC* curve that lies below the *AC* curve.



Graph 13.4: Fixed Costs and Increasing Returns to Scale

(d) The inclusion of a fixed cost *FC* causes the cost curve in panel (a) to shift up by the amount of the *FC* just as it did in exercise 13.3. It does not change the *MC* curve (for the same reason as in exercise 13.3), but it raises the *AC* curve to *AC′*. The difference between *AC* and *AC′* is the average fixed cost — which is high for low levels of *x* but diminishes as *x* gets large. Thus, *AC* and *AC′* converge as *x* increases.

(e) If the *AC* curve converges to some positive level as *x* gets large, then this is the exit price. Since *AC′* converges to *AC* as *x* gets large, the exit price does not change when fixed costs are added. (If *AC* converges to zero, then there is no exit price.) Note that if the firm produces, it will produce an infinite amount (which, as we have said repeatedly, is the reason that increasing returns to scale throughout does not make sense for competitive firms.)

(f) In this case, there is no supply curve because *MC* always lies below *AC*. The firm will either produce an infinite amount or nothing at all if price falls below the level to which *AC* converges as *x* goes to infinity.

(g) This part does not make sense for the increasing returns to scale case — because the increasing returns to scale firm will never produce a finite amount if it is a price taker.

B: Answer: Below is the answer to each of the parts of exercise 13.3 under the assumption of increasing returns to scale (which in this case implies $\alpha + \beta > 1$).

(a) This answer is the same as that in exercise 13.3 — except now $\alpha + \beta > 1$.

(b) This answer is again the same as that for exercise 13.3.

(c) While the shapes of these functions will now be different (because $\alpha+\beta>1$), the functional forms will be identical to those derived in exercise 13.3. Note, however, how we can see the difference in the shapes of the cost curves from these functional forms. When $\alpha+\beta>1$, the exponent on x in the cost function becomes less than 1 — implying a concave shape to the cost curve (as graphed in panel (a) of Graph 13.4). In the MC function, $\alpha+\beta>1$ implies that the exponent on x becomes negative — giving us the downward slope. The same is true for the AC function both with and without the fixed cost.

(d) The shape of the average cost curve can again be seen in the first derivative of AC with respect to x — which takes the same functional form as in exercise 13.3. However, the fact that $\alpha+\beta>1$ now implies that the bracketed term is also negative — which gives us an unambiguous negative sign for the derivative. Thus, the AC curve slopes down. In addition, you can see that it approaches the average cost without the FC term since FC/x^2 goes to zero as x gets large.

(e) As in exercise 13.3, the MC function is unaffected by the addition of the FC term. There is no supply curve in this case.

(f) The answer to this part is the same as in exercise 13.3.

(g) The addition of the FC has no implication for input demand and output supply functions in this case — because the increasing returns to scale firm will always produce an infinite amount of output. This is true regardless of how large the FC (as long at it is not infinite) because the average fixed cost goes to zero as output goes to infinity.

13.5 *We will often assume that a firm's long run average cost curve is U-shaped. This shape may arise for two different reasons which we explore in this exercise.*

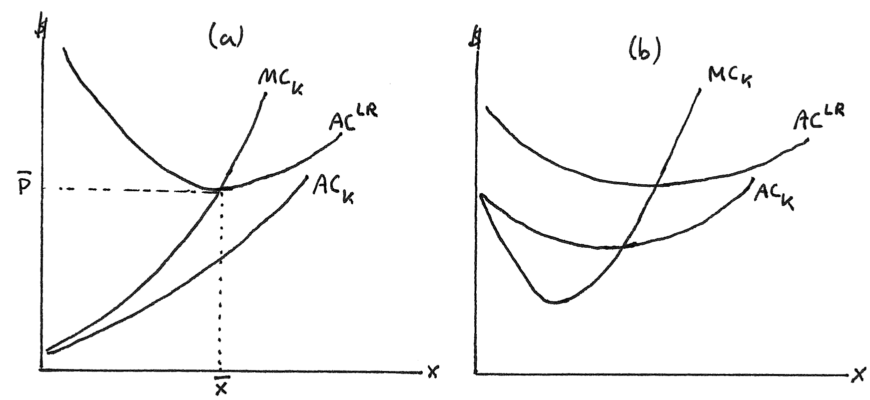
A: *Assume that the production technology uses labor ℓ and capital k as inputs, and assume throughout this problem that the firm is currently long run profit maximizing and employing a production plan that is placing it at the lowest point of its long run AC curve.*

(a) *Suppose first that the technology has decreasing returns to scale but that, in order to begin producing each year, the firm has to pay a fixed license fee F. Explain why this causes the long run AC curve to be U-shaped.*

Answer: The long run AC curve is U-shaped in this case because the fixed cost F, while only an expense in the short run and therefore not included in short run AC curves, it is a real economic cost in the long run. At low levels of output, the average fixed cost F/x is large (because x is small) — causing AC to be high. As output increases, the average fixed cost F/x falls (because x gets large) — and therefore becomes a diminishing factor in the long run AC curve. Instead, the fact that the production process has decreasing returns to scale pushes AC up as x increases.

(b) *Draw a graph with the U-shaped AC curve from the production process described in part (a). Then add to this the short run MC and AC curves. Is the short run AC curve also U-shaped?*

Answer: This is illustrated in panel (a) of Graph 13.5. The fact that the firm is currently profit maximizing at the lowest point of its long run AC curve implies that output price must be $\overline{p}$. The short run MC_k curve must then go through this lowest point on the AC^{LR} curve. Furthermore, since the production technology has decreasing returns to scale, it must be that any slice that holds capital fixed must also have decreasing marginal product of labor. Thus, the short run production function (that holds capital fixed) has decreasing returns to scale, and there are no fixed costs in the short run, only fixed expenses. This implies that the MC_k curve must be upward sloping, causing the short run AC_k curve to lie below it and be similarly upward sloping.



Graph 13.5: U-shaped Average Cost Curves

(c) *Next, suppose that there are no fixed costs in the long run. Instead, the production process is such that the marginal product of each input is initially increasing but eventually decreasing, and the production process as a whole has initially increasing but eventually decreasing returns to scale. (A picture of such a production process was given in Graph 12.16 in the previous chapter.) Explain why the long run AC curve is U-shaped in this case.*

Answer: The fact that the production process has initially increasing but eventually decreasing returns to scale implies that the long run average costs must initially fall but will eventually increase in the decreasing returns to scale portion of the production process. The reasoning is identical to that for the single input case with production frontiers that initially get steeper but eventually get shallower.

(d) *Draw another graph with the U-shaped AC curve. Then add the short run MC and AC curves.*

Answer: This is done in panel (b) of Graph 13.5. The U-shape of the short run MC_k and AC_k curves is due to the fact that, for any fixed level of capital, the short run production function has initially increasing but eventually decreasing returns to scale. That, in turn, arises from the fact that the production process has initially increasing but eventually decreasing marginal product of labor. The short run MC_k curve again intersects the lowest point of the long run AC^{LR} curve because the firm is initially profit maximizing at the lowest point of the long run AC curve.

(e) *Is it possible for short run AC curves to* not *be U-shaped if the production process has initially increasing but eventually decreasing returns to scale?*

Answer: Yes, this is possible. In order for the AC^{LR} curve to assume a U-shape (in the absence of fixed costs), the production process must have initially increasing returns to scale (that eventually turn into decreasing returns to scale). In order for the short run AC_k curve to have a U-shape, it must be that the short run production function with fixed capital initially has increasing but eventually decreasing returns to scale. But this will occur only if the marginal product of labor is initially increasing and eventually decreasing. If the marginal product of labor for the fixed capital level k is diminishing throughout, then the short run AC_k curve will be upward sloping throughout because there are no fixed costs in the short run. Since it is possible for the marginal product of each input to be decreasing but still to have increasing returns to scale of the entire production process (that varies both capital and labor), it is possible to have the U-shaped AC curve in the long run but not the short run (in the absence of fixed costs).

B: *Suppose first that the production process is Cobb-Douglas, characterized by the production function $x = f(\ell, k) = A\ell^\alpha k^\beta$ with $\alpha, \beta > 0$ and $\alpha + \beta < 1$.*

(a) *In the absence of fixed costs, you should have derived in exercise 13.2 that the long run cost function for this technology is given by*

$$C(w,r,x) = (\alpha+\beta)\left(\frac{w^\alpha r^\beta x}{A\alpha^\alpha \beta^\beta}\right)^{1/(\alpha+\beta)}. \tag{13.35}$$

If the firm has long run fixed costs F, what is its long run average cost function? Is the average cost curve U-shaped?

Answer: The long run average cost function is then simply $C(w,r,x)$ divided by x plus FC/x — which gives us

$$AC(w,r,x) = \frac{C(w,r,x)}{x} = (\alpha+\beta)\left(\frac{w^\alpha r^\beta x^{(1-\alpha-\beta)}}{A\alpha^\alpha \beta^\beta}\right)^{1/(\alpha+\beta)} + \frac{FC}{x}. \tag{13.36}$$

In B(d) of exercise 13.3, we already argued that this must be U-shaped. Its derivative is

$$\frac{\partial AC(w,r,x)}{\partial x} = \left[(1-\alpha-\beta)\left(\frac{w^\alpha r^\beta}{A\alpha^\alpha \beta^\beta}\right)^{1/(\alpha+\beta)} x^{(1-2(\alpha+\beta))/(\alpha+\beta)}\right] - \frac{FC}{x^2}. \tag{13.37}$$

which, when set to zero, gives us the output level at which the long run AC curve reaches its lowest point:

$$x = \left(\frac{A\alpha^\alpha \beta^\beta}{w^\alpha r^\beta}\right)\left(\frac{FC}{(1-\alpha-\beta)}\right)^{(\alpha+\beta)} \tag{13.38}$$

(b) *What is the short run cost curve for a fixed level of capital $\overline{k}$? Is the short run average cost curve U-shaped?*

Answer: The short run production function with fixed $\overline{k}$ is $x = f(\ell) = \left[A\overline{k}^\beta\right]\ell^\alpha$ which, when solved for ℓ, gives us the conditional labor demand of

$$\ell_{\overline{k}}(x) = \left(\frac{x}{A\overline{k}^{\beta}}\right)^{1/\alpha} \tag{13.39}$$

Multiplying by wage w, we then get the short run cost function

$$C_{\overline{k}}(x) = w\left(\frac{x}{A\overline{k}^{\beta}}\right)^{1/\alpha} \tag{13.40}$$

from which we can derive the short run $MC_{\overline{k}}$ and $AC_{\overline{k}}$ functions

$$MC_{\overline{k}}(x) = \frac{w}{\alpha}\left(\frac{x^{(1-\alpha)}}{A\overline{k}^{\beta}}\right)^{1/\alpha} \quad \text{and} \quad AC_{\overline{k}}(x) = w\left(\frac{x^{(1-\alpha)}}{A\overline{k}^{\beta}}\right)^{1/\alpha}. \tag{13.41}$$

These are both increasing in x — and thus the short run MC and AC curves slope up. They also converge to zero as x goes to zero. Thus, they give rise to a picture such as the on in panel (a) of Graph 13.5.

(c) *Now suppose that the production function is still $f(\ell,k) = A\ell^{\alpha}k^{\beta}$ but now $\alpha+\beta > 1$. Are long run average and marginal cost curves upward or downward sloping? Are short run average cost curves upward or downward sloping? What does your answer depend on?*

Answer: The long run MC curve is downward sloping because of increasing returns to scale. The long run AC curve is similarly downward sloping (and starts above MC because of the long run fixed costs). (You can see this from the equation (13.36) where the exponent on x is now negative in the first term — implying that both terms decline in x.) Whether or not the short run MC and AC curves slope down depends on whether α is less than or greater than 1. If it is less than 1, then x enters the short run MC and AC functions in equation (13.41) with positive exponent — implying that these costs increase with x. When $\alpha > 1$, on the other hand, x enters with negative exponent — causing the cost curves to fall with x. Thus, increasing returns to scale is consistent with both upward and downward sloping short run MC and AC curves — the key is whether the marginal product of labor increases or decreases.

(d) *Next, suppose that the production technology were given by the equation*

$$x = f(\ell,k) = \frac{\alpha}{1+e^{-(\ell-\beta)}+e^{-(k-\gamma)}} \tag{13.42}$$

where e is the base of the natural logarithm. (We first encountered this in exercises 12.5 and 12.6.) If capital is fixed at $\overline{k}$, what is the short run production function and what is the short run cost function?

Answer: The short run production function is

$$x = f_{\overline{k}}(\ell) = \frac{\alpha}{\left[1+e^{-(\overline{k}-\gamma)}\right]+e^{-(\ell-\beta)}} \tag{13.43}$$

The short run cost function is then simply this production function solved for ℓ. We can multiply both sides by the denominator of the right hand side, divide both sides by x and then subtract the bracketed term from both sides to get

$$e^{-(\ell-\beta)} = \frac{\alpha}{x} - \left[1+e^{-(\overline{k}-\gamma)}\right] = \frac{\alpha-\left[1+e^{-(\overline{k}-\gamma)}\right]x}{x}. \tag{13.44}$$

Taking natural logs of both sides, we can then solve for the conditional short run labor demand function

$$\ell_{\overline{k}}(w,x) = \beta - \ln\left(\frac{\alpha-\left[1+e^{-(\overline{k}-\gamma)}\right]x}{x}\right) \tag{13.45}$$

which, when multiplied by w, gives us the short run cost function

$$C_{\overline{k}}(w,x) = w\beta - w\ln\left(\frac{\alpha - \left[1+e^{-(\overline{k}-\gamma)}\right]x}{x}\right). \tag{13.46}$$

(e) *What is the short run marginal cost function?*

Answer: Taking the derivative of the short run cost function with respect to x, we get

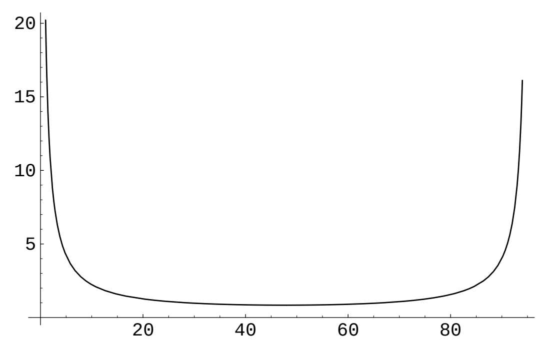
$$MC_{\overline{k}}(w,x) = \frac{w\alpha}{\left(\alpha - \left[1+e^{-(\overline{k}-\gamma)}\right]x\right)x}. \tag{13.47}$$

(f) *You should have concluded in exercise 12.6 that the long run MC function is $MC(w,r,x) = \alpha(w+r)/(x(\alpha-x))$ and demonstrated that the MC curve (and thus the long run AC curve) is U-shaped for the parameters $\alpha = 100$, $\beta = 5 = \gamma$ when $w = r = 20$. Now suppose capital is fixed at $\overline{k} = 8$. Graph the short run MC curve and use the information to conclude whether the short run AC curve is also U-shaped.*

Answer: The short run MC curve then becomes

$$MC_{\overline{k}=8} \approx \frac{2{,}000}{(100-1.05x)x} \tag{13.48}$$

which is plotted in Graph 13.6. This is obviously U-shaped — which causes the short run AC curve to be U-shaped as well.



Graph 13.6: Short Run MC when $\alpha = 100$, $\beta = \gamma = 5$, $w = r = 20$ and $\overline{k} = 8$

(g) *What characteristic of the this production function is responsible for your answer in part (f)?*

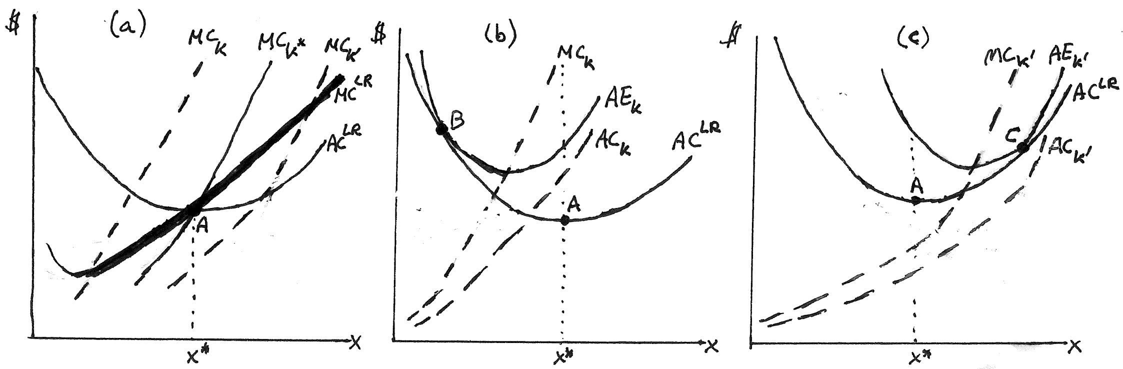
Answer: The characteristic that is causing the U-shaped curves in the short run is the the initially increasing marginal product of labor that causes the short run production function to display initially increasing and eventually decreasing returns to scale.

13.6 : *In Graph 13.3 we illustrated the relationship between short run average expenditure AE_k, short run average cost AC_k and long run average cost AC^{LR} curves for a particular level of capital. The particular level of capital chosen in Graph 13.3 is that level which makes the AE_k curve tangent to the AC^{LR} at its lowest point.*

A: *Consider a firm whose technology has decreasing returns to scale throughout and who faces a recurring fixed cost. Denote the level of capita chosen in the long run at the lowest point of the long run AC as k^*.*

(a) *Replicate the short run MC and long run AC curves from Graph 13.3.Where in your graph does the long run MC curve lie?*

Answer: This is done in panel (a) of Graph 13.7 where the dashed curves are irrelevant for now. The AC^{LR} and MC^{LR} curves are as drawn in the text. The MC_{k^*} has to cross the lowest point of the AC^{LR} curve (indicated as point A in the graph) and has to be steeper than MC^{LR} in order for supply responses to be smaller in the short run than the long run. Both short and long run marginal cost curves have to intersect AC^{LR} at A because in this graph we are assuming that the firm has exactly the "right" amount of capital (k^*) for producing x^* — the quantity at which long run average costs are at their lowest.



Graph 13.7: Short and Long Run Cost Curves

(b) *Draw a separate graph with the AC^{LR} curve. Suppose that $k < k^*$ in the short run. Illustrate where the AE_k must now lie.*

Answer: This is illustrated in panel (b) of Graph 13.7 where the AC^{LR} curve is drawn as before (reaching its lowest point at A). If $k < k^*$, then the firm's short run fixed capital level is below the quantity that would cause its average expenditure curve to be tangent at A, it must now be tangent to the AC^{LR} curve at a point like B that has to lie to the left of A. As a result, the tangency between AC^{LR} and AE_k does not occur at the lowest point of the AE_k curve.

(c) *Next illustrate where the AC_k and MC_k curves lie. Is the long run MC curve now different than in part (a)?*

Answer: This is also illustrated in pane (b) of Graph 13.7. The short run MC_k curve must intersect the AE_k curve at its lowest point — and therefore to the right of B. Since the AC_k curve does not include the fixed expense on capital, it lies below AE_k, with the short run MC_k curve intersecting at its lowest point. (It also cannot be U-shaped — because the short run production function has decreasing returns to scale without fixed costs when the long run production function has decreasing returns to scale.) Since the difference between AE_k and AC_k is the fixed expense on capital, the two curves have to converge as x gets large. The long run MC curve is unaffected because capital is assumed to be variable in the long run.

(d) *On a separate graph, repeat (b) and (c) for $k' > k^*$.*

Answer: This is illustrated in panel (c) of Graph 13.7 where the reasoning is analogous to what we just did in the previous two parts.

(e) *Illustrate the short run MC curves you drew in parts (c) and (d) in the graph you first drew in part (a). How is this graph similar to Graph 13.7 in the text?*

Answer: These are illustrated in panel (a) of Graph 13.7 as the two dashed MC curves, one on each side of the original (solid) short run marginal cost curve when capital was fixed at k^*. Note that there are many short run marginal cost curves — one for each fixed level of capital — with those corresponding to lower levels of capital lying to the left of those corresponding to higher levels of capital. Each short run marginal cost curve is steeper than the (bold) long run MC curve and therefore intersects the long run MC curve at one point. This is precisely what we concluded in Graph 13.7 of the text when we concluded that short run supply curves (which are portions of short run MC curves) are steeper than long run supply curves.

(f) True or False*: The MC_k curve crosses the AC^{LR} curve at the lowest point of the AE_k curve* only if $k = k^*$.

Answer: This is true for the case we are analyzing here. As is apparent from the graphs, the lowest point of the AE curves coincides with the lowest point of the long run AC curve only when $k = k^*$.

(g) *How would your answer to (f) change if the sentence had started with the words "If the production technology has constant returns to scale and there are no fixed costs, ...".*

Answer: The statement would then be false. If the technology has constant returns to scale and there are no long run fixed costs, the long run AC curve is flat. This implies that the AE curves corresponding to the different levels of capital will all be tangent to the horizontal long run AC curve at the lowest point of each AE curve — as illustrated in panel (c) of Graph 13.2 in the textbook. The short run marginal cost curves, which cross the short run AE curve at their lowest points, will then cross the long run AC curve as described in the statement.

(h) True or False*: Unless the production technology has constant returns to scale and no long run fixed costs, the short run AE curves are tangent at the lowest point of the long run AC curve only if $k = k^*$.*

Answer: This is true. It should be apparent from the panels in Graph 13.7 that the tangency of AE curves with the long run AC curve generally do not occur at the lowest point of the AE curves — unless $k = k^*$ as illustrated in panel (a). The only way for the tangency to occur at the lowest point of *every* AE curve is for the long run AC curve to be horizontal — and this only happens if the technology has constant returns to scale and no fixed costs.

B: *Suppose that a firm's production function is $x = f(\ell, k) = A\ell^\alpha k^\beta$ with $\alpha, \beta > 0$ and $\alpha + \beta < 1$. Suppose further that the firm incurs a recurring (long run) fixed cost FC.*

(a) *In equation (13.45) from exercise 13.5, we already provided the long run cost function for such a firm in the absence of fixed costs. What are this firm's long run marginal and average cost functions?*

Answer: The firm's long run marginal cost (which is unaffected by FC) is just the derivative of $C(w,r,x)$ with respect to x — which is

$$MC(w,r,x) = \left(\frac{w^\alpha r^\beta x^{(1-\alpha-\beta)}}{A\alpha^\alpha \beta^\beta}\right)^{1/(\alpha+\beta)}. \tag{13.49}$$

The firm's long run average cost is then $C(w,r,x)$ divided by output plus FC divided by output; i.e.

$$AC(w,r,p) = (\alpha+\beta)\left(\frac{w^\alpha r^\beta x^{(1-\alpha-\beta)}}{A\alpha^\alpha \beta^\beta}\right)^{1/(\alpha+\beta)} + \frac{FC}{x}. \tag{13.50}$$

(b) *Derive the output level x^* at which the lowest point of the long run average cost curve occurs.*

Answer: The lowest point of the U-shaped average cost curve occurs where the derivative of $AC(w,r,p)$ is zero. Thus, we first take this derivative to get

$$\frac{\partial AC(w,r,x)}{\partial x} = \left[(1-\alpha-\beta)\left(\frac{w^\alpha r^\beta}{A\alpha^\alpha\beta^\beta}\right)^{1/(\alpha+\beta)} x^{(1-2(\alpha+\beta))/(\alpha+\beta)}\right] - \frac{FC}{x^2}. \tag{13.51}$$

Next, we set this to zero and solve for x. After some algebra, this gives us

$$x^* = \left(\frac{A\alpha^\alpha\beta^\beta}{w^\alpha r^\beta}\right)\left(\frac{FC}{(1-\alpha-\beta)}\right)^{(\alpha+\beta)} \tag{13.52}$$

which is the output level at which the average cost curve reaches its lowest point.

(c) *From here on, suppose that $\alpha = 0.2$, $\beta = 0.6$, $A = 30$, $w = 20$, $r = 10$ and $FC = 1,000$. Given these values, what is x^*? How much capital k^* does the firm hire to produce x^*? (Note: The conditional input demand functions for a Cobb-Douglas production process are given in equation (13.47) of exercise 13.7.)*

Answer: Plugging these values into equation (13.52), we get $x^* = 2009.96275 \approx 2,010$. Plugging the values as well as $x^* = 2,010$ into the conditional capital demand function, we get $k^* = 300$.

(d) *What is the long run marginal cost of production at x^*? What about the long run average cost? Interpret your answer.*

Answer: Plugging the values given in part B(c) of the exercise into equations (13.49) and (13.50), we get

$$MC(x) = 0.37152239\left(x^{0.25}\right) \quad \text{and} \quad AC(x) = 0.29721791\left(x^{0.25}\right) + \frac{1000}{x}. \tag{13.53}$$

When we then plug $x^* \approx 2,010$ into these equations, we get $MC(x^*) = 2.4876 = AC(x^*)$. This simply verifies that the long run marginal and average cost curves cross at the lowest point of the long run AC curve.

(e) *For a fixed level of capital k, what are the short run MC, AC, and AE functions?*

Answer: The short run production function is

$$x = f_k(\ell) = \left[Ak^\beta\right]\ell^\alpha = \left[30k^{0.6}\right]\ell^{0.2}. \tag{13.54}$$

Solving this for ℓ, we get the short run conditional labor demand function

$$\ell_k(x) = \left(\frac{x}{\left[30k^{0.6}\right]}\right)^{1/0.2} = \frac{x^5}{24,300,000k^3}. \tag{13.55}$$

This implies a short run cost function (given $w = 20$) of

$$C_k(x) = 20\left(\frac{x^5}{24,300,000k^3}\right) = \frac{x^5}{1,215,000k^3} \tag{13.56}$$

which further implies short run MC and AC functions

$$MC_k(x) = \frac{x^4}{243,000k^3} \quad \text{and} \quad AC_k(x) = \frac{x^4}{1,215,000k^3}. \tag{13.57}$$

Finally, the AE curve includes both the expense on the fixed level of capital (which is $10k$) and the long run fixed cost (which is FC=1,000) — which implies

$$AE_k(x) = \frac{x^4}{1,215,000k^3} + \frac{10k+1000}{x}. \tag{13.58}$$

(f) *What is the short run AE, AC and MC for $x = x^*$ when capital is fixed at k^*? How do these compare to long run AC and MC of producing x^*?*

Answer: Plugging $k^* = 300$ and $x^* = 2,010$ (which we derived in part (c)) into equation (13.58) and the MC function in (13.57), we get $AE_{k^*}(x^*) = 2.4876 = MC_{k^*}(x^*)$. These are the same as the long run AC and MC — just as we would predict from our graphical analysis. The short run AC, on the other hand, is $AC_{k^*}(x^*) = 0.4975$ — below that AE and MC — again as we predict in our graphical analysis.

(g) *Now suppose capital is fixed in the short run at k = 200. How does your answer to (f) change? What if capital were instead fixed at k = 400? Interpret your answer.*

Answer: When $k = 200$, we get

$$AE_{k=200}(x^*) \approx 3.17\ ;\ \ MC_{k=200}(x^*) \approx 8.40 \ \text{ and } \ AC_{k=200}(x^*) \approx 1.68. \tag{13.59}$$

When $k = 400$, on the other hand, we get

$$AE_{k=400}(x^*) \approx 2.70\ ;\ \ MC_{k=400}(x^*) \approx 1.05 \ \text{ and } \ AC_{k=400}(x^*) \approx 0.21. \tag{13.60}$$

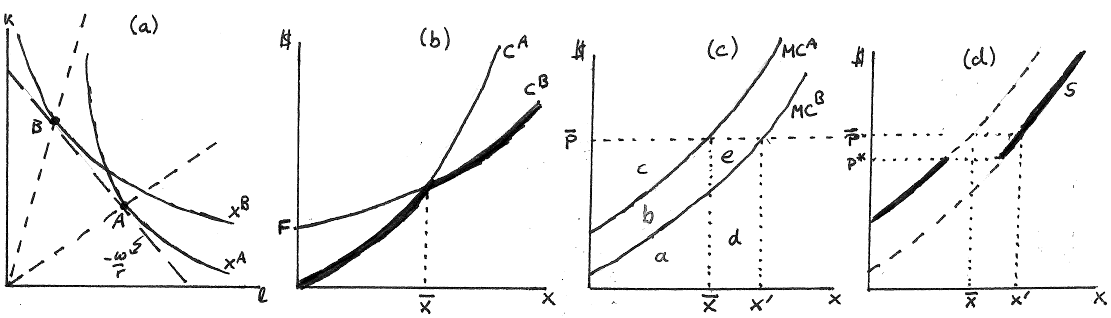
In Graph 13.7, we concluded in panel (b) that, when $k < k^*$, $MC_k(x^*) > AE_k(x^*) > AC_k(x^*)$ — which is consistent with the results in equation (13.59). In panel (c) we concluded that, when $k > k^*$, $AE_k(x^*) > AC_k(x^*) > MC_k(x^*)$ — which is also consistent with equation (13.60).

13.7 Business Application: *Switching Technologies: Suppose that a firm has two different homothetic, decreasing returns to scale technologies it could use, but one of these is patented and requires recurring license payments F to the owner of the patent. In this exercise, assume that all inputs — including the choice of which technology is used — are viewed from a long run perspective.*

A: *Suppose further that both technologies take capital k and labor ℓ as inputs but that the patented technology is more capital intensive.*

(a) *Draw two isoquants, one from the technology representing the less capital intensive and one representing the more capital intensive technology. Then illustrate the slice of each map that a firm will choose to operate on assuming the wage w and rental rate r are the same in each case.*

Answer: This is illustrated in panel (a) of Graph 13.8 where x^A labels an isoquant from the non-patented (labor intensive) technology and x^B labels an isoquant from the patented (capital intensive) technology. The isocost with slope $-w/r$ is tangent to the labor intensive technology at A and to the capital intensive technology at B. Since the production technologies are homothetic, the ray passing from the origin through A represents the slice of the labor intensive production technology along which cost minimizing input bundles lie, and the ray passing from the origin through B represents the slice of the capital intensive technology along which cost minimizing bundles lie. Thus, the ratio of capital to labor used in production is greater in the patented (capital intensive) technology.



Graph 13.8: Switching Technologies

(b) *Suppose that the patented technology is sufficiently advanced such that, for any set of input prices, there always exists an output level $\overline{x}$ at which it is (long run) cost effective to switch to this technology. On a graph with output x on the horizontal and dollars on the vertical, illustrate two cost curves corresponding to the two technologies and then locate $\overline{x}$. Then illustrate the cost curve that takes into account that a firm will switch to the patented technology at $\overline{x}$.*

Answer: This is illustrated in panel (b) of Graph 13.8. The non-patented technology gives rise to the cost curve labeled C^A and the patented technology gives rise to the cost curve C^B. Since there is an $\overline{x}$ such that it is cost effective to switch to the patented technology at $\overline{x}$, the two cost curves must intersect. The C^B curve must furthermore have positive intercept of F because use of the patented technology requires a fixed payment of F before production begins. The bold curve that connects C^A below $\overline{x}$ to C^B above $\overline{x}$ then represents the real cost curve for a firm in this industry — because the firm will, for any given output level, use the technology that minimizes costs.

(c) *What happens to $\overline{x}$ if the license cost F for using the patented technology increases? Is it possible to tell what happens if the capital rental rate r increases?*

Answer: If F increases, then C^B shifts up while C^A remains unchanged. Thus, it must be that the two cost curves intersect at a higher level of output — i.e. an increase in F causes an increase in the production level $\overline{x}$ at which a firm would switch from the non-patented to the patented technology. If the capital rental rate r increases, however, we cannot tell what will happen to $\overline{x}$. This is because now both cost curves are affected, with the upward shift in C^A by itself causing $\overline{x}$ to decrease while the upward shift of C^B by itself would cause $\overline{x}$ to increase. Which of these effects dominates when both curves shift will depend on the precise nature of the underlying technology as well as the ratio of w to r.

(d) *At $\overline{x}$, which technology must have a higher marginal cost of production? On a separate graph, illustrate the marginal cost curves for the two technologies.*

Answer: In panel (b) of Graph 13.8, it is clear that the slope of C^A is steeper at $\overline{x}$ than the slope of C^B at that output level. Thus, the marginal cost of production at $\overline{x}$ is higher under the non-patented technology than under the patented technology. This is likely to be true for all output levels — leading to marginal cost curves such as those depicted in panel (c) of Graph 13.8 where MC^A indicates the marginal cost curve under the non-patented technology and MC^B indicates the marginal cost curve under the patented technology. (It is in principle possible that these marginal cost curves cross in some places — but that would require unusually shaped cost curves in panel (b) where C^B must have an intercept of F, C^A has no such intercept and the two curves cross at $\overline{x}$.)

(e) *At $\overline{x}$, the firm is cost-indifferent between using the two technologies. Recognizing that the marginal cost curves capture all costs that are not fixed — and that total costs excluding fixed costs can be represented as areas under marginal cost curves, can you identify an area in your graph that represents the recurring fixed license fee F?*

Answer: The total cost of producing $\overline{x}$ under the non-patented technology is simply the area under the MC^A curve in panel (c) of Graph 13.8 — i.e. area $a+b$. (This is because the marginal cost of each unit of output is the additional cost incurred — and when we sum all these "additional costs" we get the total cost if there is no fixed cost of production). Similarly, the total cost minus the fixed cost F under the patented technology is the area under MC^B — i.e. area a. These areas differ by b — i.e. not counting the fixed cost F under the patented technology, the cost of producing $\overline{x}$ is smaller under the patented technology by area b. At $\overline{x}$, however, the total cost (including fixed costs) is equal for the two technologies (as seen in panel (b) of the graph); i.e. $a+b=a+F$. Thus, $F=b$.

(f) *Suppose output price p is such that it is profit maximizing* under the non-patented technology *to produce $\overline{x}$. Denote this as $\overline{p}$. Can you use marginal cost curves to illustrate whether you would produce more or less if you switched to the patented technology?*

Answer: In order for it to be profit maximizing to produce $\overline{x}$ under the non-patented technology, $\overline{p}$ must fall as depicted in panel (c) of Graph 13.8 — i.e. $\overline{p}$ must intersect MC^A at $\overline{x}$. But $\overline{p}$ intersects MC^B at x' — which implies that, were the firm to switch to the patented technology, it would produce more.

(g) *Would profit be higher if you used the patented or non-patented technology when output price is $\overline{p}$. (*Hint*: Identify the total revenues if the firm produces at $\overline{p}$ under each of the technologies. Then identify the total cost of using the non-patented technology as an area under the appropriate marginal cost curve and compare it to the total costs of using the patented technology as an area under the other marginal cost curve and add to it the fixed fee F.)*

Answer: When selling $\overline{x}$ at $\overline{p}$, total revenue is equal to $\overline{p}$ times $\overline{x}$ — which is equal to the area $a+b+c$ in panel (c) of Graph 13.8. If the firm uses the non-patented technology to produce $\overline{x}$, its total costs are equal to the area under MC^A — which is equal to area $a+b$. Thus, the profit for a firm using the non-patented technology is $(a+b+c)-(a+b)=c$. If the firm uses the patented technology at price $\overline{p}$, it produces x' and thus earns revenues of $\overline{p}$ times x' — which is equal to area $a+b+c+d+e$. Its costs (not including the fixed F) are equal to the area under MC^B — which is $a+d$. We also concluded above that the fixed F is equal to area b. Thus, total costs (including F) are $a+b+d$. Subtracting this from total revenue, we get profit of $a+b+c+d+e-(a+b+d)=c+e$. When price is $\overline{p}$, the firm would therefore earn profit of c by producing $\overline{x}$ under the non-patented technology and profit $c+e$ producing x' using the patented technology. Profit is therefore higher if the firm uses the patented technology when price is $\overline{p}$.

(h) True or False*: Although the total cost of production is the same under both technologies at output level $\overline{x}$, a profit maximizing firm will choose the patented technology if price is such that $\overline{x}$ is profit maximizing under the non-patented technology.*

Answer: This is true. We had identified $\overline{x}$ as the output level at which the two cost curves cross in panel (b) of Graph 13.8 — and thus total costs are the same under both technologies when firms produce $\overline{x}$. But we also just concluded that, at price $\overline{p}$ at which it is profit maximizing under the non-patented technology to produce $\overline{x}$, the firm can earn more profit by using the patented technology and producing x' (in panel (c) of the graph.)

(i) *Illustrate the firm's supply curve. (*Hint*: The supply curve is not continuous, and the discontinuity occurs at a price below* $\overline{p}$*.)*

Answer: This is illustrated in panel (d) of Graph 13.8. Up to some price level below $\overline{p}$, the profit maximizing firm will choose the non-patented technology. Over that range of prices, MC^A therefore forms the supply curve. But at some price — indicated as p^* in the graph — the firm will earn the same profit under both technologies but will produce more under the patented technology. We know that $p^* < \overline{p}$ because of our conclusion above that profit using the patented technology is higher at $\overline{p}$ than profit under the non-patented technology. At prices higher than p^*, the firm will then have switched to the patented technology — causing the supply curve from then on to lie on MC^B.

B: *Suppose that the two technologies available to you can be represented by the production functions* $f(\ell,k) = 19.125\ell^{0.4}k^{0.4}$ *and* $g(\ell,k) = 30\ell^{0.2}k^{0.6}$*, but technology* g *carries with it a recurring fee of* F.

(a) *In exercise 13.2 you derived the general form for the 2-input Cobb-Douglas conditional input demands and cost function.*[1] *Use this to determine the ratio of capital to labor (as a function of* w *and* r*) used under these two technologies. Which technology is more capital intensive?*

Answer: Plugging $\alpha = \beta = 0.4$ and $A = 19.125$ into the previously derived formula for input demands, we get that the f technology gives rise to conditional input demands

$$\ell_f(w,r,x) = 0.025\left(\frac{r}{w}\right)^{1/2} x^{5/4} \text{ and } k_f(w,r,x) = 0.025\left(\frac{w}{r}\right)^{1/2} x^{5/4}, \tag{13.62}$$

and plugging in $\alpha = 0.2$, $\beta = 0.6$ and $A = 30$, we get that the g technology gives rise to conditional input demands

$$\ell_g(w,r,x) = 0.00625\left(\frac{r}{w}\right)^{3/4} x^{5/4} \text{ and } k_g(w,r,x) = 0.01875\left(\frac{w}{r}\right)^{1/4} x^{5/4}. \tag{13.63}$$

The ratio of capital to labor under the technologies f an g are then (respectively)

$$\frac{k_f(w,r,x)}{\ell_f(w,r,x)} = \frac{w}{r} \text{ and } \frac{k_g(w,r,x)}{\ell_g(w,r,x)} = 3\frac{w}{r}; \tag{13.64}$$

i.e. the capital to labor ratio under technology g is three times as high as under f. (These ratios correspond to the slopes of the rays in panel (a) of Graph 13.8.) The g technology is therefore more capital intensive.

(b) *Determine the cost functions for the two technologies (and be sure to include* F *where appropriate).*

Answer: For the f technology, we plug in $\alpha = \beta = 0.4$ and $A = 19.125$ into the previously derived cost function for Cobb-Douglas production; and for the g technology we plug in $\alpha = 0.2$, $\beta = 0.6$ and $A = 30$ — and then we add the fixed technology fee F which has to be paid if g is used. This gives us

$$C_f(w,r,x) = 0.05w^{1/2}r^{1/2}x^{5/4} \text{ and } C_g(w,r,x) = 0.025w^{1/4}r^{3/4}x^{5/4} + F. \tag{13.65}$$

[1] For the Cobb-Douglas production function $x = f(\ell,k) = A\ell^{\alpha}k^{\beta}$, you should have derived the conditional input demands

$$\ell(w,r,x) = \left(\frac{\alpha r}{\beta w}\right)^{\beta/(\alpha+\beta)}\left(\frac{x}{A}\right)^{1/(\alpha+\beta)} \text{ and } k(w,r,x) = \left(\frac{\beta w}{\alpha r}\right)^{\alpha/(\alpha+\beta)}\left(\frac{x}{A}\right)^{1/(\alpha+\beta)}. \tag{13.61}$$

The cost function was previously provided in equation (13.45).

(c) *Determine the output level $\overline{x}$ (as a function of w, r and F) at which it becomes cost effective to switch from the technology f to the technology g. If F increases, is it possible to tell whether $\overline{x}$ increases or decreases? What if r increases?*

Answer: To calculate $\overline{x}$, we need to find where the cost curves intersect (as in panel (b) of Graph 13.8). We therefore set the two cost functions in equation (13.65)equal to each other and solve for x to get

$$\overline{x} = \left(\frac{40F}{2w^{1/2}r^{1/2} - w^{1/4}r^{3/4}}\right)^{4/5}. \tag{13.66}$$

If F increases, $\overline{x}$ unambiguously increases as well — which makes sense since the fixed cost of using g has increased, it will not be cost effective to switch until a higher level of output. But if r increases, we cannot tell whether $\overline{x}$ will increase or decrease. (We gave some intuition for this in the answer to (c) of part A of this exercise.)

(d) *Suppose $w = 20$ and $r = 10$. Determine the price $\overline{p}$ (as a function of F) at which a firm using technology f would produce $\overline{x}$.*

Answer: From the first cost function in equation (13.65), we can derive the marginal cost under technology f as $MC_f(w,r,x) = 0.0625w^{0.5}r^{0.5}x^{0.25}$. Plugging in $w = 20$ and $r = 10$, we then get $MC_f(20,10,x) = 0.8838835x^{0.25}$. Plugging these same input prices into equation (13.66), we get $\overline{x} = 2.0414474F^{0.8}$. Thus, the marginal cost of producing $\overline{x}$ under technology f is

$$MC_f(20,10,\overline{x}) = 0.8838835\left(2.0414474F^{0.8}\right)^{0.25} = 1.0565245F^{1/5} = \overline{p}, \tag{13.67}$$

where the last equality simply emerges from the fact that $p = MC$ for profit maximizing firms.

(e) *How much would the firm produce with technology g if it faces $\overline{p}$? Can you tell whether, regardless of the size of F, this is larger or smaller than $\overline{x}$ (which is the profit maximizing quantity when the firm used technology f and faces $\overline{p}$)?*

Answer: From the second cost function in equation (13.65) we can derive the marginal cost function under technology g as $MC_g(w,r,x) = 0.03125w^{0.25}r^{0.75}x^{0.25}$ which becomes $MC_g(20,10,x) = 0.3716272x^{0.25}$ when $w = 20$ and $r = 10$. Setting this equal to $\overline{p} = 1.0565245F^{1/5}$ from equation (13.67) and solving for x, we get $x \approx 65.33F^{0.8}$. We can then conclude that

$$x \approx 65.33F^{0.8} > 2.0414474F^{0.8} = \overline{x}; \tag{13.68}$$

i.e. regardless of what value F takes (as long as $F > 0$), the profit maximizing production level will be higher using technology g than when using technology f when the price is such that $\overline{x}$ is profit maximizing under technology f. We illustrated this intuitively in panel (c) of Graph 13.8.

(f) *The (long run) profit function for a Cobb-Douglas production function $f(\ell,k) = A\ell^{\alpha}k^{\beta}$ is*

$$\pi(w,r,p) = (1-\alpha-\beta)\left(\frac{Ap\alpha^{\alpha}\beta^{\beta}}{w^{\alpha}r^{\beta}}\right)^{1/(1-\alpha-\beta)}. \tag{13.69}$$

Can you use this to determine (as a function of p, w and r) the highest level of F at which a profit maximizing *firm will switch from f to g? Call this $\overline{F}(w,r,p)$.*

Answer: Plugging $\alpha = \beta = 0.4$ and $A = 19.125$ into the profit function, we get the profit function for the technology f; and plugging in $\alpha = 0.2$, $\beta = 0.6$ and $A = 30$, we get the profit function for technology g. This gives us

$$\pi_f(w,r,p) = 13,100\left(\frac{p^5}{w^2r^2}\right) \text{ and } \pi_g(w,r,p) = 209,952\left(\frac{p^5}{wr^3}\right). \tag{13.70}$$

For the production function g, however, we also need to take into account the fixed cost F — thus subtract F from $\pi_g(w,r,p)$. The fixed cost $\overline{F}$ at which the firm will switch to the technology g is the fixed cost at which profit is equal for both technologies. Thus, we need to solve

$$13,100\left(\frac{p^5}{w^2r^2}\right) = 209,952\left(\frac{p^5}{wr^3}\right) - F \tag{13.71}$$

for F. This gives us

$$\overline{F} = 209,952\left(\frac{p^5}{wr^3}\right) - 13,100\left(\frac{p^5}{w^2r^2}\right). \tag{13.72}$$

(g) *From your answer to (f), determine (as a function of w, r and F) the price p^* at which a profit maximizing firm will switch from technology f to technology g.*

Answer: We simply need to solve equation (13.71) for p which gives us

$$p^* = \left(\frac{Fw^3r^3}{209,952w^2 - 13,100wr}\right)^{1/5}. \tag{13.73}$$

(h) *Suppose again that $w = 20$, $r = 10$. What is p^* (as a function of F)? Compare this to $\overline{p}$ you calculated in part (d) and interpret your answer in light of what you did in A(i).*

Answer: Plugging $w = 20$ and $r = 10$ into equation (13.73), we get $p^* = 0.6288325F^{0.2}$. Comparing this to our answer in part (d), we conclude that

$$p^* = 0.6288325F^{1/5} < 1.0562545F^{1/5} = \overline{p}. \tag{13.74}$$

Thus, the price p^* at which a profit maximizing firm switches from technology f to technology g lies below the price $\overline{p}$ at which a firm using production technology f would produce $\overline{x}$ at which the cost of production is equal for the two technologies. This implies that the supply curve switches from the MC_f curve to MC_g at p^* and below $\overline{p}$ — which we illustrated intuitively in panel (c) of Graph 13.8.

(i) *Suppose (in addition to the values for parameters specified so far) that $F = 1000$. What is $\overline{p}$ and p^*? At the price at which the profit maximizing firm is indifferent between using technology f and technology g, how much does it produce when it uses f and how much does it produce when it uses g?*[2]

Answer: Plugging $F = 1000$ into the equations for $\overline{p}$ and p^*, we get $\overline{p} = 1.0562545(1000)^{1/5} \approx$ \$4.21 and $p^* = 0.6288325(1000)^{1/5} = 2.5034 \approx$ \$2.50. Plugging $\alpha = \beta = 0.4$, $A = 19.125$, $w = 20$, $r = 10$ and $p = 2.50$ into the supply function $x(w,r,p)$, we get $x_f^* \approx 64.32$; and, plugging $\alpha = 0.2$. $\beta = 0.6$, $A = 19.125$, $w = 20$, $r = 10$ and $p = 2.50$ in the supply function, we get $x_g^* \approx 2,062$.

(j) *Continuing with the values we have been using (including $F = 1000$), can you use your answer to (a) to determine how much labor and capital the firm hires at p^* under the two technologies? How else could you have calculated this?*

Answer: Plugging $r = 10$, $w = 20$ and $x_f^* = 64.32$ into the conditional labor and capital demands from equation (13.62), we get $\ell_f^* = 3.22$ and $k_f^* = 6.44$ — the cost minimizing labor and capital inputs if the firm uses the f technology to produce $x_f^* = 64.32$ (which in turn is the profit maximizing output level at $p^* = 2.50$.) Similarly, if we plug $r = 10$, $w = 20$ and $x_g^* = 2,062$ into the conditional labor and capital demands from equation (13.63), we get $\ell_g^* = 51.6$ and $k_g^* = 310$ — the cost minimizing labor and capital inputs if the firm uses the

[2] Recall from your previous work in exercise 13.1 that the supply function for a Cobb-Douglas production process $f(\ell,k) = A\ell^\alpha k^\beta$ is

$$x(w,r,p) = \left(\frac{Ap^{(\alpha+\beta)}\alpha^\alpha\beta^\beta}{w^\alpha r^\beta}\right)^{1/(1-\alpha-\beta)}. \tag{13.75}$$

g technology to produce $x_g^* = 2,062$ (which in turn is the profit maximizing output level at $p^* = 2.50$.) You could also of course have derived the unconditional labor demand and capital demand functions — either by doing the profit maximization problems or using Hotelling's lemma.

(k) *Use what you have calculated in (i) and (j) to verify that profit is indeed the same for a firm whether it uses the f or the g technology when price is p^* (when the rest of the parameters of the problem are as we have specified them in (i) and (j).) (Note: If you rounded some of your previous numbers, you will not get exactly the same profit in both cases — but if the difference is small, it is almost certainly just a rounding error.)*

Answer: Profit is simply revenue minus costs. We can then calculate the profit under each technology as

$$\pi_f = p^* x_f^* - w\ell_f^* - rk_f^* = 2.5034(64.32) - 20(3.22) - 10(6.44) \approx \$32 \tag{13.76}$$

and

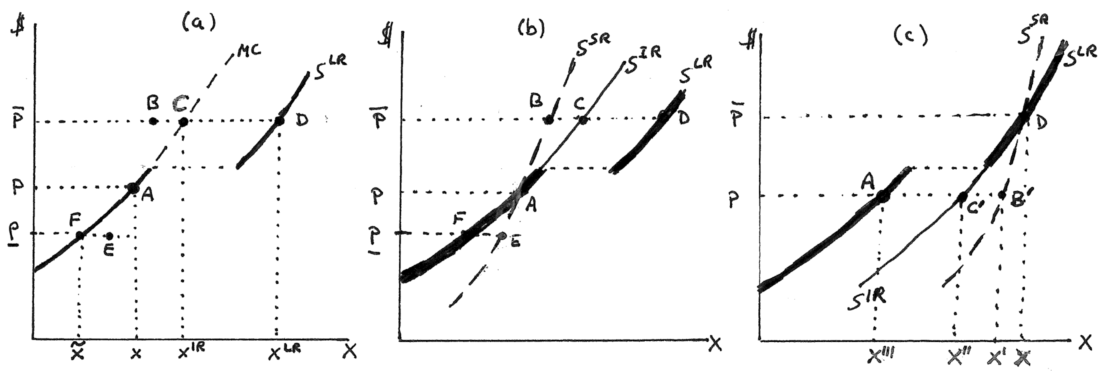
$$\pi_g = p^* x_g^* - w\ell_g^* - rk_g^* = 2.5034(2062) - 20(51.6) - 10(310) - 1000 \approx \$32. \tag{13.77}$$

13.8 Business Application: *Switching Technologies: Short Run versus Long Run: In exercise 13.7, we viewed all inputs (including the technology that is chosen) as variable — which is to say we viewed these inputs from a long run perspective.*

A: *Now consider the same set-up as in exercise 13.7 but assume throughout that labor is instantaneously variable, that capital is fixed in the short run and variable in the* intermediate *run, and that the choice of technology is fixed in the short and intermediate run but variable in the long run.*

(a) *Suppose you are currently long-run profit maximizing. Graph the (long run) supply curve you derived in part A(i) of exercise 13.7 and indicate a price p and quantity x combination that is consistent with using the non-patented technology.*

Answer: This is illustrated as point A on the long run supply curve in panel (a) of Graph 13.9. The long run supply curve, as explained in detail for the previous exercise, is composed of portions of the two long run marginal cost curves associated with the non-patented and patented technology. The lower portion of the supply curve is derived from the non-patented technology. Thus, in order for the firm to be long-run profit maximizing by using the non-patented technology, p must be sufficiently low to intersect the lower portion of the 2-part long run supply curve.



Graph 13.9: Changing Inputs and Switching Technology over Time

(b) *Next suppose that output price increases to $\overline{p}$ and that this increase is sufficient for you to wish that you in fact had rented the patented technology instead. Illustrate how your output level will adjust in the intermediate run to x^{IR}.*

Answer: The lower portion of the long run supply curve is extended by a dashed line in panel (a) of the graph — and this represents the marginal cost curve associated with the non-patented technology when both labor and capital can be varied. If there were no patented technology, the dashed line would in face be the second part of the long run supply curve. Since capital and labor are both variable along this marginal cost curve, it represents the firm's response in the intermediate run before it can change technology. Thus, C represents the intermediate run supply point for the firm when price rises to $\overline{p}$, with x^{IR} indicating the intermediate run output level.

(c) *In the short run, your firm cannot change its level of capital. Where would your short run optimal output level x^{SR} (at the new $\overline{p}$) lie relative to x and x^{IR}? How is your answer impacted by the relative substitutability of capital and labor in the non-patented technology?*

Answer: In the short run, the firm is not only confined to the non-patented technology but also cannot change capital as is assumed in the MC curve in panel (a) of Graph 13.9. The short run output response to the increase in output price to $\overline{p}$ will therefore be less than the intermediate run response. Thus, x^{SR} will lie somewhere between x and x^{IR} (which will cause the firm to locate at a point like B in the graph). The output x^{SR} will lie closer to x

if capital and labor are relatively complementary and closer to x^{IR} if capital and labor are relatively substitutable in the non-patented technology.

(d) *In the long run, where will your optimal output level x^{LR} lie?*

Answer: In the long run, the firm can not only change its capital but also its technology. Thus, the firm will adopt the patented technology and it's output response will fall on the long run supply curve at $\overline{p}$ as indicated by point D. This results in output level x^{LR} in panel (a) of the graph.

(e) *Suppose price had fallen to $\underline{p}$ instead of rising to $\overline{p}$. Indicate where your short, intermediate and long run output levels would lie.*

Answer: In the intermediate run (when the firm can adjust capital but not technology), the firm will move to point F in panel (a) of the graph — along its non-patent technology MC curve that lies on its long run supply curve. Its output therefore falls to $\widetilde{x}$ in the intermediate run. In the short run, it will not adjust by as much as it cannot yet adjust capital — ending up at a point like E with output level somewhere between x and $\widetilde{x}$. Finally, in the long run the firm has the additional option of changing technologies, but with a drop in price to $\underline{p}$, it does not wish to exercise this option. Thus, there is no change in output from the intermediate to long run, with long run output also being $\widetilde{x}$.

(f) *On a new graph, illustrate the short, intermediate and long run supply curves for your firm given you started at the original price p and the original optimal output level x.*

Answer: This is illustrated in panel (b) of Graph 13.9. The bold line segments together represent the long run supply curve previously derived in part (i) of exercise 13.7. The intermediate run supply curve S^{IR} connects the initial point A to the intermediate run point C — with the bold lower portion of the long run supply curve part of S^{IR}. Finally, the short run supply curve S^{SR} is steeper than S^{IR} and connects A to B.

(g) *What would your last graph look like if you had originally started at price $\overline{p}$ and had originally produced at the long run optimal output level x^{LR}?*

Answer: This is illustrated in panel (c) of Graph 13.9. The long run supply curve S^{LR} is the same as in panels (a) and (b). The intermediate run supply curve S^{IR} is the MC curve from the patented technology when capital and labor are allowed to adjust. As such, it is composed of the bold upper portion of the long run supply curve and its extension through C' — the point that represents the intermediate adjustment when price falls from $\overline{p}$ to p. In the short run, the firm will adjust less since it cannot change capital — and will end up at a point like B'. This forms the basis for the steeper short run supply curve S^{SR} represented by the dashed curve in panel (c). When price drops from $\overline{p}$ to p, the firm begins at D, adjusts to B' in the short run, to C' in the intermediate run and to A in the long run when it switches to the non-patented technology.

B: *Suppose, as in exercise 13.7, that the two technologies available to you can be represented by the production functions $f(\ell,k) = 19.125\ell^{0.4}k^{0.4}$ and $g(\ell,k) = 30\ell^{0.2}k^{0.6}$, but technology g carries with it a recurring fee of F. Suppose further that $w = 20$, $r = 10$ and $F = 1{,}000$.*

(a) *If $p = 2.25$ and the firm is currently long run optimizing, how much does it produce? (You can use what you learned from exercise 13.7 and employ equation (13.49).)*

Answer: We calculated in exercise 13.7 that the price at which a firm will switch from the non-patented technology f technology to the patented g technology (when $w = 20$, $r = 10$ and $F = 1{,}000$) is $p^* = 2.50$. At $p = 2.25$, the long run profit maximizing firm is therefore using the f technology. Plugging $A = 19.125$, $\alpha = \beta = 0.4$, $w = 20$ and $r = 10$ into the Cobb-Douglas supply function (equation (13.75)), we get the "long run" supply function (which allows capital and labor to vary) for technology f:

$$x_f(p) = 1.6375p^4. \tag{13.78}$$

Plugging in $p = 2.25$, we conclude that the firm is currently producing $x_f(2.25) \approx 42$ units of output.

(b) *Now suppose the output price increases from 2.25 to 2.75. How much will the firm adjust output in the short run (where neither capital nor technology can be changed)?*[3]

Answer: To answer this, we cannot use equation (13.78) because that supply function assumes capital is variable (which it is not in the short run). Instead, we need to use the short run supply function associated with technology f (because the firm cannot switch to g in the short run). But in order for us to derive the relevant short run production function, we first have to derive the amount of capital the firm is initially using when $p = 2.25$. For this, we simply plug the parameters of the f production function, the input prices and the initial output price ($p = 2.25$) into equation (13.79) to get $k \approx 3.78$. The short run supply function relevant for the firm using technology f with fixed capital $k = 3.78$ is then $f_{k=3.78}(\ell) = \left[19.125(3.78)^{0.4}\right]\ell^{0.4} \approx 32.55\ell^{0.4}$. When price increases from 2.25 to 2.75, the firm will adjust labor to insure that the new marginal revenue product of labor equals the wage $w = 20$ which is

$$MRP_\ell = 2.75\frac{\partial f_{k=3.78}(\ell)}{\partial \ell} = \frac{35.805}{\ell^{0.6}}. \tag{13.80}$$

Setting this equal to $w = 20$ and solving for ℓ, we get that labor in the short run will adjust to $\ell = 2.64$ (which, as we will show in part (e), is an increase from an initial $\ell = 1.89$.) Plugging $\ell = 2.64$ into the short run production function $f_{k=3.78}(\ell) = 32.55\ell^{0.4}$, we then get that output adjusts in the short run to $x^{SR} \approx 50$ (from the initial $x = 42$ calculated in part (a)).

(c) *How much will it increase output in the intermediate run (where capital can adjust but technology remains fixed)?*

Answer: In the intermediate run, the firm is still going to use technology f — which means we can now use the supply function derived in equation (13.78) as $x_f(p) = 1.6375p^4$. Plugging the new $p = 2.75$ into this equation, we get $x^{IR} = 93.65 \approx 94$.

(d) *How much will it adjust output in the long run?*

Answer: In the long run, the firm can adjust technology — and we showed in exercise 13.7 that a firm will, under the conditions of this problem, indeed want to switch to technology g if price rises above 2.50. At $p = 2.75$, the long run profit maximizing firm will therefore choose technology g. Plugging $A = 20$, $\alpha = 0.2$, $\beta = 0.6$, $w = 20$ and $r = 10$ into the Cobb-Douglas supply function (equation (13.75)), we get the "long run" supply function (which allows capital and labor to vary) for technology g:

$$x_g(p) = 52.488p^4. \tag{13.81}$$

Plugging in $p = 2.75$, we conclude that the firm is currently producing $x^{LR} \approx 3{,}002$ units of output. We thus went from our initial output of 42 units before the price increase to a short run increase to 50 units, an intermediate increase to 94 units and a long run jump to 3,002 units of output.

(e) *What happens to the quantity of labor and capital hired in the short, intermediate and long run?*

Answer: Using the Cobb-Douglas input demand equations from equation (13.79), we can derive the labor and capital demand functions for a firm that uses f and can adjust both capital and labor as

$$\ell_f(20,10,p) = 0.03275p^5 \ \text{ and } \ k_f(20,10,p) = 0.0655p^5. \tag{13.82}$$

Substituting in the initial price $p = 2.25$, this gives us initial labor and capital of $\ell \approx 1.89$ and $k \approx 3.78$. In the short run, capital cannot be adjusted — thus $k^{SR} = 3.78$. We already

[3] It may be helpful to know that, for Cobb-Douglas functions that take the form $f(\ell,k) = A\ell^\alpha k^\beta$, the input demand functions are

$$\ell(w,r,p) = \left(\frac{pA\alpha^{(1-\beta)}\beta^\beta}{w^{(1-\beta)}r^\beta}\right)^{1/(1-\alpha-\beta)} \quad \text{and} \quad k(w,r,p) = \left(\frac{pA\alpha^\alpha\beta^{(1-\alpha)}}{w^\alpha r^{(1-\alpha)}}\right)^{1/(1-\alpha-\beta)} \tag{13.79}$$

showed in part (b) that labor will adjust in the short run to $\ell^{SR} = 2.64$. In the intermediate run, both capital and labor can adjust but technology remains f — which means we can use the equations (13.82) and plug in the new price $p = 2.75$ to get $\ell^{IR} \approx 5.15$ and $k^{IR} \approx 10.3$. Finally, in the long run the firm switches to technology g. Using the Cobb-Douglas input demand equations from equation (13.79), we can derive the labor and capital demand functions for a firm that uses g and can adjust both capital and labor as

$$\ell_g(20,10,p) = 0.5249p^5 \text{ and } k_g(20,10,p) = 3.1493p^5. \tag{13.83}$$

Plugging the new price $p = 2.75$ into these, we get $\ell^{LR} \approx 82.55$ and $k^{LR} \approx 495.3$. In terms of labor, we therefore start with 1.89, move to 2.64 in the short run, 5.15 in the intermediate run and 82.55 in the long run. In terms of capital, we start with 3.78 which remains constant in the short run, and then move to 10.3 in the intermediate run and 495.3 in the long run.

(f) *Suppose that instead of increasing, the output price had fallen from 2.25 to 2.00. What would have happened to output in the short, intermediate and long run?*

Answer: As illustrated in part (b), the short run production function for this firm holds capital fixed at its initial quantity of $k = 3.78$ — giving us $f_{k=3.78}(\ell) = \left[19.125(3.78)^{0.4}\right]\ell^{0.4} \approx 32.55\ell^{0.4}$. When price falls, the firm can only adjust labor in the short run — and will adjust it until the marginal revenue product equals the wage. The marginal revenue product when $p = 2$ is

$$MRP_\ell = 2\frac{\partial f_{k=3.78}(\ell)}{\partial \ell} = \frac{26.04}{\ell^{0.6}}. \tag{13.84}$$

Setting this equal to $w = 20$ and solving for ℓ, we get $\ell^{SR} = 1.55$ (down from the original level of 1.89). When plugged into the short run production function, this implies a short run adjustment of output to $x^{SR} \approx 38.8$, down from the original 42. In the intermediate run, we can substitute the new price $p = 2$ into equation (13.78) to get $x^{IR} = 1.6375(2^4) = 26.2$. Nothing changes in the long run since the price is below 2.50 and thus the firm does not wish to switch technologies. Thus, output starts at 42 before the price decrease, then moves to 38.8 in the short run and to 26.2 in the intermediate and long run.

(g) *Suppose that the firm has fully adjusted to the higher output price of 2.75. Then price falls to 2.25. What happens to output in the short, intermediate and long run?*

Answer: We have already calculated that the long run profit maximizing production plan for the firm is $(\ell, k, x) = (82.55, 495.3, 3002)$ when $p = 2.75$ — and that the firm implements this production plan using technology g. When price falls to 2.25, the firm can initially only vary labor. We therefore need to use its short run production function given the fixed level of capital $k = 495.3$ which is $g_{k=495.3}(\ell) = \left[30(495.3)^{0.6}\right]\ell^{0.2} = 1241.77\ell^{0.2}$. The firm will then adjust its labor until the marginal revenue product of labor when $p = 2.25$ equals the wage. The marginal revenue product of labor is

$$MRP = 2.25\frac{\partial g_{k=495.3}(\ell)}{\partial \ell} = \frac{558.8}{\ell^{0.8}}. \tag{13.85}$$

Setting this equal to $w = 20$ and solving for ℓ, we get $\ell^{SR} = 64.24$. When plugged into the short run production function $g_{k=495.3}(\ell)$, we then get that the short run adjustment of output is to $x^{SR} \approx 2{,}855$ (down from the original 3,002). In the intermediate run, the firm can adjust capital but not technology — so we can use equation (13.78) that gives us the supply function given technology g: $x_g(p) = 52.488p^4$. Substituting in the new price $p = 2.25$, we then get $x^{IR} \approx 1{,}345$. Finally, we already calculated in part (a) what the firm will do in the long run when it can switch back to technology f — it will produce 42 units of output. Thus, output begins at 3,002, drops to 2,855 in the short run, to 1,345 in the intermediate run and to 42 in the long run.

$$x_g(p) = 52.488p^4. \tag{13.86}$$

13.9 Business and Policy Application: *Fixed amount of Land for Oil Drilling: Suppose that your oil company is part of a competitive industry and is using three rather than two inputs — labor ℓ, capital k and land L — to produce barrels of crude oil denoted by x. Suppose that the government, due to environmental concerns, has limited the amount of land available for oil drilling — and suppose that it has assigned each oil company $\overline{L}$ acres of such land. Assume throughout that oil sells at a market price p, labor at a market wage of w and capital at a rental rate r — and these prices do not change as government policy changes.*

A: *Assume throughout that the production technology is homothetic and has constant returns to scale.*

(a) *Suppose that, once assigned to an oil company, the company is not required to pay for using the land to drill for oil (but it cannot do anything else with it if it chooses not to drill.) How much land will your oil company use?*

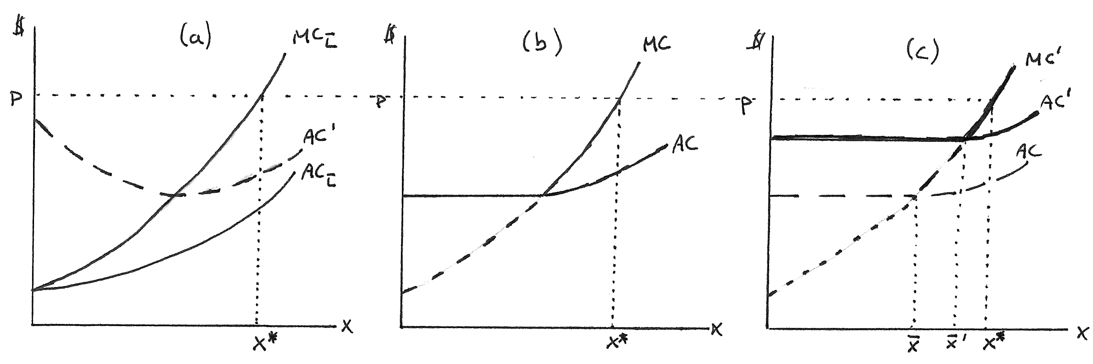
Answer: If land (up to $\overline{L}$ acres) is free to the company, it will use all of it so long as the marginal product of land is positive. This is because profit maximization implies that a firm hires inputs until the marginal revenue product equals the price of the input — and the price of land in this case is zero up to $\overline{L}$ (and effectively infinite thereafter).

(b) *While the 3-input production frontier has constant returns to scale, can you determine the effective returns to scale of production once you take into account that available land is fixed?*

Answer: Land essentially becomes a fixed input in both the short and long run. Constant returns to scale of the 3-input production frontier implies that we can double output by doubling all 3 of our inputs. But if land cannot be increased, doubling labor and capital will be insufficient to double output. The 2-input production frontier that holds land fixed at $\overline{L}$ therefore has decreasing returns to scale. (One way to think of this 2-imput production frontier is as a 3-dimensional "slice" (that holds land fixed at $\overline{L}$) of the original 4-dimensional production frontier that plotted production plans (x, ℓ, k, L)).

(c) *What do average and marginal cost curves look like for your company over the time frame when both labor and capital can be varied?*

Answer: Since the effective 2-input production frontier (that holds $\overline{L}$ fixed) has decreasing returns to scale, the average and marginal cost curves are upward sloping with marginal cost above average cost throughout. These curves are pictured as the solid curves $AC_{\overline{L}}$ and $MC_{\overline{L}}$ in panel (a) of Graph 13.10.



Graph 13.10: Land Rented for Oil Drilling

(d) *Now suppose that the government begins to charge a per-acre rental price q for use of land that is assigned to your company, but an oil company that is assigned $\overline{L}$ acres of land only has the option of renting all $\overline{L}$ acres or none at all. Given that it takes time to relocate oil drilling* *equipment, you cannot adjust to this change in the short run. Will you change how much oil you produce?*

Answer: No. In the short run, the charge for renting land is simply an expense until the company can actually vacate the land.

(e) *In the long run (when you can move equipment off land), what happens to average and marginal costs for you company? Will you change your output level?*

Answer: If your company does not exit the industry, it has to pay $q\overline{L}$ to rent all the land assigned to it — which makes it exactly like a fixed recurring license fee. Once paid, the marginal cost of drilling for oil is no different than it was when the land was free. Thus, the $MC_{\overline{L}}$ curve in panel (a) of Graph 13.10 does not change and continues to be the marginal cost curve for this company. The long run average cost curve, however, incorporates the fixed payment for the land — and thus takes on the U-shape depicted by the dashed AC' curve in the graph. The difference between $AC_{\overline{L}}$ and AC' in panel (a) is the average fixed cost of the land — which is high when oil production is low but, as an average, falls as oil production increases. Thus, $AC_{\overline{L}}$ and AC' converge as x gets large. If you continue to produce in the long run, you will produce where the price of oil p intersects the marginal cost curve that has not changed as a result of the government's policy of charging for use of the land. Thus, *if* you continue to stay in the oil business, you will not change your output level in the long run. This is depicted for the price p in panel (a) of the graph — with production x^* unchanged when the government charges a land rental fee sufficiently low to keep p above AC'. However, if p lies below the dashed AC' curve, you will exit the industry in the long run (once you can vacate the land).

(f) *Suppose the government had employed a different policy that charges a per-acre rent of q but allowed companies to rent any number of acres between 0 and $\overline{L}$. What do long run average and marginal cost curves look like in that case? Would it ever be the case that a firm will rent fewer than $\overline{L}$ acres?(*Hint*: These curves should have a flat as well as an upward sloping portion.)*

Answer: In this case, your company would initially not be limited in how much land it can use for oil drilling — not until it drills enough to run into the $\overline{L}$ constraint imposed by the government. Thus, for low levels of x, your company is operating with full (long run) discretion in terms of how much land, labor and capital to use — and thus faces a constant returns to scale production process. This implies that, for low levels of x, the average cost function takes the form it does for constant returns to scale production functions — i.e. it is constant. However, once the output level reaches the point where your company would ordinarily want to rent more than $\overline{L}$ acres of land in order to produce output level x at minimum cost, the rent on $\overline{L}$ becomes a fixed cost, land becomes a fixed input and production from here on out has decreasing returns to scale. This implies that, at some output level, the constant AC curve begins to slope up. This is pictured in panel (b) of Graph 13.10 as the initially flat and eventually upward sloping AC curve. Since constant returns to scale implies a constant marginal cost and decreasing returns to scale implies an increasing marginal cost, the MC curve in the graph is similarly flat initially but upward sloping after some output level. It therefore overlaps with AC until the land constraint binds — at which point it slopes up and lies above AC causing AC to slope up as well.

(g) *How much will you produce now compared to the case analyzed in (d)?*

Answer: You will produce exactly the same as in part (d) — because the portion of the upward sloping (solid) portion of the MC curve in panel (b) is exactly the same as the $MC_{\overline{L}}$ curve in panel (a). (In both cases, this portion of the marginal cost curve is derived from the 2-input production frontier that holds land fixed at $\overline{L}$.) Thus, as shown in Graph 13.10, price will again intersect marginal cost at x^*.

(h) *Suppose that under this alternative policy the government raises the rental price to q'. Will your company change its output level in the short run?*

Answer: No — in the short run, your company is unable to move its oil drilling equipment — and thus forced to rent the land. Since there is nothing you can do about it — i.e. you cannot affect the expense by anything that you do — it is not an economic cost and therefore affects nothing.

(i) *How do long run average and marginal cost curves change? If you continue to produce oil under the higher land rental price, will you increase or decrease your output level, or will you leave it unchanged*

Answer: Panel (c) of Graph 13.10 illustrates the new marginal and average cost curves as the bold curves labeled MC' and AC' (with the two curves overlapping along the flat hold portion). Underneath these bold curves, the AC and MC curves from panel (b) are replicated for comparison. Under the higher land rental price q', the constant returns to scale portion extends to higher level of output — because when the government charges more for land, firms will substitute away from land and toward more capital and/or labor.[4] As a result, the land constraint $\overline{L}$ does not bind until a higher output level is reached — $\overline{x}'$ as opposed to $\overline{x}$ when the land rental rate was lower. But, although the firm conserves on land when q increases, the average cost per barrel of oil still increases — which is why the flat portion of the AC' curve lies above AC. *Once the constraint of $\overline{L}$ is reached*, however, the firm operates on the 2-dimensional production frontier that holds land fixed and thus experiences decreasing returns to scale. This implies the upward sloping MC' curve — which lies on the previously derived $MC_{\overline{L}}$ curve (in panel (a)) and the MC curve in panel (b). You will continue to produce only if p lies above the lowest point of AC' as drawn in panel (c) of the graph — but this means you will produce at the intersection of price and the same marginal cost as in the previous panels. Thus, *if you continue to produce*, you will not change your output level as a result of the increase in the rental fee of land. The only possible change in your production plan would arise if p fell between the intercepts of the dashed AC and the bold AC' curves — in which case you would have produced before but would exit after the increase in q.

(j) True or False*: The land rental rate q set by the government has no impact on oil production levels so long as oil companies do not exit the industry. Explain. (*Hint*: This is true.)*

Answer: This is true as explained above. In fact, none of the policy changes we investigated has an impact on oil production unless it causes a firm to exit the industry. You can see this in Graph 13.10 by simply observing that p always intersects the same marginal cost curve to give us output level x^*.

B: *Suppose that your production technology for oil drilling is characterized by the production function $x = f(\ell, k, L) = A\ell^{\alpha}k^{\beta}L^{\gamma}$ where $\alpha+\beta+\gamma = 1$ (and all exponents are positive).*

(a) *Demonstrate that this production function has constant returns to scale.*

Answer: To demonstrate this, we simply need to show that multiplying all inputs by t results in a t-fold increase in output; i.e.

$$f(t\ell, tk, tL) = A(t\ell)^{\alpha}(tk)^{\beta}(tL)^{\gamma} = t^{(\alpha+\beta+\gamma)}A\ell^{\alpha}k^{\beta}L^{\gamma} = tA\ell^{\alpha}k^{\beta}L^{\gamma} = tf(\ell, k, L). \quad (13.87)$$

(b) *Suppose again that the government assigns $\overline{L}$ acres of land to your company for oil drilling, that there is no rental fee for the land but you cannot use the land for any other purpose. Given the fixed level of land available, what is your production function now? Demonstrate that it has decreasing returns to scale.*

Answer: The production function now is

$$x = f_{\overline{L}}(\ell, k) = \left[A\overline{L}^{\gamma}\right]\ell^{\alpha}k^{\beta}, \quad (13.88)$$

where the term in brackets simply enters as a constant. This is simply a 2-input Cobb-Douglas production function with $\alpha+\beta < 1$ — which makes it a decreasing returns to scale production function. We can demonstrate this simply by showing

$$f_{\overline{L}}(t\ell, tk) = \left[A\overline{L}^{\gamma}\right](t\ell)^{\alpha}(tk)^{\beta} = t^{(\alpha+\beta)}\left[A\overline{L}^{\gamma}\right]\ell^{\alpha}k^{\beta} < t\left[A\overline{L}^{\gamma}\right]\ell^{\alpha}k^{\beta} = tf_{\overline{L}}(\ell, k). \quad (13.89)$$

[4] For instance, you might think of a firm that produces relatively little choosing to drill horizontally from one spot rather than drilling vertically from two spots that require more land.

(c) *In exercise 13.2, you were asked to derive the (long run) cost function for a 2-input Cobb-Douglas production function. Can you use your result — which is also given in equation (13.35) of exercise 13.5 — to derive the cost function for your oil company? What is the marginal cost function associated with this?*

Answer: Since the production function $f_{\overline{L}}(\ell,k)$ is simply a 2-input Cobb-Douglas function with constant $\left[A\overline{L}^{\gamma}\right]$ (rather than just A) in the front, we can simply use the cost function previously derived and replace A with $\left[A\overline{L}^{\gamma}\right]$ to get

$$C_{\overline{L}}(w,r,x)=(\alpha+\beta)\left(\frac{w^{\alpha}r^{\beta}x}{\left[A\overline{L}^{\gamma}\right]\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)}. \tag{13.90}$$

The marginal cost function is then

$$MC_{\overline{L}}(w,r,x)=\frac{\partial C_{\overline{L}}(w,r,x)}{\partial x}=\left(\frac{w^{\alpha}r^{\beta}x^{(1-\alpha-\beta)}}{\left[A\overline{L}^{\gamma}\right]\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)}=\left(\frac{w^{\alpha}r^{\beta}x^{\gamma}}{\left[A\overline{L}^{\gamma}\right]\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)}. \tag{13.91}$$

(d) *Next, consider the scenario under which the government charges a per-acre rental fee of q but only gives you the option of renting all $\overline{L}$ acres or none at all. Write down your new (long run) cost function and derive the marginal and average cost function. Can you infer the shape of the marginal and average cost curves?*

Answer: In order to drill for oil, you now need to pay a fixed cost of $q\overline{L}$ to rent the land. Thus, the cost function now is

$$C_{\overline{L}}(w,r,q,x)=(\alpha+\beta)\left(\frac{w^{\alpha}r^{\beta}x}{\left[A\overline{L}^{\gamma}\right]\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)}+q\overline{L}. \tag{13.92}$$

This gives us a marginal cost curve

$$MC_{\overline{L}}(w,r,x)=\frac{\partial C_{\overline{L}}(w,r,q,x)}{\partial x}=\left(\frac{w^{\alpha}r^{\beta}x^{\gamma}}{\left[A\overline{L}^{\gamma}\right]\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)} \tag{13.93}$$

where the $q\overline{L}$ term drops out and the MC therefore is not a function of q. Finally, we get the average cost function

$$AC_{\overline{L}}(w,r,q,x)=\frac{C_{\overline{L}}(w,r,q,x)}{x}=(\alpha+\beta)\left(\frac{w^{\alpha}r^{\beta}x^{\gamma}}{\left[A\overline{L}^{\gamma}\right]\alpha^{\alpha}\beta^{\beta}}\right)^{1/(\alpha+\beta)}+\frac{q\overline{L}}{x}. \tag{13.94}$$

Since the marginal cost function is unchanged, the marginal cost curve with $q>0$ is the same upward sloping marginal cost curve as that with $q=0$; i.e. the land rental fee has no impact on marginal cost. (You can verify that the marginal cost curve is upward sloping by simply checking that the derivative of the marginal cost function with respect to x is positive.) The average cost function, however, does change with q — but only because of the last term $q\overline{L}/x$. For small x, this term is large — but as x increases, it converges to zero. This creates the U-shape of the AC' curve in panel (a) of Graph 13.10.

(e) *Does the (long run) marginal cost function change when the government begins to charge for use of the land in this way?*

Answer: As demonstrated above, it does not.

(f) *Now suppose that the government no longer requires your company to rent all $\overline{L}$ acres but instead agrees to rent you up to $\overline{L}$ acres at the land rental rate q. What would your conditional input demands and your (total) cost function be in the absence of the cap on how much land you can rent?*

Answer: To get the conditional input demands without a cap on how much land you can rent, you simply solve the problem

$$\min_{\ell,k,L} \; w\ell + rk + qL \text{ subject to } x = A\ell^{\alpha}k^{\beta}L^{\gamma}. \tag{13.95}$$

Solving first order conditions in the usual way, we get

$$\ell(w,r,q,x) = \left(\frac{\alpha}{w}\right)^{(1-\alpha)}\left(\frac{r}{\beta}\right)^{\beta}\left(\frac{q}{\gamma}\right)^{\gamma}\frac{x}{A} \;\; ; \;\; k(w,r,q,x) = \left(\frac{w}{\alpha}\right)^{\alpha}\left(\frac{\beta}{r}\right)^{(1-\beta)}\left(\frac{q}{\gamma}\right)^{\gamma}\frac{x}{A} \tag{13.96}$$

and

$$L(w,r,q,x) = \left(\frac{w}{\alpha}\right)^{\alpha}\left(\frac{r}{\beta}\right)^{\beta}\left(\frac{\gamma}{q}\right)^{(1-\gamma)}\frac{x}{A}. \tag{13.97}$$

Multiplying these by their input prices, adding and simplifying, we then get the cost function

$$C(w,r,q,x) = w\ell(w,r,q,x) + rk(w,r,q,x) + qL(w,r,q,x) = \frac{w^{\alpha}r^{\beta}q^{\gamma}x}{A\alpha^{\alpha}\beta^{\beta}\gamma^{\gamma}}. \tag{13.98}$$

Note that this is a constant returns to scale cost function that has the property that

$$MC = \frac{w^{\alpha}r^{\beta}q^{\gamma}}{A\alpha^{\alpha}\beta^{\beta}\gamma^{\gamma}} = AC, \tag{13.99}$$

where neither MC nor AC is dependent on x. In other words, marginal and average cost curves are overlapping and flat.

(g) *From now on, suppose that* $A = 100$, $\alpha = \beta = 0.25$, $\gamma = 0.5$, $\overline{L} = 10{,}000$. *Suppose further that the weekly wage rate is* $w = 1000$, *the weekly capital rental rate is* $r = 1000$ *and the weekly land rent rate is* $q = 1000$. *At what level of output* $\overline{x}$ *will your production process no longer exhibit constant returns to scale (given the land limit of* $\overline{L}$*)? What is the marginal and average cost of oil drilling prior to reaching* $\overline{x}$ *(as a function of* x*)?*

Answer: To determine at what level of output we reach the $\overline{L}$ constraint, we simply have to set equation (13.97) to $\overline{L}$ and solve for x. This gives us

$$\overline{x} = A\overline{L}\left(\frac{\alpha}{w}\right)^{\alpha}\left(\frac{\beta}{r}\right)^{\beta}\left(\frac{q}{\gamma}\right)^{(1-\gamma)}. \tag{13.100}$$

Plugging in the values $A = 100$, $\alpha = \beta = 0.25$, $\gamma = 0.5$, $\overline{L} = 10{,}000$ and $w = r = q = 1000$, this gives us $\overline{x} \approx 707{,}107$. Plugging the same values into equation (13.99), we get that prior to reaching $\overline{x}$, $MC = AC \approx 28.28$.

(h) *After reaching this* $\overline{x}$, *what is the marginal and average long run cost of oil drilling (as a function of* x*)? Compare the marginal cost at* $\overline{x}$ *to your marginal cost answer in (g) and explain how this translates into a graph of the marginal cost curve for the firm in this scenario.*

Answer: After $\overline{x}$, we are employing the decreasing returns to scale production function $f_{\overline{L}=10000}$ for which we calculated marginal and average costs in equations (13.93) and (13.94). Substituting the various values into these equations, we get

$$MC_{\overline{L}=10000}(x) = 0.00004x \;\text{ and }\; AC_{\overline{L}=10000}(x) = 0.00002x + \frac{10{,}000{,}000}{x}. \tag{13.101}$$

Evaluating these at $\overline{x} = 707{,}107$, we get $AC = MC \approx 28.28$ — which is exactly what we arrived at in the previous part. This justifies the picture in panel (b) of Graph 13.10 where the constant returns to scale portion meets the upward sloping portion of the MC curve at $\overline{x}$.

(i) *What happens to* $\overline{x}$ *as* q *increases? How does that change the graph of marginal and average cost curves?*

Answer: From equation (13.100), we can easily see that $\overline{x}$ increases as q increases. This changes the graph for marginal and average cost curves as illustrated in panel (c) of Graph 13.10.

(j) *If the price per barrel of oil is $p = 100$, what is your profit maximizing oil production level?*

Answer: Setting p of 100 equal to $MC_{\overline{L}} = 0.00004x$ and solving for x, we get $x = 2,500,000$. You will therefore produce 2,500,000 barrels of oil per week.

(k) *Suppose the government now raises q from 1,000 to 10,000. What happens to your production of oil? What if the government raises q to 15,000?*

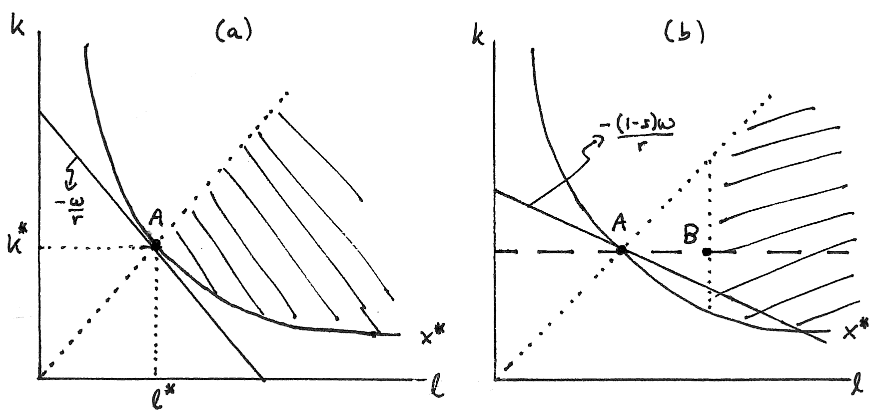
Answer: If you produce, you will still produce where p of 100 equals $MC_{\overline{L}} = 0.00004x$ — which implies you will still produce 2,500,000 barrels of oil per week. The question is whether the government has raised the land rental rate so high that it is more advantageous to exit and produce nothing. To determine at what q this happens, we have to determine at what q the lowest point of the average cost curve is equal to $p = 100$. The lowest point of average cost occurs along the flat, constant returns to scale portion where average cost is given by equation (13.99). Substituting the various values into that equation, we get $AC \approx 0.894427q^{0.5}$. Setting this equal to 100 and solving for q, we get $q = 12,500$. Thus, for any $q \leq 12,500$, output will remain unchanged at 2,500,000 barrels of oil per week, but for $q > 12,500$, output falls to zero as the firm exits the industry.

13.10 Policy and Business Application: *Minimum Wage Labor Subsidy: Suppose you run your business by using a homothetic, decreasing returns to scale production process that requires minimum wage labor ℓ and capital k where the minimum wage is w and the rental rate on capital is r.*

A: *The government, concerned over the lack of minimum wage jobs, agrees to subsidize your employment of minimum wage workers — effectively reducing the wage you have to pay to $(1-s)w$ (where $0<s<1$). Suppose your long run profit maximizing production plan before the subsidy was (ℓ^*, k^*, x^*).*

(a) *Begin with an isoquant graph that contains the isoquant corresponding to x^* and indicate on it the cost minimizing input bundle as A. What region in the graph encompasses all possible production plans that could potentially be long run profit maximizing when the effective wage falls to $(1-s)w$?*

Answer: This is illustrated in panel (a) of Graph 13.11 where the isocost with slope $-w/r$ is tangent at the initial profit maximizing production plan A. (Since A is profit maximizing, it is in addition the case that the marginal revenue product of each input is equal to that input's price.) The ray emanating from the origin and passing through A represents all cost minimizing input bundles at the original wage. When the effective wage falls, the isocosts become shallower — which implies tangencies to isoquants will lie to the right of the original ray through A. We also know that, when input prices fall, output increases — which means that the new long run profit maximizing production plan must lie above the x^* isoquant. The shaded region then represents all possible production plans that might be long run profit maximizing when the effective wage falls — i.e. all plans with a greater output level than x^* and a lower capital to labor ratio.



Graph 13.11: Minimum Wage Subsidy

(b) *On your graph, illustrate the slice of the production frontier to which you are constrained in the short run when capital is fixed. Choose a plausible point on that slice as your new short run profit maximizing production plan B. What has to be true at this point?*

Answer: This is illustrated in panel (b) of Graph 13.11. The new isocost that passes through A has slope $-(1-s)w/r$ and is shallower than the TRS at A. Thus,

$$-TRS^A = \frac{MP_\ell^A}{MP_k^A} = \frac{pMP_\ell^A}{pMP_k^A} > \frac{(1-s)w}{r}. \tag{13.102}$$

Since p, r and MP_k^A are unchanged, this implies that $pMP_\ell^A > (1-s)w$; i.e. the marginal revenue product of labor is greater than the firm's cost of labor. In the short run, the firm can only increase ℓ and not k — which means it is constrained to the horizontal (dashed) slice. It will therefore hire more labor (because the marginal revenue product is greater than the effective wage) until $pMP\ell^B = (1-s)w$ at point B.

(c) *Can you conclude anything about how the marginal product of capital changes as you switch to its new short run profit maximizing production plan?*

Answer: No, we cannot conclude whether the marginal product of capital has increased or decreased as a result of additional labor being hired. If labor and capital are relatively complementary, then MP_k would have increased; if the inputs are relatively substitutable, MP_k would decrease.

(d) *Will you hire more workers in the long run than in the short run?*

Answer: Yes — unless $MP_k^A = MP_k^B$ in which case you would hire the same amount of labor in the short run as in the long run (because the long run profit maximizing production plan would then also be equal to C). If $MP_k^B > MP_k^A$, then the firm will hire additional capital in the long run — and with additional capital will hire additional labor. If $MP_k^B < MP_k^A$, then the firm will hire less capital in the long run as it substitutes more labor for capital — and thus will again hire more labor.

(e) *Will you hire more capital in the long run than in the short run?*

Answer: You will hire more if capital and labor are relatively complementary and less if capital and labor are relatively substitutable.

(f) *Once you have located B in part (b), can you now use this to narrow down the region (that you initially indicated in part (a)) where the long run profit maximizing production plan must lie?*

Answer: This is also illustrated in panel (b) of Graph 13.11. We have just concluded that the firm will definitely hire more labor in the long run (or keep it the same) — thus we will end up with a long run profit maximizing production plan to the right of B. We also concluded that we might hire more or less capital — implying that we cannot vertically shrink the region initially identified in panel (a). (Without knowing B, the region identified in panel (a) is still the only region we can restrict the new profit maximizing production plan to — it is only once we know the short run profit maximizing plan B that we can further restrict the region.)

B: *Suppose, as in previous exercises, that your production function is $f(\ell,k) = 30\ell^{0.2}k^{0.6}$.*

(a) *Suppose that $w = 10 = r$ and $p = 5$. What is your profit maximizing production plan before the labor subsidy?*

Answer: In exercise 13.1, we derived the input demand functions for a Cobb-Douglas function of the form $f(\ell,k) = A\ell^{\alpha}k^{\beta}$ as

$$\ell(w,r,p) = \left(\frac{pA\alpha^{(1-\beta)}\beta^{\beta}}{w^{(1-\beta)}r^{\beta}}\right)^{1/(1-\alpha-\beta)} \text{ and } k(w,r,p) = \left(\frac{pA\alpha^{\alpha}\beta^{(1-\alpha)}}{w^{\alpha}r^{(1-\alpha)}}\right)^{1/(1-\alpha-\beta)}. \quad (13.103)$$

and the output supply function as

$$x(w,r,p) = \left(\frac{Ap^{(\alpha+\beta)}\alpha^{\alpha}\beta^{\beta}}{w^{\alpha}r^{\beta}}\right)^{1/(1-\alpha-\beta)}. \quad (13.104)$$

Plugging in the various values from this problem, this gives us an initially long run profit maximizing production plan of

$$(\ell^*, k^*, x^*) = (6561, 19683, 65610). \quad (13.105)$$

(b) *What is the short run profit maximizing plan after a subsidy of $s = 0.5$ is implemented.*

Answer: When $s = 0.5$, the effective new wage is $(1-s)w = 0.5(10) = 5$. The short run production function, given $k^* = 19{,}683$, is

$$x = f_{k^*}(\ell) = \left[30\left(19683^{0.6}\right)\right]\ell^{0.2} \approx 11{,}313\ell^{0.2}. \quad (13.106)$$

The short run profit maximization problem is then

$$\max_{\ell} p\left(11313\ell^{0.2}\right) - w\ell. \qquad (13.107)$$

Solving this, we get the short run labor demand function, and substituting it back into equation (13.106), we get the short run supply function:

$$\ell_{k^*}(w,p) = \left(\frac{2262.6p}{w}\right)^{5/4} \text{ and } x_{k^*}(w,p) = 11313\left(\frac{2262.6p}{w}\right)^{1/4}. \qquad (13.108)$$

We can then plug in the unchanged output price $p = 5$ and the new effective wage $(1-s)w = 5$ to get the short run profit maximizing production plan of

$$(\ell^{SR}, k^{SR}, x^{SR}) = (15605, 19683, 78024), \qquad (13.109)$$

where capital remains unchanged because it is fixed in the short run. Labor, however, increases from the initial 6,561 to 15,605, and output increases from the initial 65,610 to 78,024.

(c) *What is the new long run profit maximizing plan once capital can be adjusted?*

Answer: Plugging the new effective wage of 5 (together with the other values from this problem) in equations (13.103) and (13.104), we get the new long run profit maximizing production plan of

$$(\ell^{LR}, k^{LR}, x^{LR}) = (26244, 39366, 131220), \qquad (13.110)$$

implying that labor, capital and output increase from the original short run profit maximizing adjustment to the lower wage.

(d) *For any Cobb-Douglas function $f(\ell,k) = A\ell^{\beta\alpha}k^{\beta(1-\alpha)}$, the CES production function $g(\ell,k) = A(\alpha\ell^{-\rho} + (1-\alpha)k^{-\rho})^{-\beta/\rho}$ converges to f as ρ approaches 0. What values for A, α and β will do this for the production function $x = 30\ell^{0.2}k^{0.6}$?*

Answer: It has to be the case that $A = 30$ and that

$$\beta\alpha = 0.2 \ \ (and) \ \ \beta(1-\alpha) = 0.6. \qquad (13.111)$$

Solving these two equations for α and β, we get $\alpha = 0.25$ and $\beta = 0.8$.

(e) *Using a spreadsheet to program the output supply and input demand equations for a CES production function given in equation (13.38) in a footnote in the text, verify that your long run production plans mirror those you calculated for the Cobb-Douglas function when ρ approaches 0 and α and β are set appropriately.*

Answer: Substituting $A = 30$, $\alpha = 0.25$ and $\beta = 0.8$, and plugging in output and input prices as specified in the problem, you should get the same answers for the initial and final profit maximizing production plan as you plug in values for ρ that are very close to zero. (Your spreadsheet will not be able to calculate production plans when ρ is set to exactly zero as this would cause divisions by zero.)

(f) *Finally, derive the first order condition for the short run profit maximization problem that fixed capital using the CES production function. Then, using your spreadsheet, check to see whether those first order conditions hold when you plug in the short run profit maximizing quantity of labor that you calculated in (b).*

Answer: The short run CES production function for fixed capital $\overline{k}$ is

$$g(\ell) = A\left(\alpha\ell^{-\rho} + (1-\alpha)\overline{k}^{-\rho}\right)^{-\beta/\rho}. \qquad (13.112)$$

This implies a short run profit maximization problem

$$\max_{\ell} pA\left(\alpha\ell^{-\rho} + (1-\alpha)\overline{k}^{-\rho}\right)^{-\beta/\rho} - w\ell \qquad (13.113)$$

with first order condition

$$\alpha\beta p A\left(\alpha\ell^{-\rho}+(1-\alpha)\overline{k}^{-\rho}\right)^{-(\beta-\rho)/\rho}\ell^{-(\rho+1)}=w. \tag{13.114}$$

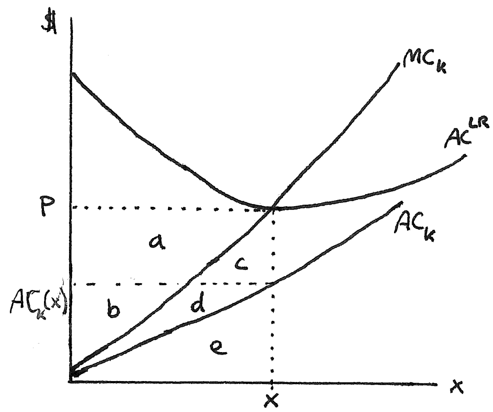
In your spreadsheet, set $A = 30$, $\alpha = 0.25$, $\beta = 0.8$, $\overline{k} = 19683$, $p = 5$ and $w = 5$; set ρ very close to zero and set ℓ equal to the short run optimal quantity calculated for the Cobb-Douglas case (i.e. 15605). You should then get that the equation holds — i.e. the CES function properly specified again yields the same solution as the Cobb-Douglas function.

13.11 Policy and Business Application: *Business Taxes*: *In this exercise, suppose that your hamburger business "McWendy's" has a homothetic decreasing returns to scale production function that uses labor ℓ and capital k to produce hamburgers x. You can hire labor at wage w and capital at rental rate r but also have to pay a fixed annual franchise fee F to the McWendy parent company in order to operate as a McWendy's restaurant. You can sell your McWendy's hamburgers at price p.*

A: *Suppose that your restaurant, by operating at its long run profit maximizing production plan (ℓ^*, k^*, x^*), is currently making zero long run profit. In each of the policy proposals in parts (b) through (h) below, suppose that prices w, r and p remain unchanged.*[5] *In each part, beginning with (b), indicate what happens to your optimal production plan in the short and long run.*

(a) *Illustrate the short run AC and MC curves as well as the long run AC curve. Where in your graph can you locate your short run profit — and what is it composed of?*

Answer: These curves are illustrated in panel (a) of Graph 13.12 where the U-shape of the long run average cost curve derives from the fixed franchise fee. Since there are no fixed costs — only fixed expenses — in the short run, and since the decreasing returns to scale of the long run production process also implies decreasing returns to scale for the short run production process, the short run MC and AC curves have to be upward sloping (with the former above the latter). The short run profit is then composed of the expense on capital rk and the fixed franchise fee F — i.e. short run profit is $rk + F$ which disappears in the long run as these turn into costs. In the graph, there are two ways of seeing this short run profit. In both cases, we begin by identifying total revenue as the area $a + b + c + d + e$ — i.e. the price p times output x. (We know price has to be at the lowest point of the long run AC curve since we know the firm is making zero long run profit.) The short run costs can then be identified as the average (short run) cost of producing x — which is $AC_k(x)$ — times the output level x, or just area $b + d + e$. Alternatively, short run costs can also be seen as the area under the short run marginal cost curve — area $c + d + e$. Subtracting short run costs from revenues, we then get that short run profit is $a + c$ or, equivalently, $a + b$. (Logically this of course implies that $b = c$.)



Graph 13.12: Short Run Profit

(b) *Suppose the government determined that profits in your industry were unusually high last year — and imposes a one-time "windfall profits tax" of 50% on your business's profits from last year.*

[5] This is only an assumption for now — which will in fact often not hold, as we will see in Chapter 14.

Answer: There is nothing you can do in your business to avoid paying this windfall profits tax — it is a sunk "cost", an expense in the short run and irrelevant in the long run. Therefore you will not change your production plan.

(c) *The government imposes a 50% tax on short run profits from now on.*

Answer: If you are currently maximizing short run profits (which you are), you will not change anything in the short run if the government takes half of your short run profit. Half of the most you can make is still more than half of less than the most you can make in your business. In the long run, however, your profit will now no longer be positive — which means you will exit in the long run and stop producing (absent any changes in prices in the industry).

(d) *The government instead imposes a 50% tax on long run profits from now on.*

Answer: Your long run profit is zero — so the government will not collect any taxes from you. If what you were doing before was profit maximizing, you are still profit maximizing by doing exactly the same as you were doing before. This would be true even if your long run profits were positive. If you now only make half as much long run profit, you are still making a positive profit — which means you are still making more in this business than you could anywhere else.

(e) *The government instead taxes franchise fees causing the blood sucking McWendy's parent company to raise its fee to $G > F$.*

Answer: This will increase your long run costs — which means that, since you were making zero long run profit before, you will now be making negative long run profit. In the absence of any other changes (such as a change in price), you will therefore exit and stop producing hamburgers.

(f) *The government instead imposes a tax t on capital (which is fixed in the short run) used by your restaurant — causing you to have to pay not only r but also tr to use one unit of capital.*

Answer: Since capital is fixed in the short run, nothing changes for you in the short run (assuming you still are committed to the capital you have for now). Put differently, the tax payment tr is a short run expense, not a cost. In the long run, however, your average and marginal cost curves increase. If you were to continue to produce, this implies you will produce less (as p intersects MC at a lower quantity) — but you will in fact exit in the long run because your long run profit — which was zero before the increase in costs — must now be negative.

(g) *Instead of taxing capital, the government taxes labor in the same way as it taxed capital in part (f).*

Answer: Since labor is variable in the short run, this tax is an immediate cost since you affect you overall tax payment by changing how many workers you hire. Thus, the MC_k shifts up. If you continue to produce in the short run, you will then produce less because the new MC_k intersects price at a lower quantity. It is not clear, however, whether you will not fully shut down even in the short run. The crucial question is whether the tax on labor is sufficiently high for short run profit (which was positive at the outset) to fall below zero. If so, you will shut down. Depending on how large the tax rate t, you will therefore either produce less or not at all in the short run. In the long run, however, you will exit (unless something else changes) — because your previously zero long run profit is now negative.

(h) *Finally, instead of any of the above, the government imposes a "health tax" t on hamburgers — charging you \$t for every hamburger you sell.*

Answer: The answer is similar to that given to part (g) — in the short run, you may stay open and produce fewer hamburgers or you may shut down depending on whether short run profits under the lower production level remain positive. In the long run, however, you will exit (unless something else changes).

B: *In previous exercises, we gave the input demand functions for a a firm facing prices (w,r,p) and technology $f(\ell,k)=A\ell^{\alpha}k^{\beta}$ (with $\alpha,\beta>0$ and $\alpha+\beta<1$) in equation (13.50) and the long run output supply function in equation (13.49) — both given in footnotes to earlier end-of-chapter exercises in this chapter.*

(a) *When you add a recurring fixed cost F, how are these functions affected? (*Hint*: You will have to restrict the set of prices for which the functions are valid — and you can use the profit function given in exercise 13.7) to do this strictly in terms of A, α, β and the prices* (w,r,p)*.) What are the short run labor demand and output supply functions for a given* $\bar{k}$*?*

Answer: We now have to include the role of the recurring fixed cost F in the long run input demand and output supply functions. But we know from our graphical work that this simply causes these functions to become "shorter" — i.e. these functions remain valid but only for the set of input and output prices at which long run profit (which incorporates F) is not negative. In the absence of F, the profit function for Cobb-Douglas production functions was given in exercise 13.7 as

$$\pi(w,r,p)=(1-\alpha-\beta)\left(\frac{Ap\alpha^{\alpha}\beta^{\beta}}{w^{\alpha}r^{\beta}}\right)^{1/(1-\alpha-\beta)}. \tag{13.115}$$

To say that long run profit is positive at a given set of prices (p,w,r) is therefore to say that

$$(1-\alpha-\beta)\left(\frac{Ap\alpha^{\alpha}\beta^{\beta}}{w^{\alpha}r^{\beta}}\right)^{1/(1-\alpha-\beta)}\geq F \tag{13.116}$$

or, rearranging terms, that

$$p\geq\left(\frac{w^{\alpha}r^{\beta}}{A\alpha^{\alpha}\beta^{\beta}}\right)\left(\frac{F}{1-\alpha-\beta}\right)^{(1-\alpha-\beta)}. \tag{13.117}$$

We can thus write the long run labor demand, capital demand and output supply functions as

$$\ell(w,r,p)=\left(\frac{pA\alpha^{(1-\beta)}\beta^{\beta}}{w^{(1-\beta)}r^{\beta}}\right)^{1/(1-\alpha-\beta)}\text{ if } p\geq\left(\frac{w^{\alpha}r^{\beta}}{A\alpha^{\alpha}\beta^{\beta}}\right)\left(\frac{F}{1-\alpha-\beta}\right)^{(1-\alpha-\beta)} \tag{13.118}$$

$$k(w,r,p)=\left(\frac{pA\alpha^{\alpha}\beta^{(1-\alpha)}}{w^{\alpha}r^{(1-\alpha)}}\right)^{1/(1-\alpha-\beta)}\text{ if } p\geq\left(\frac{w^{\alpha}r^{\beta}}{A\alpha^{\alpha}\beta^{\beta}}\right)\left(\frac{F}{1-\alpha-\beta}\right)^{(1-\alpha-\beta)} \tag{13.119}$$

$$x(w,r,p)=\left(\frac{Ap^{(\alpha+\beta)}\alpha^{\alpha}\beta^{\beta}}{w^{\alpha}r^{\beta}}\right)^{1/(1-\alpha-\beta)}\text{ if } p\geq\left(\frac{w^{\alpha}r^{\beta}}{A\alpha^{\alpha}\beta^{\beta}}\right)\left(\frac{F}{1-\alpha-\beta}\right)^{(1-\alpha-\beta)}, \tag{13.120}$$

with all three equations equaling zero if the condition does not hold.

In the short run, however, F plays no role since it is not an economic cost. The short run production function given fixed $\bar{k}$ is $f_{\bar{k}}(\ell)=\left[A\bar{k}^{\beta}\right]\ell^{\alpha}$. Solving the short run profit maximization problem

$$\max_{\ell}\ p\left[A\bar{k}^{\beta}\right]\ell^{\alpha}-w\ell, \tag{13.121}$$

we get the short run labor demand function

$$\ell_{\bar{k}}(w,p)=\left(\frac{\alpha p\left[A\bar{k}^{\beta}\right]}{w}\right)^{1/(1-\alpha)}. \tag{13.122}$$

Substituting this back into the short run production function, we get the short run supply function

$$x_{\bar{k}}(w,p)=\left[A\bar{k}^{\beta}\right]^{1/(1-\alpha)}\left(\frac{\alpha p}{w}\right)^{\alpha/(1-\alpha)}. \tag{13.123}$$

Each of the following correspond to the respective parts (b) through (h) in part A of the question:

(b) *For each of (b) through (h) in part A of the exercise, indicate whether (and how) the functions you derived in part (a) are affected.*

Answer: Beginning with A(b), the following are answers corresponding to each part. In (b), none of the functions are affected since last year's profits do not enter any of them.

(c) The short run functions are not affected when a 50% tax on short run profits is imposed. To see more clearly why, you can write the short run profit maximization problem to include the 50% short run profits tax — and you would get

$$\max_{\ell} \; 0.5\left(pA\bar{k}^{\beta}\ell^{\alpha} - w\ell\right), \quad (13.124)$$

which has first order conditions identical to those in the original problem in equation (13.121).

In the long run, however, the 50% short run profit tax becomes a recurring fixed cost of doing business and would thus increase the F term in equations (13.118), (13.119) and (13.120). While this does not affect the functions themselves directly, it affects the range of prices under which the functions are not simply equal to zero (because the firm exits). If long run profit is originally zero, for instance, the 50% short run profit tax would then cause the input demand and output supply functions to go to zero because the inequality in each expression no longer holds.

(d) This affects none of the functions. You can again see that the long run functions are unaffected by realizing that a tax on long run profits drops out as we solve the profit maximization problem

$$\max_{\ell,k} (1-t)\left(pf(\ell,k) - w\ell - rk\right) \quad (13.125)$$

where t stands for the tax rate applied to long run profit. Put differently, since the government taking *a fraction* of long run profit does not cause long run profit to become negative, this tax will never cause the inequality in equations (13.118), (13.119) and (13.120) to not hold.

(e) Since F does not appear in the short run equations, the new fixed cost G will also not appear — leaving the short run curves unaffected. F does, however, appear in equations (13.118), (13.119) and (13.120) — or, to be more precise, it appears in the inequality that restricts the prices for which the functions are applicable. As F increases, the inequality will no longer hold for some range of prices at the lower end — thus raising the price at which the firm exits. If, for instance, the firm was initially making zero long run profit, it would exit with an increase in F to G because the inequality in (13.118), (13.119) and (13.120) no longer holds.

(f) Since r appears in equations (13.118), (13.119) and (13.120), we know that the long run functions are affected. They are affected in two ways: First, the equations themselves are affected, with an increase in r causing a decrease in ℓ, k, and x; and second, the inequality is affected in the sense that the inequality now no longer holds for some range of prices at the lower end. This implies that, in the long run, the firm will reduce its output and its demand for labor and capital — and it will reduce these to zero if the inequality no longer holds. For instance, if long run profit is initially zero, the firm will exit (unless something else changes). In the short run, however, r does not appear in either the labor demand or output supply equations — and thus nothing changes in the short run.

(g) The impact on the long run will mirror what we just described in (f) for a capital tax. In the short run, however, there was no impact of the tax on capital because r did not enter the short run labor demand or output supply functions — but w does appear in these, which implies that the labor tax has an immediate short run impact. In particular, an increase in w causes an immediate decrease in both labor demand and output supply.

(h) The tax on hamburgers will also have short and long run impacts on our derived functions by changing the output price from p to $(p-t)$. This lower output price will shift short run supply and short run labor demand in the respective short run functions, reducing the quantity in each. In the long run, p appears in both the initial equation as well as the inequality of expressions (13.118), (13.119) and (13.120). In the equations to the left of each expression, p changes to $(p-t)$ — causing a drop in each. In the inequalities on the right, p

changes to $(p - t)$ on the left-hand side of the inequality, or — alternatively, we can rewrite the inequality as

$$p \geq \left(\frac{w^{\alpha} r^{\beta}}{A\alpha^{\alpha}\beta^{\beta}}\right)\left(\frac{F}{1-\alpha-\beta}\right)^{(1-\alpha-\beta)} + t. \tag{13.126}$$

This implies that the price for which the input demand and output supply functions are valid increases — again “shortening” the input demand and output demand curves. If, for instance, the firm was making zero profit before the implementation of the tax, it will exit after the implementation (unless something else changes).