SOLUTIONS

16

General Equilibrium

Solutions for *Microeconomics: An Intuitive Approach with Calculus*

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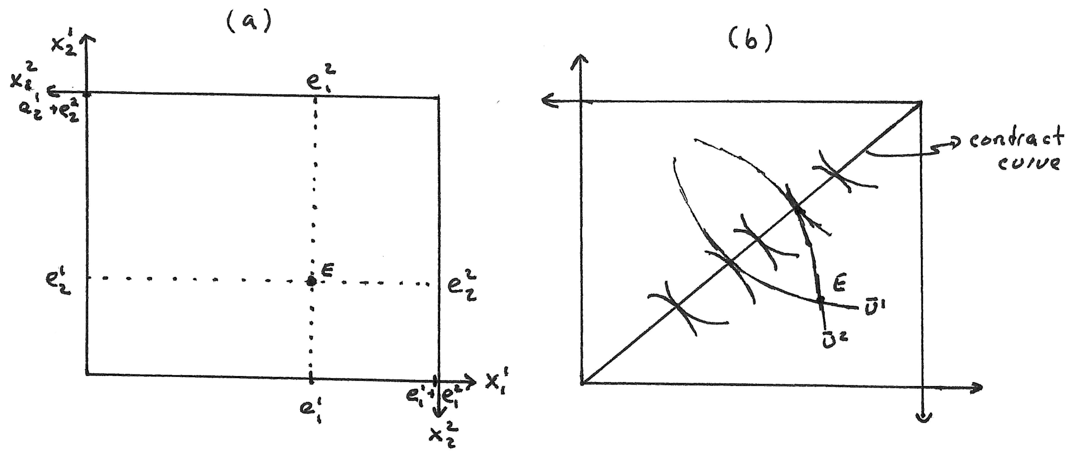
- Each end-of-chapter exercise begins on a new page. This is to facilitate maximum flexibility for instructors who may wish to share answers to some but not all exercises with their students.
- If you are assigning only the A-parts of exercises in *Microeconomics: An Intuitive Approach with Calculus*, you may wish to instead use the solution set created for the companion book *Microeconomics: An Intuitive Approach*.
- *Solutions to Within-Chapter Exercises are provided in the student* Study Guide.

16.1 *Consider a 2-person/2-good exchange economy in which person 1 is endowed with* (e_1^1, e_2^1) *and person 2 is endowed with* (e_1^2, e_2^2) *of the goods* x_1 *and* x_2.

A: *Suppose that tastes are homothetic for both individuals.*

(a) *Draw the Edgeworth Box for this economy, indicating on each axis the dimensions of the the box.*

Answer: This is illustrated in panel (a) of Graph 16.1 where the width of the box is $(e_1^1 + e_1^2)$ and the height is $(e_2^1 + e_2^2)$.



Graph 16.1: Contract Curve with Homothetic Tastes

(b) *Suppose that the two individuals have identical tastes. Illustrate the contract curve — i.e. the set of all efficient allocations of the two goods.*

Answer: Homothetic tastes have the characteristic that the *MRS* is the same along any ray from the origin. Consider the ray that passes from the lower left to the upper right corners of the box — i.e. from the origin for person 1 to the origin for person 2. If tastes are homothetic for each of the two individuals, this means that, for each individual, it is the case that the *MRS* is constant along this ray. And if their tastes are identical, then their *MRS*'s are the same along that ray — i.e. on each point of the ray, the indifference curves that pass through that point are tangent to one another. Since the contract curve is the set of allocations where the indifference curves are tangent, this ray is then the contract curve. It is depicted in panel (b) of Graph 16.1.

(c) True or False*: Identical tastes in the Edgeworth Box imply that there are no mutually beneficial trades.*

Answer: This is false. In panel (b) of Graph 16.1, for instance, the indifference curves $\overline{u}^1$ and $\overline{u}^2$ contain the endowment bundle E — with allocations inside the lens created by these indifference curves representing mutually beneficial trades. The only way that there are no mutually beneficial trades when both individuals have identical homothetic tastes is if the endowment bundle falls on the contract curve — i.e. on the line connecting the origins for the two individuals.

(d) *Now suppose that the two individuals have different (but still homothetic) tastes.* True or False*: The contract curve will lie to one side of the line that connects the lower left and upper right corners of the Edgeworth Box — i.e. it will never cross this line inside the Edgeworth Box.*

Answer: This is true (almost). If the two individuals' tastes are not identical, then their indifference curves are not likely to be tangent on the line connecting the lower left and upper right corners of the box. Take one point on that line — it is likely the case that the indifference curve for person 1 is steeper or shallower than that for individual 2 at this point. Suppose first that individual 1's indifference curve is shallower. Then the two indifference curves form a lens shape — with the entire area of the lens lying *above* the line connecting the corners of the box. Since the slopes of the indifference curves are constant along this line, this same lens shape will appear above the line for *any* allocation on the line. This implies that the tangencies of indifference curves (which form the contract curve) must also lie *above* the line (because these tangencies will be found within the lens shapes formed from indifference curves that cross on the line.) The reverse will be true if individual 1's indifference curve is steeper along the line than indifference curves for individual 2 — with the entire contract curve now lying *below* the line. The reason the answer is true (*almost*) is that it is still possible that the marginal rates of substitution for the two individuals are in fact equal along the line connecting the lower left and upper right corners of the box. For instance, it might be that the tastes have different degrees of substitutability (and are therefore different) but still have the same marginal rates of substitution on that line. In that case, the contract curve lies on the line connecting the lower left and upper right corners. Thus, homothetic tastes imply that the contract curve lies either on the line connecting the corners or all to one side of that line.

B: *Suppose that the tastes for individuals 1 and 2 can be described by the utility functions* $u^1 = x_1^{\alpha} x_2^{(1-\alpha)}$ *and* $u^2 = x_1^{\beta} x_2^{(1-\beta)}$ *(where* α *and* β *both lie between 0 and 1). Some of the questions below are notationally a little easier to keep track off if you also denote* $E_1 = e_1^1 + e_1^2$ *as the economy's endowment of* x_1 *and* $E_2 = e_2^1 + e_2^2$ *as the economy's endowment of* x_2.

(a) *Let* $\overline{x}_1$ *denote the allocation of* x_1 *to individual 1, and let* $\overline{x}_2$ *denote the allocation of* x_2 *to individual 1. Then use the fact that the remainder of the economy's endowment is allocated to individual 2 to denote individual 2's allocation as* $(E_1 - \overline{x}_1)$ *and* $(E_2 - \overline{x}_2)$ *for* x_1 *and* x_2 *respectively. Derive the contract curve in the form* $\overline{x}_2 = x_2(\overline{x}_1)$ *— i.e. with the allocation of* x_2 *to person 1 as a function of the allocation of* x_1 *to person 1.*

Answer: You can derive this either by setting the MRS for individual 1 equal to the MRS for individual 2 — or you can solve the problem

$$\max_{\overline{x}_1,\overline{x}_2} \overline{x}_1^{\alpha}\overline{x}_2^{(1-\alpha)} \text{ subject to } u^2 = (E_1 - \overline{x}_1)^{\beta}(E_2 - \overline{x}_2)^{(1-\beta)} \tag{16.1}$$

where person 1's utility is maximized subject to getting person 2 to a particular indifference curve u^2. Either way, you will get to the point where you have an expression that sets the marginal rates of substitution equal to one another — i.e.

$$\begin{aligned}\frac{\partial u^1(\overline{x}_1,\overline{x}_2)/\partial x_1}{\partial u^1(\overline{x}_1,\overline{x}_2)/\partial x_2} = \frac{\alpha\overline{x}_2}{(1-\alpha)\overline{x}_1} = \frac{\beta(E_2-\overline{x}_2)}{(1-\beta)(E_1-\overline{x}_1)} = \\ = \frac{\partial u^2((E_1-\overline{x}_1),(E_2-\overline{x}_2))/\partial x_1}{\partial u^2((E_1-\overline{x}_1),(E_2-\overline{x}_2))/\partial x_2}.\end{aligned} \tag{16.2}$$

Solving the middle of this expression for $\overline{x}_2$, we then get the contract curve

$$x_2(\overline{x}_1) = \frac{(1-\alpha)\beta E_2 \overline{x}_1}{\alpha(1-\beta)E_1 + (\beta-\alpha)\overline{x}_1}. \tag{16.3}$$

(b) *Simplify your expression under the assumption that tastes are identical — i.e.* $\alpha = \beta$. *What shape and location of the contract curve in the Edgeworth Box does this imply?*

Answer: Replacing β with α, the expression then simplifies to

$$x_2(\overline{x}_1) = \frac{(1-\alpha)\alpha E_2 \overline{x}_1}{\alpha(1-\alpha)E_1 + (\alpha-\alpha)\overline{x}_1} = \frac{E_2}{E_1}\overline{x}_1. \tag{16.4}$$

This is simply the equation of a line with zero vertical intercept and slope E_2/E_1 — which is the slope of the ray that passes from the lower left to the upper right corner of the Edgeworth Box. Thus, when tastes are identical, we get that the contract curve is the line that connects the origins for the two individuals in the Edgeworth Box — exactly as we did for homothetic tastes in part A of the question (and as depicted in panel (b) of Graph 16.1.)

(c) *Next, suppose that $\alpha \neq \beta$. Verify that the contract curve extends from the lower left to the upper right corner of the Edgeworth Box.*

Answer: Evaluating the contract curve equation (16.3) at $\overline{x}_1 = 0$, we get $x_2(0) = 0$ — i.e. the contract curve passes through the lower left hand corner of the Edgeworth Box. Evaluating the contract curve at $\overline{x}_1 = E_1$, we get

$$x_2(E_1) = \frac{(1-\alpha)\beta E_2 E_1}{\alpha(1-\beta)E_1 + (\beta-\alpha)E_1} = \frac{(1-\alpha)\beta E_2 E_1}{(1-\alpha)\beta E_1} = E_2; \tag{16.5}$$

i.e. the contract curve passes through the upper right corner of the box where individual 1 gets all the goods in the economy.

(d) *Consider the slopes of the contract curve when $\overline{x}_1 = 0$ and when $\overline{x}_1 = E_1$. How do they compare to the slope of the line connecting the lower left and upper right corners of the Edgeworth Box if $\alpha > \beta$? What if $\alpha < \beta$?*

Answer: The slope of the contract curve is the derivative of equation (16.3) with respect to x_1 —

$$\begin{aligned}\frac{\partial x_2(\overline{x}_1)}{\partial x_1} &= \frac{(1-\alpha)\beta E_2}{\alpha(1-\beta)E_1 + (\beta-\alpha)\overline{x}_1} - \frac{(\beta-\alpha)(1-\alpha)\beta E_2 \overline{x}_1}{\left[\alpha(1-\beta)E_1 + (\beta-\alpha)\overline{x}_1\right]^2} = \\ &= \frac{\alpha\beta(1-\alpha)(1-\beta)E_1 E_2}{\left[\alpha(1-\beta)E_1 + (\beta-\alpha)\overline{x}_1\right]^2}.\end{aligned} \tag{16.6}$$

Evaluated at $\overline{x}_1 = 0$ and at $\overline{x}_1 = E_1$, we get

$$\frac{\partial x_2(0)}{\partial x_1} = \frac{\beta(1-\alpha)E_2}{\alpha(1-\beta)E_1} \text{ and } \frac{\partial x_2(E_1)}{\partial x_1} = \frac{\alpha(1-\beta)E_2}{\beta(1-\alpha)E_1}. \tag{16.7}$$

The slope of the line connecting the two corners of the Edgeworth box is E_2/E_1. If $\alpha = \beta$, both derivatives in expression (16.7) are equal to E_2/E_1 — i.e. the slope of the contract curve is exactly the slope of the line connecting the corners (as we concluded already above). If $\alpha > \beta$, then

$$\frac{\beta(1-\alpha)}{\alpha(1-\beta)} < 1 \text{ and } \frac{\alpha(1-\beta)}{\beta(1-\alpha)} > 1 \tag{16.8}$$

implying that

$$\frac{\partial x_2(0)}{\partial x_1} < \frac{E_2}{E_1} \text{ and } \frac{\partial x_2(E_1)}{\partial x_1} > \frac{E_2}{E_1}. \tag{16.9}$$

The reverse relationship holds when $\alpha < \beta$.

(e) *Using what you have concluded, graph the shape of the contract curve for the case $\alpha > \beta$ and for the case when $\alpha < \beta$?*

The contract curves consistent with these relationships are graphed in Graph 16.2.

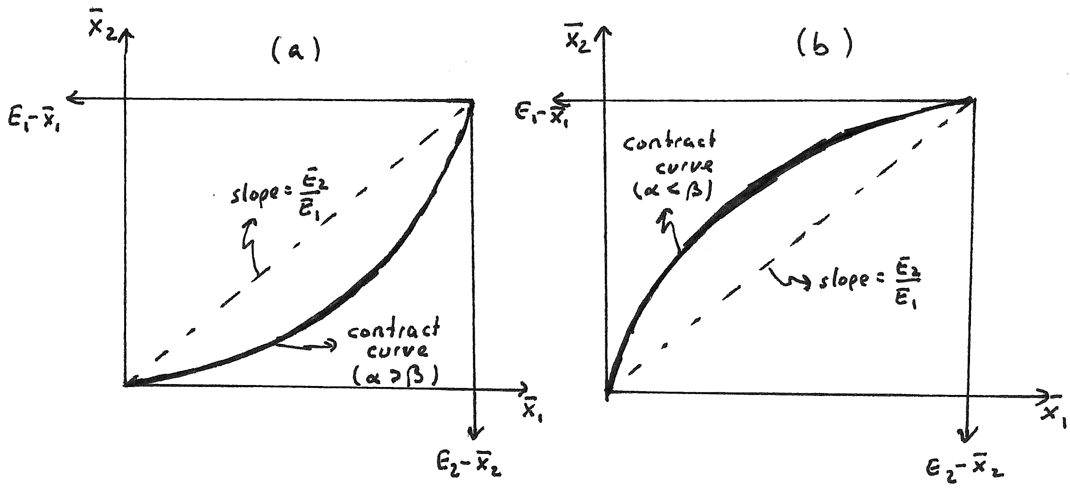
(f) *Suppose that the utility function for the two individuals instead took the more general constant elasticity of substitution form $u = (\alpha x_1 + (1-\alpha)x_2)^{-1/\rho}$. If the tastes for the two individuals are identical, does your answer to part (b) change?*

Answer: No, the answer does not change. The MRS for this utility function (derived in Chapter 5) is

$$MRS = -\left(\frac{\alpha}{(1-\alpha)}\right)\left(\frac{x_2}{x_1}\right)^{\rho+1}. \tag{16.10}$$

Using our notation and setting the MRS's equal to each other for the two individuals, we then get

$$\left(\frac{\alpha}{(1-\alpha)}\right)\left(\frac{\overline{x}_2}{\overline{x}_1}\right)^{\rho+1} = \left(\frac{\alpha}{(1-\alpha)}\right)\left(\frac{(E_2-\overline{x}_2)}{(E_1-\overline{x}_1)}\right)^{\rho+1} \tag{16.11}$$



Graph 16.2: Contract Curves when (a) $\alpha > \beta$ and when (b) $\alpha < \beta$

which we can solve for $\overline{x}_2$ to get the contract curve

$$x_2(\overline{x}_1) = \left(\frac{E_2}{E_1}\right)\overline{x}_1; \tag{16.12}$$

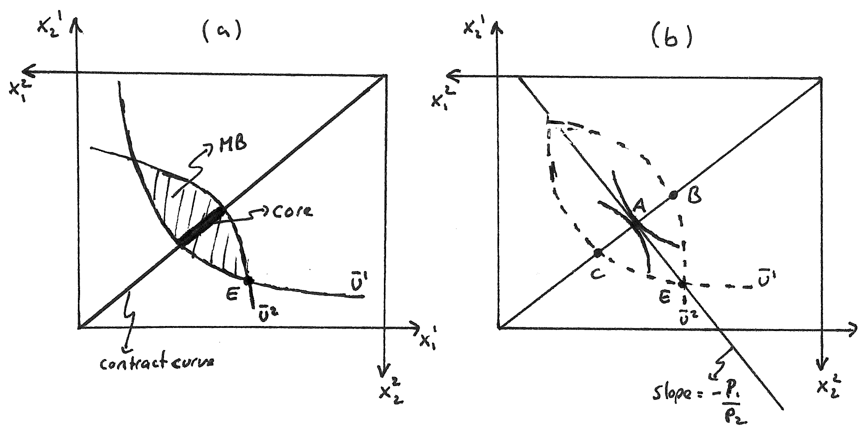
i.e. the contract curve again has zero vertical intercept and slope E_2/E_1, the slope of the ray that connects the two corners of the Edgeworth Box.

16.2 *Consider again, as in exercise 16.1, a 2-person/2-good exchange economy in which person 1 is endowed with (e_1^1, e_2^1) and person 2 is endowed with (e_1^2, e_2^2) of the goods x_1 and x_2.*

A: *Suppose again that tastes are homothetic, and assume throughout that tastes are also identical.*

(a) *Draw the Edgeworth Box and place the endowment point to one side of the line connecting the lower left and upper right corners of the box.*

Answer: This is done in panel (a) of Graph 16.3.



Graph 16.3: Core and Equilibria with Identical Homothetic Tastes

(b) *Illustrate the contract curve (i.e. the set of efficient allocations) you derived in exercise 16.1. Then illustrate the set of mutually beneficial trades as well as the set of core allocations.*

Answer: These are also illustrated in panel (a) of Graph 16.3 where $\overline{u}^1$ and $\overline{u}^2$ are the utility levels obtained by individuals 1 and 2 in the absence of trading — i.e. when they just consume their endowment.

(c) *Why would we expect these two individuals to arrive at an allocation in the core by trading with one another?*

Answer: The allocations in the core satisfy two properties: (1) both individuals are better off than at their endowments and (2) there are no further gains from trade — i.e. no further reason to trade. We would expect the individuals to trade until (2) is satisfied, and we would expect neither to agree to trade unless he/she is better off. Thus, it is in the region where (1) and (2) are satisfied that we would expect trading to stop.

(d) *Where does the competitive equilibrium lie in this case? Illustrate this by drawing the budget line that arises from equilibrium prices.*

Answer: This is illustrated in panel (b) of Graph 16.3 where the budget line (with slope $-p_1/p_2$) through the endowment point E causes the two individuals to trade from E to A.

(e) *Does the equilibrium lie in the core?*

Answer: Yes, it does, as illustrated in panel (b) by the fact that A lies on the section of the contract curve that falls between the (dotted) indifference curves which pass through the endowment E.

(f) *Why would your prediction when the two individuals have different bargaining skills differ from this?*

Answer: When individuals behave competitively (as they do in reaching A), they essentially have the same bargaining skills (in the sense that they are not bargaining but simply taking prices as given). In an economy with only two individuals, however, such "competitive" or "price taking" behavior is not necessarily to be expected. Rather, individuals would employ their bargaining skills to get to the best possible allocation in the core. If individual 1 has greater bargaining power, for instance, the bargaining process would lead to an allocation that lies between A and B on the contract curve, whereas if person 2 has greater bargaining skills, we would expect to end up between A and C.

B: *Suppose, as in exercise 16.1, that the tastes for individuals 1 and 2 can be described by the utility functions $u^1 = x_1^{\alpha} x_2^{(1-\alpha)}$ and $u^2 = x_1^{\beta} x_2^{(1-\beta)}$ (where α and β both lie between 0 and 1).*

(a) *Derive the demands for x_1 and x_2 by each of the two individuals as a function of prices p_1 and p_2 (and as a function of their individual endowments).*

Answer: Solving the usual utility maximization problems (with income represented by the value of endowments), we get

$$x_1^1(p_1,p_2) = \frac{\alpha(p_1 e_1^1 + p_2 e_2^1)}{p_1} \text{ and } x_2^1(p_1,p_2) = \frac{(1-\alpha)(p_1 e_1^1 + p_2 e_2^1)}{p_2} \tag{16.13}$$

for individual 1 and

$$x_1^2(p_1,p_2) = \frac{\beta(p_1 e_1^2 + p_2 e_2^2)}{p_1} \text{ and } x_2^2(p_1,p_2) = \frac{(1-\beta)(p_1 e_1^2 + p_2 e_2^2)}{p_2} \tag{16.14}$$

for individual 2.

(b) *Let p_1^* and p_2^* denote equilibrium prices. Derive the ratio p_2^*/p_1^*.*

Answer: In equilibrium, the sum of the demands for good 1 has to be equal to the supply (or economy-wide endowment) of good 1. (The same has to be true for good 2, but we only have to solve one of the "demand equal to supply" equations because of Walras' Law.) Put differently, the equilibrium prices have to satisfy the condition that

$$x_1^1(p_1,p_2) + x_1^2(p_1,p_2) = e_1^1 + e_1^2. \tag{16.15}$$

Using the demand equations derived in part (a), we can write this as

$$\frac{\alpha(p_1 e_1^1 + p_2 e_2^1)}{p_1} + \frac{\beta(p_1 e_1^2 + p_2 e_2^2)}{p_1} = e_1^1 + e_1^2 \tag{16.16}$$

which can be solved for p_2/p_1 to give the equilibrium price ratio

$$\frac{p_2^*}{p_1^*} = \frac{(1-\alpha)e_1^1 + (1-\beta)e_1^2}{\alpha e_2^1 + \beta e_2^2}. \tag{16.17}$$

(c) *Derive the equilibrium allocation in the economy — i.e. derive the amount of x_1 and x_2 that each individual will consume in the competitive equilibrium (as a function of their endowments).*

Answer: The demand equations from part (a) can be re-written in terms of relative prices —

$$x_1^1(p_1,p_2) = \alpha e_1^1 + \alpha e_2^1\left(\frac{p_2}{p_1}\right) \text{ and } x_2^1(p_1,p_2) = (1-\alpha)e_1^1\left(\frac{p_1}{p_2}\right) + (1-\alpha)e_2^1 \tag{16.18}$$

for individual 1 and

$$x_1^2(p_1,p_2) = \beta e_1^2 + \beta e_2^2\left(\frac{p_2}{p_1}\right) \text{ and } x_2^2(p_1,p_2) = (1-\beta)e_1^2\left(\frac{p_1}{p_2}\right) + (1-\beta)e_2^2 \tag{16.19}$$

for individual 2. Substituting in our equilibrium price ratio from equation (16.17) (or its inverse when called for), these become

$$\begin{aligned} x_1^1 &= \alpha e_1^1 + \alpha e_2^1\left(\frac{(1-\alpha)e_1^1 + (1-\beta)e_1^2}{\alpha e_2^1 + \beta e_2^2}\right) \\ x_2^1 &= \frac{(1-\alpha)e_1^1(\alpha e_2^1 + \beta e_2^2)}{(1-\alpha)e_1^1 + (1-\beta)e_1^2} + (1-\alpha)e_2^1 \end{aligned} \tag{16.20}$$

for individual 1 and

$$x_1^2 = \beta e_1^2 + \beta e_2^2 \left(\frac{(1-\alpha)e_1^1 + (1-\beta)e_1^2}{\alpha e_2^1 + \beta e_2^2} \right)$$
$$x_2^2 = \frac{(1-\beta)e_1^2(\alpha e_2^1 + \beta e_2^2)}{(1-\alpha)e_1^1 + (1-\beta)e_1^2} + (1-\beta)e_2^2 \quad (16.21)$$

for individual 2.

(d) *Now suppose that $\alpha = \beta$ — i.e. tastes are the same for the two individuals. From your answer in (c), derive the equilibrium allocation to person 1.*

Answer: Replacing β in the equations of expression (16.20), we get

$$x_1^1 = \alpha e_1^1 + (1-\alpha)e_2^1 \left(\frac{e_1^1 + e_1^2}{e_2^1 + e_2^2} \right) = \alpha e_1^1 + (1-\alpha)e_2^1 \left(\frac{E_1}{E_2} \right) \quad (16.22)$$

and

$$x_2^1 = \alpha e_1^1 \left(\frac{e_2^1 + e_2^2}{e_1^1 + e_1^2} \right) + (1-\alpha)e_2^1 = \alpha e_1^1 \left(\frac{E_2}{E_1} \right) + (1-\alpha)e_2^1. \quad (16.23)$$

(e) *Does your answer to (d) satisfy the condition you derived in exercise 16.1B(b) for pareto efficient allocations (i.e. allocations on the contract curve)?*[1]

Answer: Yes. Plugging equation (16.22) in for x_1 in our expression for the contract curve, we get

$$\overline{x}_2(x_1) = \left(\frac{E_2}{E_1} \right) \left[\alpha e_1^1 + (1-\alpha)e_2^1 \left(\frac{E_1}{E_2} \right) \right] = \alpha e_1^1 \left(\frac{E_2}{E_1} \right) + (1-\alpha)e_2^1, \quad (16.24)$$

which is equal to the equilibrium allocation of good 2 to individual 1 derived in expression (16.23).

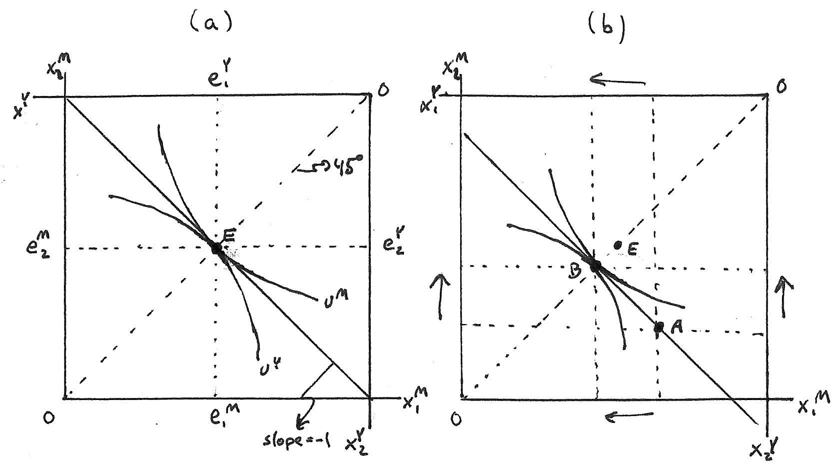
[1] You should have derived the equation describing the contract curve as $\overline{x}_2(x_1) = (E_2/E_1)x_1$.

16.3 *Suppose you and I have the same homothetic tastes over x_1 and x_2, and our endowments of the two goods are $E^M = (e_1^M, e_2^M)$ for me and $E^Y = (e_1^Y, e_2^Y)$ for you.*

A: *Suppose throughout that, when $x_1 = x_2$, our MRS is equal to -1.*

(a) *Assume that $e_1^M = e_2^M = e_1^Y = e_2^Y$. Draw the Edgeworth box for this case and indicate where the endowment point $E = (E^M, E^Y)$ lies.*

Answer: This is done in panel (a) of Graph 16.4 where the Edgeworth Box is drawn as a square (because the overall endowments of x_1 are equal to those of x_2), with the endowment bundle E located in the center (since we are endowed with equal amounts of everything.)



Graph 16.4: Equal Endowments and Same Tastes

(b) *Draw the indifference curves for both of us through E. Is the endowment allocation efficient?*

Answer: This is also done in panel (a). Since our endowment lies on the 45-degree line and our MRS along the 45-degree line is always -1, the indifference curves through E are tangent to one another. This implies that the endowment allocation is efficient — because there is no lens shape between our indifference curves that would give us room to make both of us better off (or at least one better off without making the other worse off).

(c) *Normalize the price of x_2 to 1 and let p be the price of x_1. What is the equilibrium price p^*?*

Answer: The equilibrium price must pass through E and induce budget constraints for me and you such that both of us optimize at the same point within the Edgeworth Box. In this case, the only way to do this is to let $p^* = 1$ — resulting in a budget with slope -1 through E. Since the MRS at E is -1 for both of us, we will both choose to remain at our endowment bundle at this price. This is also illustrated in panel (a) of Graph 16.4.

(d) *Where in the Edgeworth Box is the set of all efficient allocations?*

Answer: The set of all efficient allocations lies on the 45-degree line — because along the 45 degree line, our MRS's are equal to 1 and thus equal to one another, implying indifference curves that are tangent to one another. This is the contract curve.

(e) *Pick another efficient allocation and demonstrate a possible way to re-allocate the endowment among us such that the new efficient allocation becomes an equilibrium allocation supported by an equilibrium price. Is this equilibrium price the same as p^* calculated in (c)?*

Answer: This is illustrated in panel (b) of Graph 16.4 where we consider the equilibrium if the endowment is redistributed so that it moves from E to A. The only place where the indifference curves are tangent to one another is on the 45-degree line where their slope is -1. Thus, the new equilibrium price must again be 1 — and the new budget must pass through the new endowment A as drawn. This will cause us to trade from A to B along the budget line with slope $p=1$ — with me giving up x_1 to get more x_2 and you giving up an equal amount of x_2 to get more x_1 (as indicated by the arrows on the axes.)

B: *Suppose our tastes can be represented by the CES utility function* $u(x_1,x_2)=\left(0.5x_1^{-\rho}+0.5x_2^{-\rho}\right)^{-1/\rho}$.

(a) *Let p be defined as in A(c). Write down my and your budget constraint (assuming again endowments $E^M=(e_1^M,e_2^M)$ for me and $E^Y=(e_1^Y,e_2^Y)$.)*

Answer: The value of or endowments has to be equal to the value of what we consume. For me, this implies

$$pe_1^M+e_2^M=px_1^M+x_2^M, \tag{16.25}$$

and for you it means

$$pe_1^Y+e_2^Y=px_1^Y+x_2^Y. \tag{16.26}$$

(b) *Write down my optimization problem and derive my demand for x_1 and x_2.*

Answer: My optimization problem is then

$$\max_{x_1,x_2}\left(0.5x_1^{-\rho}+0.5x_2^{-\rho}\right)^{-1/\rho} \text{ subject to } pe_1^M+e_2^M=px_1+x_2 \tag{16.27}$$

where, for now, we suppress the M superscripts on the x variables. Setting up the Lagrangian and solving in the usual way, we get

$$x_1^M=\frac{pe_1^M+e_2^M}{p+p^{1/(\rho+1)}} \text{ and } x_2^M=\frac{p^{1/(\rho+1)}e_1^M+e_2^M}{p+p^{1/(\rho+1)}}. \tag{16.28}$$

(c) *Similarly, derive your demand for x_1 and x_2.*

Answer: Repeating the steps in the previous part for you, we get

$$x_1^Y=\frac{pe_1^Y+e_2^Y}{p+p^{1/(\rho+1)}} \text{ and } x_2^Y=\frac{p^{1/(\rho+1)}e_1^Y+e_2^Y}{p+p^{1/(\rho+1)}}. \tag{16.29}$$

(d) *Derive the equilibrium price. What is that price if, as in part A, $e_1^M=e_2^M=e_1^Y=e_2^Y$?*

Answer: In equilibrium, the price has to be such that demand is equal to supply in both markets. Because of Walras' Law, we only have to solve for p in one of the markets though — and either one will work. Choosing the market for x_1, it must therefore be the case that $x_1^M+x_1^Y=e_1^M+e_1^Y$ or, plugging in our demands from the previous parts,

$$\frac{pe_1^M+e_2^M}{p+p^{1/(\rho+1)}}+\frac{pe_1^Y+e_2^Y}{p+p^{1/(\rho+1)}}=e_1^M+e_1^Y. \tag{16.30}$$

Multiplying both sides by the denominators on the left hand side, we get

$$pe_1^M+e_2^M+pe_1^Y+e_2^Y=\left(e_1^M+e_1^Y\right)\left(p+p^{1/(\rho+1)}\right) \tag{16.31}$$

and, rearranging terms,

$$p\left(e_1^M+e_1^Y\right)+\left(e_2^M+e_2^Y\right)=p\left(e_1^M+e_1^Y\right)+p^{1/(\rho+1)}\left(e_1^M+e_1^Y\right). \tag{16.32}$$

Subtracting out the first term on each side and then solving for p, we get

$$p^*=\left(\frac{e_2^M+e_2^Y}{e_1^M+e_1^Y}\right)^{(\rho+1)}. \tag{16.33}$$

When $e_1^M = e_2^M = e_1^Y = e_2^Y$, this simplifies to $p^* = 1$ — consistent with what we did in part A.

(e) *Derive the set of pareto efficient allocations assuming* $e_1^M = e_2^M = e_1^Y = e_2^Y$. *Can you see why, regardless of how we might redistribute endowments, the equilibrium price will always be* $p = 1$*?*

Answer: Let $e = e_1^M = e_2^M = e_1^Y = e_2^Y$. Then the economy is endowed with $2e$ of each good, which implies that, for any allocation (x_1^M, x_2^M) that I get, what's left over for you is $(2e - x_1^M), (2e - x_2^M)$. The pareto efficient set of (x_1^M, x_2^M) (with its implied consumption levels for you) is then defined as the set where our MRS's are equal to one another. The MRS for me at a bundle (x_1^M, x_2^M) is

$$MRS^M = -\frac{\partial u(x_1^M, x_2^M)/\partial x_1}{\partial u(x_1^M, x_2^M)/\partial x_2} = -\left(\frac{x_2^M}{x_1^M}\right)^{(\rho+1)} \tag{16.34}$$

and the MRS for you at the left-over bundle $((2e - x_1^M), (2e - x_2^M))$ is

$$MRS^Y = -\frac{\partial u((2e - x_1^M), (2e - x_2^M))/\partial x_1}{\partial u((2e - x_1^M), (2e - x_2^M))/\partial x_2} = -\left(\frac{2e - x_2^M}{2e - x_1^M}\right)^{(\rho+1)}. \tag{16.35}$$

Setting MRS^M equal to MRS^Y and solving for x_2^M, we get

$$x_2^M = x_1^M; \tag{16.36}$$

i.e. the contract curve is a straight line with slope 1 and intercept 0 — the 45-degree line in the Edgeworth Box. Since all efficient allocations happen on this line, and since equilibria are efficient, we know that any competitive equilibrium lies on the 45-degree line. This further implies that, when we plug $x_1^M = x_2^M$ and $2e - x_1^M = 2e - x_2^M$ into the equations for marginal rates of substitution, we get $MRS^M = -1 = MRS^Y$ in any equilibrium, which can only hold if the slope of the budget is -1. And that can only be true if $p = 1$.

16.4 *Suppose, as in exercise 16.3, that you and I have the same homothetic tastes over x_1 and x_2, and our endowments of the two goods are $E^M = (e_1^M, e_2^M)$ for me and $E^Y = (e_1^Y, e_2^Y)$ for you.*

A: *Suppose also, again as in exercise 16.3, that whenever $x_1 = x_2$, $MRS = -1$.*

(a) *First, consider the case where $e_1^M + e_1^Y = e_2^M + e_2^Y$.* True or False*: As long as the two goods are not perfect substitutes, the contract curve consists of the 45 degree line within the Edgeworth Box.*

Answer: This is true. When $e_1^M + e_1^Y = e_2^M + e_2^Y$, the Edgeworth box is a square — with the 45 degree line running from one corner to the other. Thus, the 45-degree line is a ray emanating from both origins — and since we have the same tastes, it must be that we have the same homothetic tastes with $MRS = -1$ along the 45 degree line, we have the same MRS at all bundles on the 45 degree line. Thus, our indifference curves are tangent to one another.

(b) *What does the contract curve look like for perfect substitutes?*

Answer: When x_1 and x_2 are perfect substitutes for both of us and we share the same tastes, then, for any bundle in the Edgeworth box, your indifference curve that passes through that bundle lies on top of my indifference curve that passes through that bundle. Thus, our indifference curves are "tangent" to one another at every point in the Edgeworth Box — which implies the contract curve is the entire Edgeworth Box.

(c) *Suppose next, and for the rest of part A of this question, that $e_1^M + e_1^Y > e_2^M + e_2^Y$. Where does the contract curve now lie? Does your answer depend on the degree of substitutability between the two goods?*

Answer: The contract curve will now lie on the ray that connects the two origins of the Edgeworth Box. This is because we know that our tastes are homothetic — and thus our indifference curves have the same slope along any ray from the origin. The ray connecting the two origins is the same ray (with the same slope) for both of us — thus our MRS along the ray is the same — which implies our indifference curves area tangent to one another along the ray.

(d) *Pick some arbitrary bundle (on either side of the 45-degree line) in the Edgeworth Box and illustrate an equilibrium price. Where will the equilibrium allocation lie?*

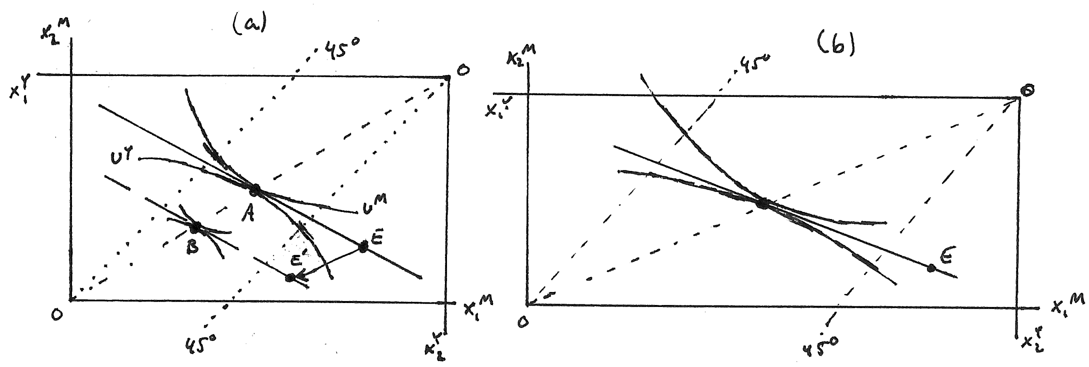
Answer: This is illustrated in panel (a) of Graph 16.5 where the box is now a rectangle with longer base than height (because $e_1^M + e_1^Y > e_2^M + e_2^Y$). The 45 degree lines are now no longer the same for the two of us — and are indicated separately. Along these lines, our $MRS = -1$ — which means that our indifference curves will be shallower in between the 45 degree lines. Since all the efficient allocations lie on the ray connecting the two origins, the equilibrium will lie on that ray — and thus must involve a price for x_1 (relative to x_2 that is now lower than before.) With endowment E, the equilibrium allocation is then A.

(e) *If you move the endowment bundle, will the equilibrium price change? What about the equilibrium allocation?*

Answer: If we move the endowment bundle in any way other than to move it along the budget formed by the current price in panel (a) of Graph 16.5, the equilibrium allocation will change from A but the equilibrium price will be unchanged. In the graph, we change E to E' — resulting in a new equilibrium allocation B. But since all the efficient allocations lie on the ray connecting the two origins, the new equilibrium must again lie on that same ray. And since the MRS for you and me are the same all along that ray, it must mean that the budget line which is tangent to our indifference curves has the same slope as before — i.e. the equilibrium price is unchanged.

(f) True or False*: As the economy's endowment of x_1 grows relative to its endowment of x_2, p falls.*

Answer: This is true. In panel (b) of Graph 16.5, we stretch the Edgeworth box (by increasing the x_1 endowments) — which stretches the area between the 45 degree lines. Our MRS on our 45 degree lines will still be -1, but now we have to slide down further along an indifference curve to get to the ray that connects the two origins in our graph. This implies that the slopes of the indifference curves along that ray become shallower as we stretch the Edgeworth box — which in turn implies that any budget which makes bundles along that ray optimal must become shallower. And that implies that the equilibrium price is falling as the box is stretched. This should make some intuitive sense: As the box is stretched in this



Graph 16.5: Increasing x_1 endowments relative to x_2 endowments

way, good x_2 becomes more scarce relative to good x_1 — which should make x_1 relatively less valuable than x_2.

(g) True or False*: As the goods become more complementary, the equilibrium price falls in an economy with more x_1 endowment than x_2 endowment.*

Answer: This is also true. As the goods become more complementary, the slope of an indifference curve changes more rapidly as we slide down from the point on the 45 degree line. Thus, the more complementary the goods, the shallower the slope of the indifference curves by the time we reach the ray that connects the two origins in our Edgeworth Box in Graph 16.5. Since the equilibrium occurs along that ray, this implies that the budget line has to get shallower as the goods become more complementary — which in turn implies that the equilibrium price for p_1 must fall.

B: *Suppose, as in exercise 16.3, that our tastes can be represented by the CES utility function* $u(x_1, x_2) = \left(0.5x_1^{-\rho} + 0.5x_2^{-\rho}\right)^{-1/\rho}$.

(a) *Derive the contract curve and compare it to your graphical answer in part A(c). Does the shape of the contract curve depend on the elasticity of substitution?*

Answer: Let $E_1 = e_1^M + e_1^Y$ represent the economy's endowment of x_1, and let $E_2 = e_2^M + e_2^Y$ be the economy's endowment of x_2. For any allocation (x_1^M, x_2^M) that I get, what's left over for you is then $((E_1 - x_1^M), (E_2 - x_2^M))$. The pareto efficient set of (x_1^M, x_2^M) (with its implied consumption levels for you) is then defined as the set where our MRS's are equal to one another. The MRS for me at a bundle (x_1^M, x_2^M) is

$$MRS^M = -\frac{\partial u(x_1^M, x_2^M)/\partial x_1}{\partial u(x_1^M, x_2^M)/\partial x_2} = -\left(\frac{x_2^M}{x_1^M}\right)^{(\rho+1)} \tag{16.37}$$

and the MRS for you at the left-over bundle $((E_1 - x_1^M), (E_2 - x_2^M))$ is

$$MRS^Y = -\frac{\partial u((E_1 - x_1^M), (E_2 - x_2^M))/\partial x_1}{\partial u((E_1 - x_1^M), (E_2 - x_2^M))/\partial x_2} = -\left(\frac{E_2 - x_2^M}{E_1 - x_1^M}\right)^{(\rho+1)}. \tag{16.38}$$

Setting MRS^M equal to MRS^Y and solving for x_2^M, we get

$$x_2^M = \left(\frac{E_2}{E_1}\right) x_1^M = \left(\frac{e_2^M + e_2^Y}{e_1^M + e_1^Y}\right) x_1^M; \tag{16.39}$$

i.e. the contract curve is a straight line with slope (E_2/E_1) and intercept 0. When $E_1 = E_2$, this is simply the 45 degree line that connects the two origins in the Edgeworth Box. When $E_2 < E_1$ — i.e. when the economy has more endowment of x_1 than of x_2, the slope of the contract curve becomes shallower (as E_2/E_1 falls) — and it once again connects the two origins of the Edgeworth Box. Note that ρ does not appear in our equation for the contract curve — which implies that the elasticity of substitution does not affect the shape of the contract curve in this example.

(b) *If you have not done so already in exercise 16.3, derive my and your demand functions, letting p denote the price of x_1 and letting the price of x_2 equal 1. Then derive the equilibrium price.*

Answer: In exercise 16.3, we derived my demands as

$$x_1^M = \frac{pe_1^M + e_2^M}{p + p^{1/(\rho+1)}} \text{ and } x_2^M = \frac{p^{1/(\rho+1)}e_1^M + e_2^M}{p + p^{1/(\rho+1)}}, \tag{16.40}$$

and your demands as

$$x_1^Y = \frac{pe_1^Y + e_2^Y}{p + p^{1/(\rho+1)}} \text{ and } x_2^Y = \frac{p^{1/(\rho+1)}e_1^Y + e_2^Y}{p + p^{1/(\rho+1)}}. \tag{16.41}$$

We then derived the equilibrium price as

$$p^* = \left(\frac{e_2^M + e_2^Y}{e_1^M + e_1^Y}\right)^{(\rho+1)}. \tag{16.42}$$

(c) *Does the equilibrium price depend on* how *the overall endowment in the economy is distributed?*

Answer: No, the equilibrium price only depends on how large the x_2 endowment is relative to the x_1 endowment for the economy as a whole — not how this endowment is distributed between us. This is consistent with what we derived in part A of the question where we argued that a redistribution of endowments does not change the equilibrium price but does change the equilibrium allocation. (This was illustrated in panel (a) of Graph 16.5.)

(d) *What happens to the equilibrium price as the economy's endowment of x_1 grows? Compare this to your intuitive answer in A(f).*

Answer: As the economy's endowment of x_1 grows, the denominator in our equation for p^* grows while the numerator stays the same — which implies that p^* falls. This is consistent with what we concluded in A(f) — as x_2 becomes scarcer relative to x_1, p^* falls.

(e) *Suppose $e_1^M + e_1^Y = e_2^M + e_2^Y$. Does the equilibrium price depend on the elastic of substitution?*

Answer: When $e_1^M + e_1^Y = e_2^M + e_2^Y$, our equation for p^* simply becomes $p^* = 1$. Since ρ does not enter our equation for p^*, we can conclude that p^* does not depend on the elasticity of substitution in this example.

(f) *Suppose $e_1^M + e_1^Y > e_2^M + e_2^Y$. Does this change your answer to (e)?*

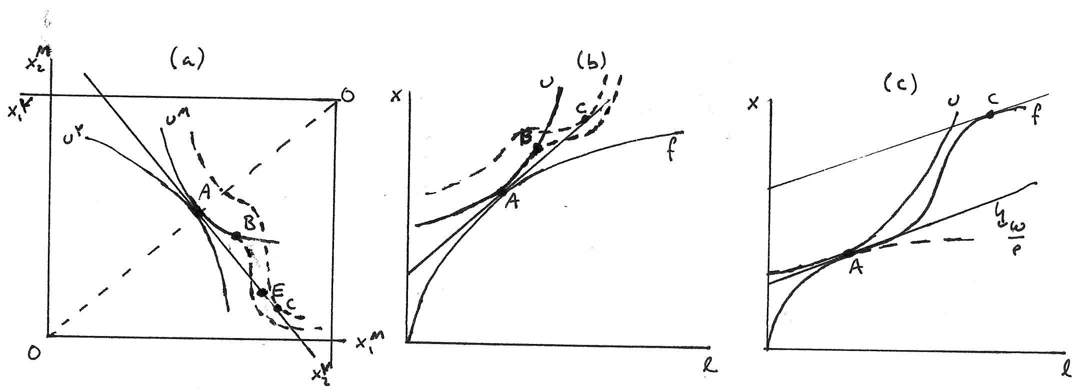
Answer: Now the numerator and denominator in our equation for p^* do not cancel — which implies that p^* *does* depend on ρ and thus on the elasticity of substitution. With $e_1^M + e_1^Y > e_2^M + e_2^Y$, the term $(e_2^M + e_2^Y)/(e_1^M + e_1^Y)$ is less than 1. Thus, as ρ increases, p^* falls — implying a shallower budget constraint in our Edgeworth box. This should make sense: As ρ increases, the two goods become more complementary — which implies that the MRS changes more quickly as you slide down an indifference curve. Thus, by the time you get to A in panel (a) of Graph 16.5, the slope of the indifference curve will be shallower than it would be for a lower ρ — and it is at A that the budget has to be tangent.

16.5 *In this exercise we explore some technical aspects of general equilibrium theory in exchange economies and Robinson Crusoe economies. Unlike in other problems, parts A and B are applicable to both those focused on A-Section material and those focused on B-Section material. Although the insights are developed in simple examples, they apply more generally in much more complex models.*

A: The role of convexity in Exchange Economies: *In each part below, suppose you and I are the only individuals in the economy, and pick some arbitrary allocation E in the Edgeworth Box as our initial endowment. Assume throughout that your tastes are convex and that the contract curve is equal to the line connecting the lower left and upper right corners of the box.*

(a) *Begin with a depiction of an equilibrium. Can you introduce a non-convexity into my tastes such that the equilibrium disappears (despite the fact that the contract curve remains unchanged?)*

Answer: This is done in panel (a) of Graph 16.6 where the equilibrium budget passes through *E* and is tangent to both solid (and convex) indifference curves at *A*. Thus, *A* is an equilibrium allocation. However, if I permit my indifference curves to have non-convexities, I can maintain the tangency at *A* but lose the equilibrium at *A* by having my indifference curve continue along the dashed curve beginning at *B* and moving right. Notice that *A* is still efficient — but, when faced with the budget line that previously supported *A* as an equilibrium, I now no longer optimize at *A* but rather at *C* which lies on a higher dashed (and non-convex) indifference curve.



Graph 16.6: Convexity Assumptions in General Equilibrium

(b) True or False*: Existence of a competitive equilibrium in an exchange economy cannot be guaranteed if tastes are allowed to be non-convex.*

Answer: This is true, as we have just shown.

(c) *Suppose an equilibrium does exist even though my tastes exhibit some non-convexity.* True or False*: The first welfare theorem holds even when tastes have non-convexities.*

Answer: The allocation *A* in panel (a) of Graph 16.6 would continue to be an equilibrium so long as the non-convexity that is introduced is not sufficiently pronounced so as to cause the indifference curve that is tangent at *A* to cross the budget line. Thus, had we drawn the non-convexity in a less pronounced manner, the budget line through *A* and *E* would still have been such that I optimize at *A* — and thus *A* would have continued to be an equilibrium. We can conclude that, *if an equilibrium exists in the presence of non-convex tastes*, then it will indeed still be efficient. The first welfare theorem therefore holds in the presence of non-convexities.

(d) True or False*: The second welfare theorem holds even when tastes have non-convexities.*

Answer: The second welfare theorem says that any efficient allocation can be an equilibrium allocation so long as endowments can be appropriately redistributed. We have just shown in panel (a) of Graph 16.6 an example of an efficient allocation A that cannot be supported as an equilibrium no matter where we move the endowment. This is because, in order to support A as an equilibrium, the budget line *has to be* the line that is draw in the graph — because that is the only budget that will cause *you* to optimize at A. But that line crosses the dashed extension of my indifference curve that is tangent at A — implying that I will not optimize at A if my tastes are the non-convex kind in the graph. Thus, we have identified a case where an efficient allocation cannot become an equilibrium allocation regardless of where we put the endowment. The statement is therefore false — the second welfare theorem may not hold when tastes have non-convexities.

B: The role of convexity in Robinson Crusoe Economies: *Consider a Robinson Crusoe economy. Suppose throughout that there is a tangency between the worker's indifference curve and the production technology at some bundle A.*

(a) *Suppose first that the production technology gives rise to a convex production choice set. Illustrate an equilibrium when tastes are convex. Then show that A may no longer be an equilibrium if you allow tastes to have non-convexities even if the indifference curve is still tangent to the production choice set at A.*

Answer: This is illustrated in panel (b) of Graph 16.6. The solid indifference curve is tangent to the convex production choice set at A, with both tangent to the isoprofit/budget line (that has slope w/p). When viewed as a budget line, the worker is doing the best he can by choosing A, and when viewed as an isoprofit line, the firm is doing the best it can at A, with the wage/price ration w/p supporting A as an equilibrium. But we can take the same indifference curve, keep it tangent to the budget at A, but then change its shape from B on to take the shape of the dashed curve. When we do this, we introduce a non-convexity — and, as a result, the worker is no longer doing the best he can by choosing A when confronting the budget formed by the former equilibrium wage/price ratio. In particular, the worker would now be better off optimizing at C — but that lies outside the production frontier and is therefore not an equilibrium. Thus, by introducing the non-convexity, A ceases to be a competitive equilibrium in this economy.

(b) *Next, suppose again that tastes are convex but now let the production choice set have non-convexities. Show again that A might no longer be an equilibrium (even though the indifference curve and production choice set are tangent at A).*

Answer: This is shown in panel (c) of Graph 16.6 where the production frontier f is tangent to the indifference curve u — thus making A an efficient production plan. The budget that is tangent to both the production frontier and the indifference curve at A — with slope w/p — causes the worker to optimize at A where his indifference curve is tangent. However, the firm would not be optimizing at A — because it can reach a higher isoprofit curve and would maximize profit at C instead. The production plan A would be optimal for the firm (and would thus be an equilibrium) if the production frontier took on the dashed shape following A — i.e. if the production choice set were convex. But A is lost as an equilibrium because of the non-convexity of the solid production choice set.

(c) True or False*: A competitive equilibrium may not exist in a Robinson Crusoe economy that has non-convexities in either tastes or production.*

Answer: This is true as shown in the previous two parts.

(d) True or False*: The first welfare theorem holds even if there are non-convexities in tastes and/or production technologies.*

Answer: This is true. The non-convexities may cause there to be no equilibrium, but *if there is an equilibrium*, it will again have the feature that the indifference curve is tangent to the production frontier at that point — which will make it efficient. You can see this in panels (b) and (c) if you imagine the non-convexity that was introduced as being less pronounced. In panel (b), A would remain an equilibrium so long as the dashed portion of the indifference curve does not cross the budget line to the right of B — which is certainly possible even if there were a less pronounced non-convexity. And that equilibrium would be efficient. Similarly, in panel (c) you can imagine a non-convexity in the production choice set either to the left of A or some distance to the right of A — and you can imagine such a non-convexity to not be sufficiently pronounced so as to cross the isoprofit line that is tangent at A. In that case, *A* would remain as an equilibrium — and it would be efficient. Thus, the first welfare theorem holds — every equilibrium (that exists) is indeed efficient.

(e) True or False*: The second welfare theorem holds regardless of whether there are non-convexities in tastes or production.*

Answer: This is false. In panel (b) of the graph, we have shown an efficient point *A* that cannot be an equilibrium because the budget line that must support it crosses the indifference curve that is tangent at *A*. In panel (c) we have shown another efficient point *A* that cannot be supported as an equilibrium because the isoprofit line that is needed to support it as an equilibrium crosses the production frontier because of a non-convexity. We have therefore shown that, when there are non-convexities, there may be efficient outcomes that cannot be supported as equilibria.

(f) *Based on what you have done in parts A and B, evaluate the following: "Non-convexities may cause a non-existence of competitive equilibria in general equilibrium economies, but if an equilibrium exists, it results in an efficient allocation of resources. However, only in the absence of non-convexities can we conclude that there always exists some lump-sum redistribution such that any efficient allocation can also be an equilibrium allocation." (Note: Your conclusion on this holds well beyond the examples in this problem — for reasons that are quite similar to the intuition developed here.)*

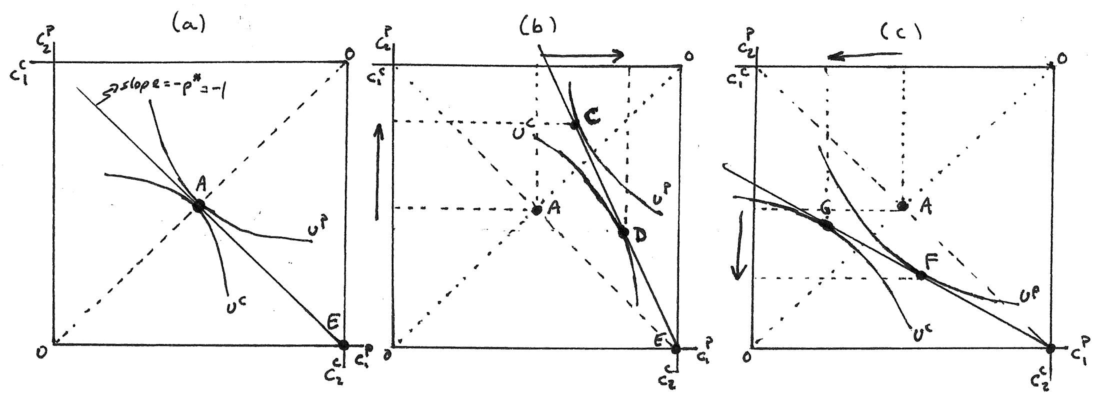
Answer: The statement is fully consistent with everything we have done in this exercise. We have shown — in both exchange and Robinson Crusoe economies — that non-convexities may lead to a non-existence of equilibria; that *if* equilibria exist, they will be efficient (i.e. the first welfare theorem holds); but not all efficient outcomes can be supported as equilibria (i.e. the second welfare theorem fails in the presence of non-convexities).

16.6 Everyday Application: *Children, Parents, Baby Booms and Baby Busts: Economists often think of parents and children trading with one another across time. When children are young, parents take care of children; but when parents get old, children often come to take care of their parents. We will think of this in a 2-period model in which children earn no income in period 1 and parents earn no income in period 2. For purposes of this problem, we will assume that parents have no way to save in period 1 for the future and children have no way to borrow from the future when they are in period 1. Thus, parents and children have to rely on one another.*

A: *Suppose that, during the periods when they earn income (i.e. period 1 for parents and period 2 for children), parents and children earn the same amount y. Suppose further that everyone has homothetic tastes with $MRS = -1$ when $c_1 = c_2$.*

(a) *Suppose first that there is one parent and one child. Illustrate an Edgeworth Box with current consumption c_1 on the horizontal and future consumption c_2 on the vertical axes. Indicate where the endowment allocation lies.*

Answer: This is illustrated in panel (a) of Graph 16.7 where the Edgeworth box is a square since overall resources for consumption are the same now and in the future. The endowment allocation E lies at the bottom right corner where the parent has all consumption now and the child has all consumption in the future.



Graph 16.7: Baby Booms and Baby Busts

(b) *Given that everyone has homothetic tastes (and assuming that consumption now and in the future are not perfect substitutes), where does the region of mutually beneficial trades lie?*

Answer: If tastes are homothetic and not perfect substitutes, then consumption now and consumption in the future are essential goods for parents and children. (In other words, their indifference curves do not cross the axes.) Thus, all allocations inside the box are preferred to allocations on the axes in general — and to the endowment allocation in particular. Thus, the entire inside of the Edgeworth Box is the mutually beneficial regions.

(c) *Let p be the price of current consumption in terms of future consumption (and let the price of future consumption be normalized to 1.) Illustrate a competitive equilibrium.*

Answer: This is also done in panel (a) of Graph 16.7. Since our tastes are homothetic with $MRS = -1$ along the 45-degree line, we know that all efficient allocations (where parent and child indifference curves are tangent to one another) lie on the 45-degree line that connects the two corners of the box. Since the two further have symmetric endowments, the equilibrium will be the midpoint of the box, labeled A, where the equilibrium price of -1 forms the budget that is tangent to both indifference curves at A.

(d) *Suppose that there are now two identical children and one parent. Keep the Edgeworth Box the same dimensions as in (a). However, because there are now two children, every action on* *the a child's part must be balanced by twice the opposite action from the one parent that is being modeled in the Edgeworth Box. Does p go up or down? (*Hint*: An equilibrium is now characterized by the parent moving twice as far on the equilibrium budget as each child.)*

Answer: This is illustrated in panel (b) of Graph 16.7 where the equilibrium price increases and forms a steeper budget line. The original equilibrium allocation *A* is no longer an equilibrium because the one parent's actions do not offset the two children's. Instead, and equilibrium now has the feature that the parent has to move twice as far on the equilibrium budget line as the children — because there are two children and only one parent. Relative to how far we moved along the budget line to get to the original equilibrium *A*, we therefore now have to find a way to move the parent farther while moving the child not as far until we create a sufficient gap such that the child only moved half the distance of the parent. By increasing the slope of the budget, we accomplish precisely that because the steeper portions of the parent's indifference curve lie above the 45 degree line while the steeper portions of the child's indifference curves lie below the 45 degree line. The graph then illustrates a sufficiently high price such that *C* — the parent's optimum — is twice as far down the budget line as *D* — the child's optimum.

(e) *What happens to child consumption now and parent consumption in the future?*

Answer: As indicated by the arrows in panel (b) of the graph, parent consumption in the future increases while child consumption now decreases.

(f) *Instead, suppose that there are two parents and one child. Again show what happens to the equilibrium price p.*

Answer: This is illustrated in panel (c) of Graph 16.7 where the reverse happens. Since there are now two parents and only 1 child, an equilibrium requires that parents move only half the distance on the equilibrium budget line as the child. Thus, relative to *A*, we have to reduce the movement along the budget line for parents and increase it for children. By making the budget line shallower, we do precisely that — because the shallower parts of the parental indifference curves lie below the 45 degree line while the shallower parts of child indifference curves lie above the 45 degree line. (Note that *G* in the graph refers to the child's allocation and *F* refers to the parents' allocation.)

(g) *What happens to child consumption now and parent consumption in the future?*

Answer: As indicated by the arrows in panel (c) of the graph, child consumption now increases and parent consumption in the future falls.

(h) *Would anything have changed in the original one-child/one-parent equilibrium had we assumed two children and two parents instead?*

Answer: No. So long as we increase both sides of the "market" by the same factor, the competitive equilibrium price remains unchanged.

(i) *While it might be silly to apply a competitive model to a single family, we might interpret the model as representing generations that compete for current and future resources. Based on your analysis above, will parents enjoy a better retirement if their children were part of a baby boom or a baby bust? Why?*

Answer: The model suggests parents will enjoy a better retirement if their children were part of a baby boom. This is because children have to compete for parental resources that are scarcer during baby booms — and thus will end up paying a higher price in terms of future support for their parents in order to obtain current resources.

(j) *Will children be more spoiled if they are part of a baby boom or a baby bust? Why?*

Answer: Children will be more spoiled if they are part of a baby bust. In this case, children are scarce relative to parents — which implies that parents have to compete for the support of their children. Thus, they will spoil them now.

(k) *Consider two types of government spending: (1) spending on social security benefits for retirees, and (2) investments in a clean environment for future generations. When would this model predict will the environment do better: During baby booms or during baby busts?*

Answer: The model would predict that the environment will do better during baby busts — because it is then that children are scarce and parents compete to invest in their children's generation to assure their own retirement security.

B: *Suppose the set-up is as described in A and A(a), with* $y = 100$*, and let tastes be described by the utility function* $u(c_1, c_2) = c_1 c_2$.

(a) *Is it true that, given these tastes, the entire inside of the Edgeworth Box is equal to the area of mutually beneficial allocations relative to the endowment allocation?*

Answer: Yes. This is because utility is equal to zero for both if either c_1 or c_2 are zero — which they are in the endowment allocations. For any positive combination of c_1 and c_2 — i.e. for any allocation inside the Edgeworth Box — utility is positive.

(b) *Let* p *be defined as in A(c). Derive the parent and child demands for* c_1 *and* c_2 *as a function of* p.

Answer: The parental optimization problem is then

$$\max_{c_1, c_2} c_1 c_2 \text{ subject to } 100p + 0 = pc_1 + c_2 \tag{16.43}$$

while the child optimization problem is

$$\max_{c_1, c_2} c_1 c_2 \text{ subject to } 0 + 100 = pc_1 + c_2. \tag{16.44}$$

Solving these in the usual way, we get demands of

$$\begin{gathered} c_1^P = 50 \text{ and } c_2^P = 50p \text{ for parents, and} \\ c_1^C = \frac{50}{p} \text{ and } c_2^C = 50 \text{ for children.} \end{gathered} \tag{16.45}$$

(c) *Derive the equilibrium price* p^* *in the case where there is one parent and one child.*

Answer: Setting demand for c_1 (by the one parent plus the one child) equal to supply (which is 100), we get the equation

$$50 + \frac{50}{p} = 100 \tag{16.46}$$

which solves to give us $p^* = 1$.

(d) *What is the equilibrium allocation of consumption across time between parent and child?*

Answer: Plugging $p = 1$ into our demand equations, we get equilibrium allocations of $(c_1^P, c_2^P) = (50, 50)$ for the parent and $(c_1^C, c_2^C) = (50, 50)$ for the child; i.e. consumption is fully equalized between parent and child across time.

(e) *Suppose there are 2 children and one parent. Repeat (c) and (d).*

Answer: The economy now has an endowment of 100 now and 200 in the future, and it has twice the child demand as before. Demand for current consumption being equal to supply then implies

$$50 + 2\left(\frac{50}{p}\right) = 100 \tag{16.47}$$

which solves to give us $p^* = 2$. Thus, the fact that consumption now has become relatively more scarce has increased its equilibrium price from 1 to 2. Plugging this into the demand equations for parents and children, we get $(c_1^P, c_2^P) = (50, 100)$ for the parent and $(c_1^C, c_2^C) = (25, 50)$ for the children. Thus children lose current consumption and parents gain future consumption. (The fact that parental consumption now and child consumption in the future is unchanged results from the particular elasticity of substitution that is present in the Cobb-Douglas utility function.)

(f) *Suppose there are 2 parents and one child. Repeat (c) and (d).*

Answer: The economy now has an endowment of 200 now and 100 in the future, and it has twice the parent demand as before. Demand for current consumption being equal to supply then implies

$$2(50) + \frac{50}{p} = 200 \tag{16.48}$$

which solves to give us $p^* = 1/2$. Thus, the fact that consumption now has become relatively less scarce has lowered its equilibrium price from 1 to 1/2. Plugging this into the demand equations for parents and children, we get $(c_1^P, c_2^P) = (50, 25)$ for the parent and $(c_1^C, c_2^C) = (100, 50)$ for the children. Thus children gain current consumption and parents lose future consumption. (Again, the fact that parental consumption now and child consumption in the future is unchanged results from the particular elasticity of substitution that is present in the Cobb-Douglas utility function.)

(g) *Suppose there are 2 children and 2 parents. Repeat (c) and (d).*

Answer: The economy now has an endowment of 200 now and 200 in the future, and it has twice the child demand as well as twice the parent demand as before. Demand for current consumption being equal to supply then implies

$$2(50) + 2\left(\frac{50}{p}\right) = 100 \tag{16.49}$$

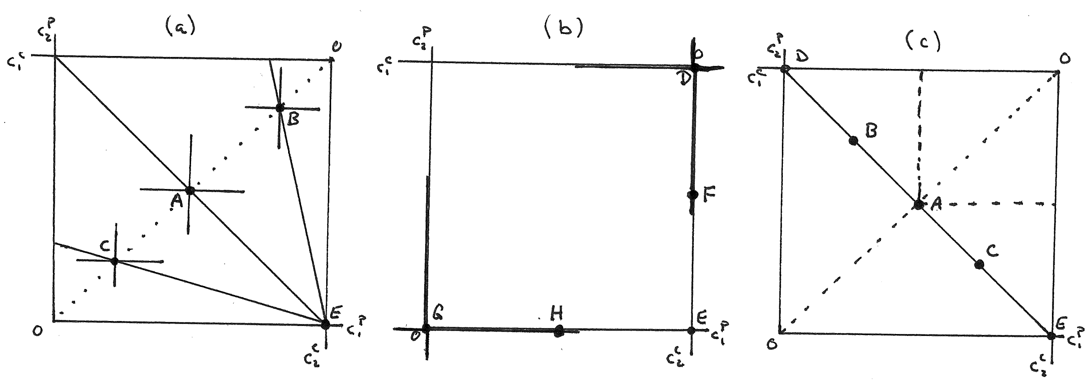
which solves to give us $p^* = 1$. Thus, nothing changes from the case where there was a single child and a single parent — with both parents and children getting an equilibrium allocation of (50,50).

16.7 Everyday Application: *Parents, Children and the Degree of Substitutability across Time: Consider again exactly the same scenario as in exercise 16.6.*

A: *This time, however, suppose that parent and child tastes treat consumption now and consumption in the future as perfect complements.*

(a) *Illustrate in an Edgeworth Box an equilibrium with a single parent and a single child.*

Answer: Perhaps the most obvious equilibrium is the one with price equal to 1 and thus a budget line that runs from E in the lower right corner to the upper left corner of the box — with the equilibrium allocation at A, pictured in panel (a) of Graph 16.8.



Graph 16.8: Parents and Children: Part 1

(b) *Is the equilibrium you pictured in (a) the only equilibrium? If not, can you identify the set of all equilibrium allocations?*

Answer: It is not the only equilibrium — in fact, panel (a) of Graph 16.8 picture two others, with allocations at B and at C. Because of the sharp corners on indifference curves for perfect complements, any budget line with negative slope can be fit to any "tangency" of the two indifference curves on the 45 degree line. Thus, all allocations on the 45 degree line have some budget line that passes through the endowment allocation E *and* is "tangent" to both indifference curves on that point of the 45 degree line. The entire 45-degree line in the box is therefore the set of possible equilibrium allocations.

(c) *Now suppose that there were two children and one parent. Keep the Edgeworth Box with the same dimensions but model this by recognizing that, on any equilibrium budget line, it must now be the case that the parent moves twice as far from the endowment E as the child (since there are two children and thus any equilibrium action by a child must be half the equilibrium action by the parent). Are any of the equilibrium allocations for parent and child that you identified in (b) still equilibrium allocations? (*Hint*: Consider the corners of the box.)*

Answer: For any budget line that intersects the 45-degree line inside the box, both parent and child will optimize on the 45 degree line. But with two children and one parent, that cannot be an equilibrium — because the parent's action must be twice the children's in the opposite direction in order for demand to equal supply. Thus, none of the efficient allocations on the 45 degree line inside the box can be an equilibrium allocation. However, suppose that $p = \infty$. Then the budget line becomes vertical and passes through E. The parent will optimize at the top corner (point D in panel (b) of Graph 16.8), and the children don't care where on the budget they optimize because all the bundles on that budget lie on the same indifference curve. Thus, it is not inconsistent with optimization to assume that the children will choose F — halfway up the budget and halfway to D where the parent optimizes. Thus, children consume nothing now and give half of what they earn in the future to the parent, and parents consume everything now and half of everything (i.e. half of what each of the two children earns) in the future.

(d) *Suppose instead that there are two parents and one child. How does your answer change?*

Answer: No equilibrium allocation can lie on the 45 degree line for the same reason as in the previous case — and now we end up with the child optimizing at G in panel (b) of Graph 16.8 and the two parents optimizing at H, with $p = 0$. Thus, parents consume half their income now and nothing in the future, while children consume half of each parents' income now and everything in the future.

(e) *Repeat (a) through (d) for the case where consumption now and consumption in the future are perfect substitutes for both parent and child.*

Answer: When consumption across time is perfectly substitutable, the indifference curves have slope -1 at every allocation in the Edgeworth Box. Thus any equilibrium allocation inside the box must lie on the line connecting the upper left to the lower right corners of the box — the line pictured in panel (c) of Graph 16.8. Neither parent nor child cares where on that line they consume — and thus any split of the economy's endowment that falls on this line will be an equilibrium allocation with $p = 1$. For instance, when there is one child and one parent, A is a possible equilibrium allocation, as is C and B. When there are two children and 1 parent, any allocation that has the parent's bundle twice as far from E as the children's works — for instance A for the parent and C for the children. When there are two parents and one child, then any allocation that has the child twice as far as the parents from E works. In all cases, the equilibrium price continues to be $p = 1$ — because it makes no sense for individuals to trade on other terms when consumption now is the same as consumption in the future.

(f) *Repeat for the case where consumption now and consumption in the future are perfect complements for parents and perfect substitutes for children.*

Answer: Consider first the case of one parent and one child. For any budget with positive slope (not equal to infinity), the parent will optimize on the 45-degree line. For any price not equal to 1, the child will choose a corner solution (since consumption now and in the future are the same for her). Thus, the only way the child will trade to permit the parent to get to the 45 degree line is if $p = 1$ and the budget line takes the shape graphed in panel (c) of Graph 16.8. The equilibrium allocation is then A — where the parent's indifference curve is drawn as a dotted L-shape. Next, suppose there are two children. Nothing has changed in terms of the children's willingness to trade to an interior solution only at $p = 1$ and in terms of the parent's optimal bundle falling on the 45 degree line for any positive price. Thus, p will remain 1, the parent will optimize at A and the children will each optimize at C — halfway between A and E. Finally, suppose there are two parents and one child. Again, for the same reasons as before, price has to remain 1, and the parents' optimization has to lead to A. Thus, parents end up at A and the child ends up at the top left corner D — twice as far from E as the two parents.

(g) True or False*: The more consumption is complementary for the parent relative to the child, and the more children there are per parent, the more gains from trade will accrue to the parent.*

Answer: This is roughly true, as illustrated in the previous parts of the question. For instance, when parent viewed consumption as perfectly complementary across time while children viewed it as substitutable (in panel (c) of Graph 16.8), the children gain no utility from trading while the parent(s) get all gains from trade. Similarly, we saw in this and the previous exercise that more gains typically accrue to the party that is in control of the goods that are scarcer. Parents are in control of consumption now — which is relatively more scarce the more children there are per parent.

B: *Suppose that parent and child tastes can be represented by the CES utility function* $u(c_1, c_2) = \left(0.5c_1^{-\rho} + 0.5c_2^{-\rho}\right)^{-1/\rho}$. *Assume that the income earned by parents in period 1 and by children in period 2 is 100.*

(a) *Letting p denote the price of consumption now with price of future consumption normalized to 1, derive parent and child demands for current and future consumption as a function of ρ and p.*

Answer: We want to maximize utility (which is the same for parents and children) subject to the budget constraint — which is $100p = pc_1 + c_2$ for parents (who are endowed with 100 now) and $100 = pc_1 + c_2$ for children (who are endowed with 100 in the future). Solving this in the usual way, we get

$$c_1^P = \frac{100p^{1/(\rho+1)}}{p^{\rho/(\rho+1)}+1} \text{ and } c_2^P = \frac{100p}{p^{\rho/(\rho+1)}+1} \text{ for parents, and} \quad (16.50)$$

$$c_1^C = \frac{100}{p+p^{1/(\rho+1)}} \text{ and } c_2^C = \frac{100}{p^{\rho/(\rho+1)}+1} \text{ for children.} \quad (16.51)$$

(b) *What is the equilibrium price — and what does this imply for equilibrium allocations of consumption between parent and child across time. Does any of your answer depend on the elasticity of substitution?*

Answer: This solves slightly more easily if we set demand and supply in the c_2 market equal to one another (rather than setting it equal to one another in the c_1 market. Of course the latter would give the same answer even if it is slightly more burdensome to get there.) Thus, we need to solve

$$\frac{100p}{p^{\rho/(\rho+1)}+1} + \frac{100}{p^{\rho/(\rho+1)}+1} = 100. \quad (16.52)$$

Dividing by 100, multiplying by the denominator on the left hand side, and simplifying, we get

$$p = p^{\rho/(\rho+1)} \text{ or } 1 = \rho^{-1/(\rho+1)} \quad (16.53)$$

which solves to $p = 1$. The answer therefore does not depend on ρ and thus is independent of the elasticity of substitution. (This is because the indifference curves for the utility function always have $MRS = -1$ along the 45 degree line no matter what elasticity of substitution is assumed.)

(c) *Next, suppose there are 2 children and only 1 parent. How does your answer change?*

Answer: We now have to sum twice the child demands with the parent demand for c_2 and set it equal to overall consumption in the future — which is 200 when there are two children. This implies we need to solve

$$\frac{100p}{p^{\rho/(\rho+1)}+1} + 2\left(\frac{100}{p^{\rho/(\rho+1)}+1}\right) = 200 \quad (16.54)$$

which solves to

$$p = 2^{\rho+1}. \quad (16.55)$$

The equilibrium price now depends on ρ and thus on the elasticity of substitution. As ρ increases — which implies the elasticity of substitution falls — price increases. In the limit, as ρ approaches infinity — and consumption becomes perfectly complementary across time — price rises to infinity. This is exactly what we concluded in part A for perfect complements. As ρ falls to -1 — and consumption becomes perfectly substitutable across time, on the other hand, price becomes 1 — again exactly as we concluded in part A.

(d) *Next, suppose there are 2 parents and only 1 child. How does your answer change?*

Answer: We now have to sum twice the parent demands with the child demand for c_2 and set it equal to overall consumption in the future — which is 100 when there is only one child. This implies we need to solve

$$2\left(\frac{100p}{p^{\rho/(\rho+1)}+1}\right) + \frac{100}{p^{\rho/(\rho+1)}+1} = 100 \quad (16.56)$$

which solves to

$$p = \left(\frac{1}{2}\right)^{\rho+1}. \tag{16.57}$$

The equilibrium price again depends on ρ and thus on the elasticity of substitution. As ρ increases — which implies the elasticity of substitution falls — price falls. In the limit, as ρ approaches infinity — and consumption becomes perfectly complementary across time — price falls to zero. This is exactly what we concluded in part A for perfect complements. As ρ falls to -1 — and consumption becomes perfectly substitutable across time, on the other hand, price becomes 1 — again exactly as we concluded in part A.

(e) *Explain how your answers relate to the graphs you drew for the extreme cases of both parent and child preferences treating consumption as perfect complements over time.*

Answer: We already did this in our answer above. We showed that, as tastes become perfectly complementary, then p approaches infinity if there are two children and one parent and to zero if there are two parents and one child. We illustrated precisely this extreme case in panel (b) of Graph 16.8.

(f) *Explain how your answers relate to your graphs for the case where consumption was perfectly substitutable across time for both parents and children.*

Answer: Again, we already did this above. We showed that, when consumption is perfectly substitutable across time, then price will be 1 regardless of the number of children relative to parents.

16.8 Business Application: *Valuing Land in Equilibrium: Suppose we consider a Robinson Crusoe economy with one worker who has preferences over leisure and consumption and one firm that uses a constant returns to scale production process using inputs land and labor.*

A: *Suppose that the worker owns the fixed supply of land that is available for production. Throughout the problem, normalize the price of output to 1.*

(a) *Explain why we can normalize one of the three prices in this economy (where the other two prices are the wage w and the land rental rate r).*

Answer: In general equilibrium models where everything is owned by someone (and thus everything is "endogenous"), we can only derive relative prices — i.e. the slopes of budgets and isoprofits which determine behavior that is immune to raising or lowering both numerator and denominator. Thus, we are in this exercise expressing the wage and the land rental rate relative to the price of output when we set the output price to 1.

(b) *Assuming the land can fetch a positive rent per unit, how much of it will the worker rent to the firm in equilibrium (given his tastes are only over leisure and consumption)?*

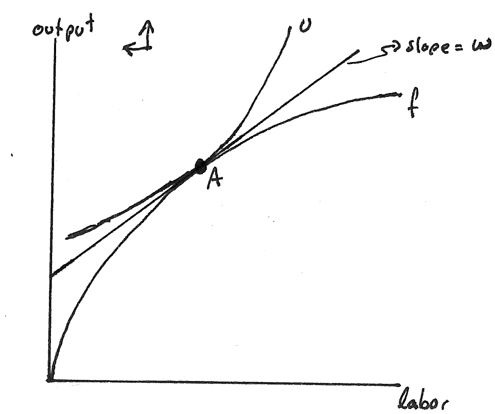
Answer: Given that land does not give the worker "utility", the only use for it that the worker has is to use it as an income producing asset. Thus, he will rent all of it to the firm (since he is acting as a price taker).

(c) *Given your answer to (b), explain how we can think of the production frontier for the firm as simply a single-input production process that uses labor to produce output?*

Answer: If the firm's production process has constant returns to scale, this implies that its short run production frontier (holding land fixed) has diminishing marginal product of labor. Because the worker will provide all his land to the land market on which the firm rents land, and because the equilibrium land rental rate will adjust to insure that all of the land is in fact rented by the firm, we can think of the firm as operating with a fixed land input and simply look at the slice of its production frontier that varies labor.

(d) *What returns to scale does this single input production process have? Draw the production frontier in a graph with labor on the horizontal and output on the vertical axis.*

Answer: We already said this production frontier has diminishing marginal product of labor — which implies a concave shape associated with decreasing returns to scale. This is pictured as the frontier f in Graph 16.9.



Graph 16.9: Valuing Land in General Equilibrium

(e) *What do the worker's indifference curves in this graph look like? Illustrate the worker's optimal bundle if he took the production frontier as his constraint.*

Answer: The worker's indifference curves are shaped as illustrated by the curve u in Graph 16.9 — with higher indifference curves lying to the northwest of the graph. If the worker took the production frontier as his constraint, he would choose bundle A.

(f) *Illustrate the budget for the worker and the isoprofit for the firm that lead both worker and firm to choose the bundle you identified in (e) as their optimum. What is the slope of this budget/isopfrofit? Does the budget/isopfrofit have a positive vertical intercept?*

Answer: This budget/isoprofit is a line that is tangent to both the indifference curve u and the production frontier f at A where these are tangent to one another. Its slope is usually w/p but, since we normalized p to 1, it is simply w. Its intercept is positive because of the concave shape of f.

(g) *In the text, we interpreted this intercept as profit which the worker gets as part of his income because he owns the firm. Here, however, he owns the land which the firm uses. Can you re-interpret this positive intercept in the context of this model (keeping in mind that the true underlying production frontier for the firm has constant returns to scale)? If land had been normalized to 1 unit, where would you find the land rental rate r in your graph?*

Answer: A constant returns to scale firm has to make zero profit in a competitive equilibrium. The positive intercept in the graph can therefore not be profit — but it must be payment for the fixed input that the firm is renting from the worker. Thus, if we divide the intercept by the total number of units of land that the worker has, we will get the land rental rate r. If land units are normalized so that the total number of land units owned by the worker is equal to 1, then the intercept is simply r.

B: *Suppose that the worker's tastes can be represented by the utility function $u(x,(1-\ell)) = x^{\alpha}(1-\ell)^{(1-\alpha)}$ (where x is consumption, ℓ is labor, and where the leisure endowment is normalized to 1.) Suppose further that the firm's production function is $f(y,\ell) = y^{0.5}\ell^{0.5}$ where y represents the number of acres of land rented by the firm and ℓ represents the labor hours hired.*

(a) *Normalize the price of output to be equal to 1 for the remainder of the problem and let land rent and the wage be equal to r and w. Write down the firm's profit maximization problem, taking into account that the firm has to hire both land and labor.*

Answer: The firm's profit maximization problem is then

$$\max_{y,\ell} \pi = y^{0.5}\ell^{0.5} - w\ell - ry \tag{16.58}$$

(b) *Take the first order conditions of the firm's profit maximization problem. The worker gets no consumption value from his land — and therefore will rent his whole unit of land to the firm. Thus, you can replace land in your first order conditions with 1. Then solve each first order condition for ℓ and from this derive the relationship between w and r.*

Answer: The first order conditions are

$$\frac{\partial \pi}{\partial y} = \frac{\ell^{0.5}}{2y^{0.5}} - r = 0 \text{ and } \frac{\partial \pi}{\partial \ell} = \frac{y^{0.5}}{2\ell^{0.5}} - w = 0. \tag{16.59}$$

which become

$$\frac{\ell^{0.5}}{2} = r \text{ and } \frac{1}{2\ell^{0.5}} = w \tag{16.60}$$

when y is set to 1 (as it has to be in equilibrium). This gives us two labor demand equations — one as a function of r and the other as a function of w:

$$\ell = 4r^2 \text{ and } \ell = \frac{1}{4w^2}. \tag{16.61}$$

Setting these equal to each other, we can conclude that, in equilibrium, $r = 1/(4w)$.

(c) *The worker earns income from his labor and from renting his land to the firm. Express the worker's budget constraint in terms of w and then solve for his labor supply function in terms of w.*

Answer: The worker will earn $w(1-\ell)$ in the labor market and $r = 1/(4w)$ in the land market. His budget constraint is therefore

$$x = w\ell + \frac{1}{4w}. \tag{16.62}$$

We can then write his optimization problem as

$$\max_{x,\ell} x^{\alpha}(1-\ell)^{(1-\alpha)} \text{ subject to } x = w\ell + \frac{1}{4w}. \tag{16.63}$$

From the first order conditions, we get

$$x = \frac{\alpha w(1-\ell)}{1-\alpha} \tag{16.64}$$

and plugging this into the budget constraint, we can solve for the labor supply as

$$\ell = \alpha - \frac{1-\alpha}{4w^2}. \tag{16.65}$$

(d) *Derive the equilibrium wage in your economy by setting labor supply equal to labor demand (which you implicitly derived in (b) from one of your first order conditions).*

Answer: Labor demand is $\ell = 1/(4w^2)$ — so equilibrium implies

$$\alpha - \frac{1-\alpha}{4w^2} = \frac{1}{4w^2}. \tag{16.66}$$

Solving this for w, we get the equilibrium wage

$$w^* = \frac{1}{2}\left(\frac{2-\alpha}{\alpha}\right)^{0.5}. \tag{16.67}$$

(e) *What's the equilibrium rent of land?*

Answer: In (b) we concluded that $r = 1/(4w)$ in equilibrium. Plugging in w^* and simplifying, we get

$$r^* = \frac{1}{2}\left(\frac{\alpha}{2-\alpha}\right)^{0.5}. \tag{16.68}$$

(f) *Now suppose we reformulate the problem slightly: Suppose the firm's production function is $f(y,\ell) = y^{(1-\beta)}\ell^{\beta}$ (where y is land and ℓ is labor) and the worker's tastes can be represented by the utility function $u(x,(L-\ell)) = x^{\alpha}(L-\ell)^{(1-\alpha)}$, where L is the worker's leisure endowment. Compare this to the way we formulated the Robinson Crusoe economy in the text. If land area is in fixed supply at 1 unit, what parameter in our formulation in the text must be set to 1 in order for our problem to be identical to the one in the text.*

Answer: We would need to set A equal to 1 in the text.

(g) True or False*: By turning land into a fixed input, we have turned the constant returns to scale production process into one of decreasing returns to scale.*

Answer: This is true. The original production function $f(y,\ell) = y^{(1-\beta)}\ell^{\beta}$ is constant returns to scale (since the Cobb-Douglas exponents sum to 1). But if land is in fixed supply at 1 unit, then the production function becomes $f(\ell) = \ell^{\beta}$ — which is a decreasing returns to scale production function (since $\beta < 1$).

(h) *Suppose that, as in the earlier part of the problem, $\beta = 0.5$ and the worker's leisure endowment is normalized to $L = 1$. Use the solution for the equilibrium wage in the text to derive the equilibrium wage now, again normalizing the output price to 1.*

Answer: The equation for the equilibrium price derived in the text is

$$w^* = \beta A \left(\frac{1-\alpha(1-\beta)}{\alpha \beta L} \right)^{(1-\beta)} p. \tag{16.69}$$

Setting $L = 1$, $p = 1$, $A = 1$ and $\beta = 0.5$, we then get

$$w^* = 0.5 \left(\frac{1-\alpha(1-0.5)}{\alpha 0.5} \right)^{(1-0.5)} = \frac{1}{2} \left(\frac{2-\alpha}{\alpha} \right)^{0.5}. \tag{16.70}$$

(i) *Use the profit function in equation (16.32) of the text to determine the profit of the firm (given the equilibrium wage and given the parameter values used here). Compare this to the equilibrium land rent you derived in (e). Explain your result intuitively.*

<u>Answer</u>: Plugging $\beta = 0.5$, $A = 1$ and $p = 1$ into the profit function of the text, it simplifies to $\pi = 1/(4w)$. Plugging in the equilibrium wage w^*, this gives us

$$\pi^* = \frac{1}{2} \left(\frac{\alpha}{2-\alpha} \right)^{0.5} \tag{16.71}$$

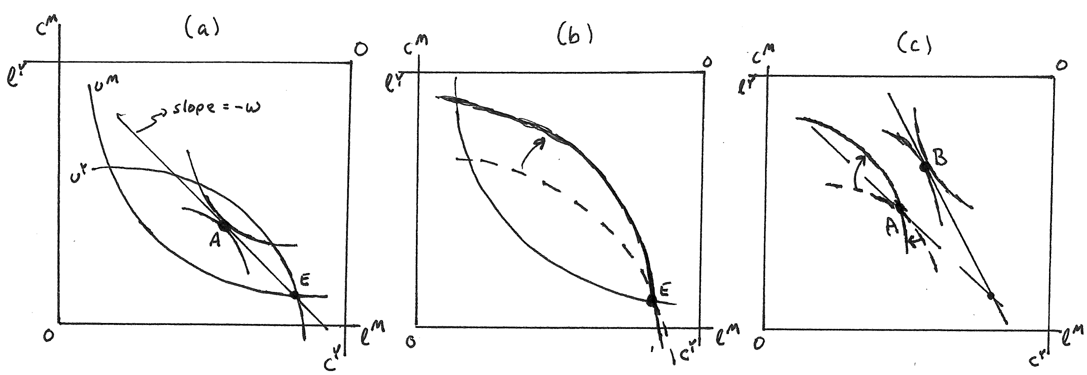
which is identical to the equilibrium land rent we calculated in part (e). This should make intuitive sense once you realize that the "profit" we derived here is simply the equilibrium rent the firm gets on its scarce land resource. In the earlier parts of the problem, we assumed that the worker owns the land and rents it to a firm with a constant returns to scale production technology. Such a firm will make zero profit — and will pay rent for land to the worker who takes this into account as he decides how much to work (which in turn factors into the equilibrium wage). In the later parts of the problem (as formulated in the text), we assume the worker owns a firm with decreasing returns to scale. But the decreasing returns to scale arise simply from the fact that the constant returns to scale firm has a fixed level of land at its disposal — and thus is a decreasing returns to scale firm that earns positive profit. Since everything else is identical between the two scenarios, the profit in the latter scenario is the equilibrium land rent in the former scenario (where the firm makes zero profit). In both cases, the worker owns something valuable in addition to his leisure: either the land (which gives him an equilibrium land rent) or the firm that owns the land — which then gives him profit that the firm earns because it owns the land and formally is not paying anything for it.

16.9 Business Application: *Hiring an Assistant: Suppose you are a busy CEO — with lots of consumption but relatively little leisure. I, on the other hand, have only a part-time job and therefore lots of leisure with relatively little consumption.*

A: *You decide that the time has come to hire a personal assistant — someone who can do some of the basics in your life so that you can have a bit more leisure time.*

(a) *Illustrate our current situation in an Edgeworth Box with leisure on the horizontal and consumption on the vertical axis. Indicate an endowment bundle that fits the description of the problem and use indifference curves to illustrate a region in the graph where both of us would benefit from me working for you as an assistant.*

Answer: This is illustrated in panel (a) of Graph 16.10 where the endowment allocation E has me (on the lower axes) with lots of leisure but little consumption and you (on the upper axes) with the reverse. The mutually beneficial region is formed by the lens made from our indifference curves that pass through E. Both of us would prefer any allocation in that lens shape to the endowment bundle E.



Graph 16.10: Cheerfulness in Office Assistants

(b) *Next, illustrate what an equilibrium would look like. Where in the graph would you see the wage that I am being paid?*

Answer: This is also illustrated in panel (a) of Graph 16.10 where the budget line that passes through A and E has slope $-w$ (where w is the wage when the price of consumption is normalized to 1).

(c) *Suppose that anyone can do the tasks you are asking of your assistant — but some will do it cheerfully and others will do it with attitude. You hate attitude — and therefore would prefer someone who is cheerful. Assuming you can read the level of cheerfulness in me, what changes in the Edgeworth box as your impression of me changes?*

Answer: As you think I am more cheerful, you will be willing to trade more of your consumption for an increase in your leisure. Thus, your indifference curves become steeper.

(d) *How do your impressions of me — i.e. how cheerful I am — affect the region of mutually beneficial trades?*

Answer: This is illustrated in panel (b) of Graph 16.10 where your original indifference curve through E is illustrated as a dashed indifference curve and your new indifference curve (that contains E) as my cheerfulness increases is illustrated as a bold curve. This increases the lens formed by our indifference curves through E — and thus the mutually beneficial region.

(e) *How does increased cheerfulness on my part change the equilibrium wage?*

Answer: This is illustrated in panel (c) of Graph 16.10 where A is the original equilibrium at low levels of cheerfulness and B is the new equilibrium at higher levels of cheerfulness. As my cheerfulness increases, your indifference curve through A becomes steeper — rotating from the dashed curve to the solid one. Thus, A can't be an equilibrium anymore because you now want more of me but I am not willing to offer any more at the original wage. Thus, the wage must increase in order to get me to offer more of myself and you to reduce your demand for me. This leads us to the steeper budget through B — with a higher wage. Cheerfulness is rewarded in the competitive market.

(f) *Your graph probably has the new equilibrium (with increased cheerfulness) occurring at an indifference curve for you that lies below (relative to your axes) the previous equilibrium (where I was less cheerful). Does this mean that you are worse off as a result of me becoming more cheerful?*

Answer: No, it does not. It is indeed true that your indifference curve through B in panel (c) of Graph 16.10 lies below A (relative to your axes). But this does not mean you are less happy — because my cheerfulness is what made your indifference curves get steeper. In terms of some of the earlier problems in our development of consumer theory, cheerfulness is a third good you care about — and as it changes in the problem, you switch to a different "slice" of your 3-dimensional indifference surfaces. Increased cheerfulness switches you to a slice where you are happier for any level of consumption and leisure than you were before — and so an indifference curve with more cheerfulness can lie below one with less cheerfulness and still be preferred.

B: *Suppose that my tastes can be represented by $u(c,\ell) = 200\ln\ell + c$ while yours can be represented by $u(c,\ell,x) = 100x\ln\ell + c$ where ℓ stands for leisure, c stands for consumption and x stands for cheerfulness of your assistant. Suppose that, in the absence of working for you, I have 50 leisure hours and 10 units of consumption while you have 10 leisure hours and 100 units of consumption.*

(a) *Normalize the price of c as 1. Derive our leisure demands as a function of the wage w.*

Answer: My budget constraint is $w\ell + c = 50w + 10$ while yours is $w\ell + c = 10w + 100$. Maximizing our utilities subject to these constraints, we get (by solving this in the usual way)

$$\ell^M = \frac{200}{w} \text{ for me and } \ell^Y = \frac{100x}{w} \text{ for you.} \tag{16.72}$$

(b) *Calculate the equilibrium wage as a function of x.*

Answer: The sum of our leisure demands has to be equal to the leisure supply of 60 in equilibrium — i.e.

$$\frac{200}{w} + \frac{100x}{w} = 60 \tag{16.73}$$

which implies that the equilibrium wage is

$$w^* = \frac{10+5x}{3}. \tag{16.74}$$

(c) *Suppose $x = 1$. What is the equilibrium wage, and how much will I be working for you?*

Answer: Substituting $x = 1$ into our equation for w^*, we get an equilibrium wage of 5. Plugging this wage into our leisure demand equations, we get that you will have 20 hours of leisure and I will have 40 — which is 10 less for me and 10 more for you than what we were endowed with. Thus, I'll be working for you for 10 hours.

(d) *How does your MRS change as my cheerfulness x increases?*

Answer: Your MRS is

$$MRS^Y = -\frac{\partial u(c,\ell,x)/\partial\ell}{\partial u(c,\ell,x)/\partial x} = -\frac{100x}{\ell}. \tag{16.75}$$

Thus, for any bundle (ℓ, c), the MRS gets larger in absolute value as x increases — i.e your indifference curves become steeper as my cheerfulness increases.

(e) *What happens to the equilibrium wage as x increases to 1.2? What happens to the equilibrium number of hours I work for you? What if I get grumpy and x falls to 0.4?*

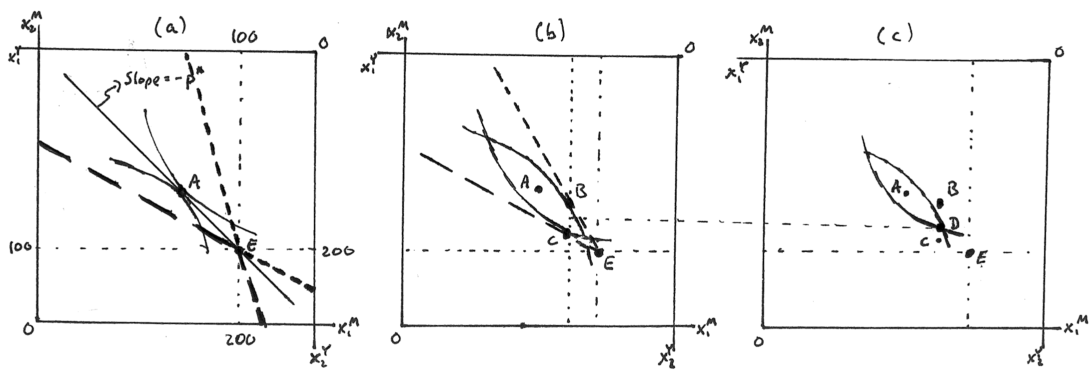
Answer: When x goes to 1.2, the equilibrium wage rises to 5.33 and the number of hours I work for you increases to 12.5. When x falls to 0.4, the equilibrium wage falls to 4 but you no longer hire me and we simply consume at our endowment bundles.

16.10 Policy Application: *Distortionary Taxes with Redistribution*: *Consider a 2-person exchange economy in which I own 200 units of x_1 and 100 units of x_2 while you own 100 units of x_1 and 200 units of x_2.*

A: *Suppose you and I have tastes that are quasilinear in x_1, and suppose that I sell x_1 to you in the competitive equilibrium without taxes.*

(a) *Illustrate the no-tax competitive equilibrium in an Edgeworth Box.*

Answer: This is illustrated in panel (a) of Graph 16.11 where the square box has sides of 300 (since the endowment of the two of us together is 300 of each good). The initial equilibrium allocation is A where our indifference curves are tangent to one another and separated by a budget line (whose slope is the equilibrium price of x_1 in terms of x_2) that runs through out endowment allocation E.



Graph 16.11: Distortionary Taxes and Redistribution

(b) *Suppose the government imposes a per-unit tax t (paid in terms of x_2) on all units of x_1 that are traded. This introduces a difference of t between the price received by sellers and the price paid by buyers. How does the tax result in kinked budget constraints for us?*

Answer: I am a buyer if my consumption of x_1 occurs to the right of E and a seller if it occurs to the left of E. Thus, to the left of E my budget is shallower than to the right of E (since prices will be lower for sellers than for buyers because of the tax). That produces a kink at E. This is illustrated in panel (a) of Graph 16.11 as the long-dashded kinked budget. You are a buyer in the Edgeworth box if you consume to the left of E and a seller if you consume to the right of E. For you, the budget in the Edgeworth box is therefore steep to the left of E and shallow to the right of E, with a kink again at E. This is illustrated in panel (a) as the short-dashed budget constraint.

(c) True or False*: The tax can never be so high that I will turn from being a seller to being a buyer.*

Answer: This is true. My tastes are quasilinear in x_1 — which implies that the MRS is the same for any vertical line in the graph. This tells me that my indifference curves are shallower than at A everywhere to the left of A and steeper everywhere to the right of A. But in order for me to become a buyer, I would have to optimize on the steep portion of my long-dashed budget — which would have to mean a steeper MRS to the left of A. Similarly, you will never become a seller as a result of a tax.

(d) *Illustrating the tax in the Edgeworth box will imply we face different budget lines in the box — but demand and supply of x_1 still has to equalize. Illustrate this and show how a difference between economy's endowment of x_2 and the amounts consumed by us emerges. What's that difference?*

Answer: This is illustrated in panel (b) of Graph 16.11 where you face the short-dashed budget and I face the long-dashed budget. (We don't draw the budgets to the right of E since we have concluded that we will never optimize there.) As a tax wedge is introduced between the buyer and seller price, I will optimize to the right of A (since my seller budget is shallower and my indifference curves to the right of A are shallower). You will also optimize to the left of A — because your buyer budget becomes steeper as do your indifference curves to the right of A. The prices will have reached equilibrium when demand and supply of x_1 are equalized — i.e. when your optimum at B lies on the same vertical line in the Edgeworth box as my optimum C. We therefore trade less as a result of the tax. The vertical difference between B and C is the tax revenue paid to the government.

(e) *Suppose the government simply takes the x_2 revenue it collects, divides it into two equal piles and gives it back to us. In a new Edgeworth box, illustrate our indifference curves through the final allocation that we will consume. How can you tell that the combination of the tax and transfer of x_2 is inefficient?*

Answer: If the government divides the revenue between us, we will end up at an allocation halfway between B and C — labeled D in panel (c) of Graph 16.11. The indifference curves through D have the same slopes as our indifference curves through B and C — because our tastes are quasilinear in x_1 (which implies the MRS is unaffected by increases or decreases of x_2 in our consumption bundles.) Thus, our indifference curves are not tangent to one another — which implies that there is a lens shape between our indifference curve with all allocations in that lens shape preferred by both of us to D. The inefficiency results from the fact that you and I are optimizing at different prices due to the tax distortion — and it has nothing to do with the fact that we paid taxes. Even when the revenue is returned to us, the inefficiency remains — because the inefficiency emerges from the substitution effects unleashed by the tax distortions.

B: *Suppose that our endowments are as specified at the beginning. My tastes can be represented by the utility function $u^M(x_1,x_2) = x_2 + 50\ln x_1$ and yours by the utility function $u^Y(x_1,x_2) = x_2 + 150\ln x_1$.*

(a) *Derive our demand functions and use them to calculate the equilibrium price p defined as the price of x_1 given that the price of x_2 is normalized to 1.*

Answer: My optimization problem is

$$\max_{x_1,x_2} x_2 + 50\ln x_1 \text{ subject to } 200p + 100 = px_1 + x_2 \tag{16.76}$$

and yours is

$$\max_{x_1,x_2} x_2 + 150\ln x_1 \text{ subject to } 100p + 200 = px_1 + x_2. \tag{16.77}$$

Solving these, we get demands of

$$x_1^M = \frac{50}{p} \text{ and } x_2^M = 200p + 50 \text{ for me, and} \tag{16.78}$$

$$x_1^Y = \frac{150}{p} \text{ and } x_2^Y = 100p - 50 \text{ for you.} \tag{16.79}$$

Setting $(x_1^M + x_1^Y)$ equal to the supply of x_1 (which is 300), we can solve for p as $p^* = 2/3$. This implies the equilibrium allocation

$$(x_1^M, x_2^M) = (75, 183.33) \text{ and } (x_1^Y, x_2^Y) = (225, 116.67). \tag{16.80}$$

(b) *How much of x_1 do we trade among each other?*

Answer: I am endowed with 200 units of x_1 but only consume 75 in equilibrium — which implies I sell 125 units. Similarly, you are endowed with 100 units of x_1 but consume 225 in equilibrium — which implies you buy 125 units.

(c) *Now suppose that a per-unit tax t (payable in terms of x_2) is introduced. Let p be the price buyers will end up paying, and let $(p-t)$ be the price sellers receive. Derive the equilibrium levels of p and $(p-t)$ as a function of t. (*Hint*: You will need to solve a quadratic equation using the quadratic formula — and the larger of the two solutions given by the formula is the correct one.)*

Answer: Our demands now become

$$x_1^M = \frac{50}{(p-t)} \text{ and } x_2^M = 200(p-t)+50 \text{ for me, and} \tag{16.81}$$

$$x_1^Y = \frac{150}{p} \text{ and } x_2^Y = 100p-50 \text{ for you.} \tag{16.82}$$

(Mine change because I am the seller and we defined my price as $(p-t)$ while defining the buyers price — which applies to you — as p. We could equally well have defined the seller's price as p and the buyer's price as $(p+t)$ and then defined demands accordingly.) Setting $(x_1^M + x_1^Y)$ equal to the supply of x_1 (which is 300), we get

$$\frac{50}{(p-t)} + \frac{150}{p} = 300 \tag{16.83}$$

which simplifies to

$$p^2 - \left(\frac{2}{3}+t\right)p + \frac{1}{2}t = 0. \tag{16.84}$$

Using the quadratic formula and accepting the larger of the two solutions, we get

$$p = \frac{\frac{2}{3}+t+\sqrt{t^2-\frac{2}{3}t+\frac{4}{9}}}{2} \tag{16.85}$$

and

$$(p-t) = \frac{\frac{2}{3}-t+\sqrt{t^2-\frac{2}{3}t+\frac{4}{9}}}{2}. \tag{16.86}$$

(d) *Consider the case of $t=0.25$. Illustrate that the post-tax allocation is inefficient.*

Answer: When $t=0.25$ is substituted into our equations for p and $(p-t)$, we get $p=0.75$ and $(p-t)=0.50$. Substituting this into our demands (using the buyer price p for you and the seller price $(p-t)$ for me), we get the allocation

$$(x_1^M, x_2^M) = (100, 150) \text{ and } (x_1^Y, x_2^Y) = (200, 125) \tag{16.87}$$

with tax revenues of 25 in terms of x_2. My MRS is $MRS^M = -50/x_1$ and yours is $MRS^Y = -150/x_1$. Given the allocation of x_1 above, this implies that $MRS^M = -50/100 = -0.5$ and $MRS^Y = -150/200 = 0.75$. Thus, our indifference curves are not tangent to one another in the Edgeworth box — implying that we are not at an efficient allocation. (This is not surprising — we are optimizing to budget constraints with slope 0.5 for me and 0.75 for me — which results in the equilibrium MRS's we calculated.)

(e) *Suppose the government distributes the x_2 revenue back to us — giving me half of it and you the other half. Does your previous answer change?*

Answer: Our marginal rates of substitution are not a function of x_2 — because our tastes are quasilinear. Thus, our answer above is unaffected except for the fact that the allocation now has 12.5 more units of x_2 for each of us.

(f) *Construct a table relating t to tax revenues, buyer price p, seller price $(p-t)$, my consumption level of x_1 and your consumption level of x_1 in 0.25 increments from 0 to 1.25. (This is most easily done by putting the relevant equations into an excel spreadsheet and changing t.)*

Answer: This is done in Table 16.1.

Tax Rates and Tax Revenues						
	$t = 0.00$	$t = 0.25$	$t = 0.50$	$t = 0.75$	$t = 1.00$	$t = 1.25$
Tax Rev	0.00	25.00	34.86	30.70	17.71	0.00
p	0.67	0.75	0.88	1.06	1.27	1.50
$(p-t)$	0.67	0.50	0.38	0.31	0.27	0.25
x_1^M	75.00	100.00	130.28	159.07	182.29	200.00
x_1^Y	225.00	200.00	169.72	140.93	117.71	100.00

Table 16.1: Distortionary Tax Outcomes

(g) *Would anything in the table change if the government takes the x_2 revenue it collects and distributes it between us in some way?*

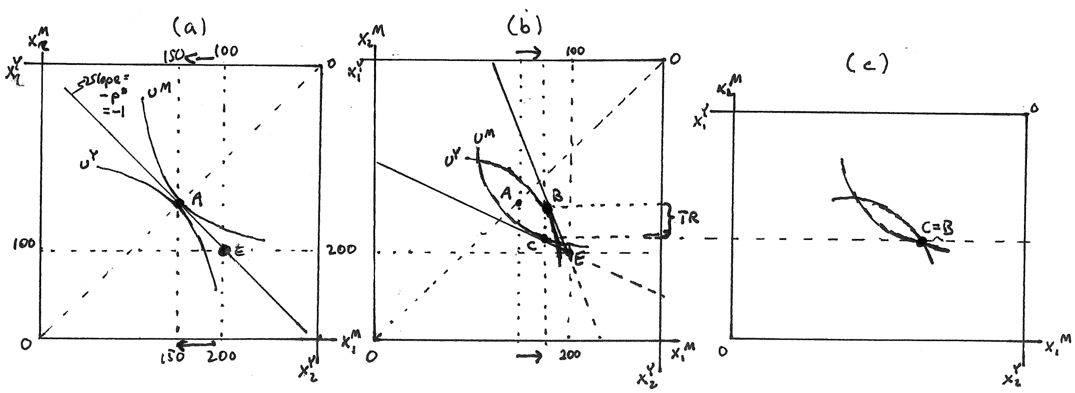
Answer: Aside from the tax revenue which would now be zero everywhere given that it is being passed back to us, nothing would change in the table because of our use of tastes that are quasilinear in x_1.

16.11 Policy Application: *Distortionary Taxes in General Equilibrium: Consider, as in exercise 16.10, a 2-person exchange economy in which I own 200 units of x_1 and 100 units of x_2 while you own 100 units of x_1 and 200 units of x_2.*

A: *Suppose you and I have identical homothetic tastes.*

(a) *Draw the Edgeworth Box for this economy and indicate the endowment allocation E.*

Answer: This is illustrated in panel (a) of Graph 16.12 where the box takes on the shape of a square since the economy's endowment of both goods is 300.



Graph 16.12: Distortionary Taxes

(b) *Normalize the price of good x_2 to 1. Illustrate the equilibrium price p^* for x_1 and the equilibrium allocation of goods in the absence of any taxes. Who buys and who sells x_1?*

Answer: This is also done in panel (a) of Graph 16.12 where A is the equilibrium allocation (which appears in the center of the box on the 45 degree line because of our identical homothetic tastes and endowments.) Thus, I sell 50 units of x_1 to you for the price of $p^* = 1$.

(c) *Suppose the government introduces a tax t levied on all transactions of x_1 (and paid in terms of x_2). For instance, if one unit of x_1 is sold from me to you at price p, I will only get to keep $(p-t)$. Explain how this creates a kink in our budget constraints.*

Answer: This implies that the price p paid by the buyer is greater than the price $(p-t)$ received by the seller. On my budget constraint, I am a seller to the left of E and a buyer to the right of E — implying that my budget has shallower slope $-(p-t)$ to the left of E and steeper slope $-p$ to the right of E, with a kink at E. The same is true for you — except that "right" and "left" are reversed when we flip your axes to create the Edgeworth Box. The portions along which I am a seller and you are a buyer of x_2 are illustrated as the solid lines in panel (b) of Graph 16.12, with the remaining portion of the constraints dashed to the right of E.

(d) *Suppose a post-tax equilibrium exists and that price increases for buyers and falls for sellers. In such an equilibrium, I will still be selling some quantity of x_1 to you. (Can you explain why?) How do the relevant portions of the budget constraints you and I face look in this new equilibrium, and where will we optimize?*

Answer: This is illustrated in panel (b) of Graph 16.12 where the steeper (solid) constraint is yours (with the higher post-tax price) and the shallower one is mine (with the lower pre-tax price). In equilibrium, it will still have to be the case that the amount of x_1 I sell to you is equal to the amount of x_1 you buy. Thus, in equilibrium, our two budgets have to be such that your optimum B lies right above my optimum C in the Edgeworth Box. We know that this will be to the right of the original equilibrium A — because your budget is steeper than before and mine is shallower than before. The fact that it is shallower for me means that I will be optimizing on a shallower ray from the origin (given that my tastes are homothetic), and the fact that it is steeper for you implies you will be optimizing on a steeper ray from your origin. Thus, the amount we trade will fall by the amount of the arrows in the graph. (The reason we know that I will still be selling (or at least not buying) x_1 under the tax is as follows: My budget under the tax has a kink at E — and becomes steeper to the right of E. Given that my tastes are homothetic, it cannot be that I optimize on that steeper portion — because the steeper parts of my indifference curves lie to the left of E).

(e) *When we discussed price changes with homothetic tastes in our development of consumer theory, we noted that there are often competing income (or wealth) and substitution effects. Are there such competing effects here relative to our consumption of* x_1*? If so, can we be sure that the quantity we trade* in equilibrium *will be less when* t *is introduced?*

Answer: Both of us experience a negative wealth effect — me because what I am selling has fallen in price, you because what you are buying has increased in price. Thus, the wealth effect says "consume less of x_1" for both of us. But the substitution effects operate in opposite directions for the two of us. For me, the price of x_1 falls as a result of the tax — which means the substitution effect will tell me to consume *more* of x_1. For you, on the other hand, the price of x_1 has increased — with the substitution effect therefore telling you to consume *less* of x_1. The wealth and substitution effects therefore point in opposite directions for me but not for you. This implies you will consume less x_1 under the tax, which means *in equilibrium* the prices have to adjust such that I will sell you less (and therefore consume more) even though the wealth effect tells me to consume less. (This implies that the equilibrium that we assume exists (with price increasing for buyers and falling for sellers) requires that the goods are sufficiently substitutable to create the necessary substitution effect.)

(f) *You should see that, in the new equilibrium, a portion of* x_2 *remains not allocated to anyone. This is the amount that is paid in taxes to the government. Draw a new Edgeworth box that is adjusted on the* x_2 *axes to reflect the fact that some portion of* x_2 *is no longer allocated between the two of us. Then locate the equilibrium allocation point that you derived in your previous graph. Why is this point not efficient?*

Answer: The portion of x_2 that remains not allocated in our tax-equilibrium in panel (b) of the graph is the vertical difference between B and C — labeled TR in the graph. Thus, the amount that gets allocated is TR less of x_2 than what is available — because the difference is collected by the government. If we shrink the Edgeworth Box by that vertical amount, we get the box illustrated in panel (c) of Graph 16.12. By shrinking the height of the box, we move B on top of C and now see even more clearly than in panel (b) that this allocation is not efficient. The reason it is inefficient is that both you and I would prefer to divide everything that was not taken by the government differently — with all the allocations in the lens shape between our indifference curves through $B = C$ all preferred by both of us. We could thus make everyone better off by moving the allocation into that lens shape without taking any of the tax revenue the government has raised back.

(g) True or False*: The deadweight loss from the distortionary tax on trades in* x_1 *results from the fact that our marginal rates of substitution are no longer equal to one another after the tax is imposed and* not *because the government raised revenues and thus lowered the amounts of* x_2 *consumed by us.*

Answer: This is true. The inefficiency we show in panel (c) arises from the fact that there is a lens shape between our indifference curves — and that lens shape arises from the fact that our marginal rates of substitution are not equal to one another (which is due to the fact that the prices we face as buyers and sellers is different when the government uses price-distorting taxes). The fact that the box has shrunk is not evidence of an inefficiency — because the government now has the difference and may well be doing some very useful things with the money. The problem is that what remains is not allocated efficiently due to distorted prices.

(h) True or False*: While the post-tax equilibrium is not efficient, it does lie in the region of mutually beneficial trades.*

Answer: This is true. In panel (b), the indifference curves through B and C still lie above E for both of us — i.e. trade is still making us better off than we would be without trade, just worse off than we would be if we could trade without price distortions. (Even if it is not obvious from the graph that our indifference curves through B and C lie above E, it should intuitively make sense that this has to be the case: After all, even in the presence of the distortionary tax, no one is forcing us to trade with one another — and we would not do so if trade made us worse off than we would be if we simply consumed our endowments.)

(i) *How would taxes that redistribute endowments (as envisioned by the Second Welfare Theorem) be different than the price distorting tax analyzed in this problem?*

Answer: Redistributions of endowments would involve lump sum taxes and subsidies that do not distort prices — because they would simply shift E around in the box. From the new E, markets could act as before — finding the competitive equilibrium price and causing the individuals to optimize where their indifference curves are tangent to one another and the resulting allocation is therefore efficient.

B: *Suppose our tastes can be represented by the utility function $u(x_1, x_2) = x_1 x_2$. Let our endowments be specified as at the beginning of the problem.*

(a) *Derive our demand functions for x_1 and x_2 (as functions of p — the price of x_1 when the price of x_2 is normalized to 1).*

Answer: My budget constraint is $px_1 + x_2 = 200p + 100$ while yours is $px_1 + x_2 = 100p + 200$. Solving our utility maximization problems subject to these constraints in the usual way, we get

$$x_1^M = \frac{100p + 50}{p} \text{ and } x_2^M = 100p + 50 \text{ for me, and} \tag{16.88}$$

$$x_1^Y = \frac{50p + 100}{p} \text{ and } x_2^Y = 50p + 100 \text{ for you.} \tag{16.89}$$

(b) *Derive the equilibrium price p^* and the equilibrium allocation of goods.*

Answer: To derive the equilibrium price, we can sum the demands for x_1 and set them equal to 300 — the amount of x_1 that the economy is endowed with. Solving for p, we get $p^* = 1$. Substituting back into the demand equations from above, we get $x_1^M = x_2^M = x_1^Y = x_2^Y = 150$.

(c) *Now suppose the government introduces a tax t as specified in A(c). Given that I am the one that sells and you are the one that buys x_1, how can you now re-write our demand functions to account for t? (*Hint*: There are two ways of doing this — either define p as the pre-tax price and let the relevant price for the buyer be $(p + t)$ or let p be defined as the post-tax price and let the relevant price for the seller be $(p - t)$.)*

Answer: Letting p indicate the price paid by you and $(p - t)$ be equal to the price received by me (as the seller), we can substitute $(p - t)$ into my demand equations to get

$$x_1^M(t) = \frac{100(p - t) + 50}{(p - t)} \text{ and } x_2^M(t) = 100(p - t) + 50 \tag{16.90}$$

Your demand functions would remain the same as before.

(d) *Derive the new equilibrium pre- and post-tax prices in terms of t. (*Hint*: You should get to a point where you need to solve a quadratic equation using the quadratic formula that gives two answers. Of these two, the larger one is the correct answer for this problem.)*

Answer: We again set demand for x_1 equal to supply to get the equation

$$x_1^M(t) + x_1^Y = \frac{100(p - t) + 50}{(p - t)} + \frac{50p + 100}{p} = 300. \tag{16.91}$$

Multiplying both sides by $(p - t)p$, taking all terms to one side, summing like terms and dividing by 50, we get

$$3p^2 - 3(t + 1)p + 2t = 0. \tag{16.92}$$

Applying the quadratic formula (and accepting the higher of the two solutions), we get

$$p = \frac{3(t+1)+\sqrt{9(t+1)^2-4(3)(2t)}}{6} = \frac{(t+1)+\sqrt{t^2-\frac{2}{3}t+1}}{2} \tag{16.93}$$

which is the post-tax equilibrium price that buyers pay. The pre-tax price that sellers receive is then simply t less; i.e.

$$(p-t) = \frac{(1-t)+\sqrt{t^2-\frac{2}{3}t+1}}{2}. \tag{16.94}$$

(e) *How much of each good do you and I consume if $t = 1$?*

Answer: Plugging $t = 1$ into our equations for p and $(p-t)$, we get $p \approx 1.5774$ and $(p-t) \approx 0.5774$. Plugging these into our demand equations, we get

$$x_1^M \approx 186.60, x_2^M \approx 107.74, x_1^Y \approx 113.40 \text{ and } x_2^Y \approx 178.87. \tag{16.95}$$

(f) *How much revenue does the government raise if $t = 1$?*

Answer: The tax revenue must be the difference between the 300 units of x_2 that were available in the economy and the sum of our consumption levels of x_2; i.e. tax revenue must be $300-(107.74+178.87) = 13.39$. We can verify that this is the case by multiplying $t = 1$ times the quantity of x_1 that is sold by me to you in equilibrium — i.e. $(1)(200-186.60) = 13.40$. (The difference between the two values for tax revenue is rounding error.)

(g) *Show that the equilibrium allocation under the tax is inefficient.*

Answer: To show that the equilibrium allocation is inefficient, all we have to show is that our marginal rates of substitution at the equilibrium consumption bundles are not the same. For the utility function we are using, the MRS is given by

$$MRS = -\frac{\partial u/\partial x_1}{\partial u/\partial x_2} = -\frac{x_2}{x_1}. \tag{16.96}$$

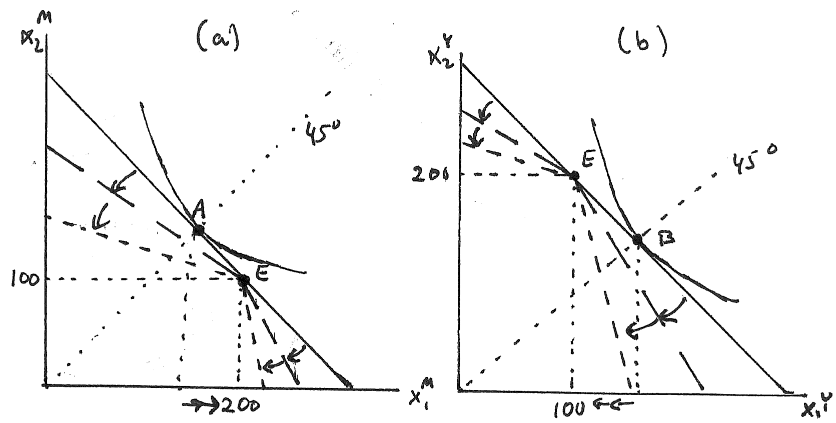
Plugging in our consumption levels from equation (16.95), we get $MRS^M \approx 0.5774$ and $MRS^Y \approx 1.5774$ for you — which are, of course, equal to the negative $(p-t)$ and p values we calculated earlier and that form the slopes of our two equilibrium budget constraints.

16.12 Policy Application: *The Laffer Curve in General Equilibrium*: *Consider, as in exercise 16.11, an exchange economy in which I own 200 units of x_1 and 100 units of x_2 while you own 100 units of x_1 and 200 units of x_2.*

A: *Suppose again that we have identical homothetic tastes.*

(a) *In exercise 16.11, you illustrated the impact of a tax t (defined in A(c) of exercise 16.11) in the Edgeworth Box. Begin now with a graph of just my endowment and my budget constraint (outside the Edgeworth Box). Illustrate how this constraint changes as t increases assuming that equilibrium price falls for sellers and rises for buyers.*

Answer: This is done in panel (a) of Graph 16.13 where my endowment is E and the solid budget is the one in effect without any tax and has (by what we did in the previous problem) a slope of -1. As t increases, the price I can receive as a seller falls and the price I have to pay as a buyer increases. Thus, the budget to the left of E (where I am a seller) becomes shallower and the budget to the right of E (where I am a buyer) becomes steeper.



Graph 16.13: Behind the Laffer Curve

(b) *Repeat (a) for you.*

Answer: This is done in panel (b) of Graph 16.13 where your original endowment is E and your optimal bundle in the no-tax equilibrium is B. Your budget constraint changes in the same way as mine, except that you start with an optimum to the right of E while I start with an optimum to the left of E.

(c) True or False*: As t increases, you will reduce the amount of x_1 you buy, and — for sufficiently high t, you will stop buying x_1 altogether.*

Answer: This is true. First, we can know that you will not optimize to the left of E. This is because we know that your indifference curves all have slope of -1, the original equilibrium price, along the 45 degree line — and the only place your indifference curves become shallower is below the 45 degree line. In order for you to optimize to the left of E as t increases, however, we need a tangency with a shallower portion of the indifference curves — and since all those are located below the 45 degree line while the budgets to the left of E are located above the 45 degree line, no such tangency can exist. Second, we can use wealth and substitution effects to confirm that consumption of x_1 will fall for you as price increases when t goes up. The wealth effect is negative because such a price increase makes you less well off — and the substitution effect is similarly negative because it always tells us to buy less of what has become relatively more expensive. Thus, as t goes up and brings the buyer price up with it, you will reduce your purchases of x_1 from me — but you will never become a seller of x_1.

(d) True or False*: As t increases, I will reduce the amount of x_1 I sell and, for sufficiently high t, I will stop selling altogether.*

Answer: First, for reasons similar to those stated for you, we can exclude the possibility of me ever becoming a buyer as t increases. Second, if all we knew was that the price of x_1 increases, we could not be sure in my case whether I will consume more or less x_1 because wealth and substitution effects now point in opposite directions. (The wealth effect is again negative for me, but the substitution effect is positive — because price is falling for me, the substitution effect tells me to consume more x_1.) But the problem assumes that the equilibrium price rises for buyers and falls for sellers — and since we know that you will buy less from me from the previous part, it must be that I will sell you less (and thus consume more myself). This is indicated with the arrows on the horizontal axis in panel (a) that run in the opposite direction to the arrows on your horizontal axis in panel (b).

(e) *Can you explain from what you have done how a Laffer curve emerges from it? (Recall that the Laffer curve plots the relationship of t on the horizontal axis to tax revenue on the vertical — and Laffer's claim is that this relationship will have an inverse U-shape.)*

Answer: For a fixed level of trade between us, tax revenues would increase as t increases. However, with trade between us falling as t increases — and with trade eventually going to zero, it must be that tax revenues fall with additional increases in t after some point. We therefore get zero tax revenues when $t = 0$ — and when t gets sufficiently high to eliminate trade, tax revenue returns back to zero.

(f) True or False *The equilibrium allocation in the Edgeworth box will lie in the core so long as t is not sufficiently high to stop trade in x_1.*

Answer: This is false. As we showed in the previous exercise, the equilibrium under the tax will not be efficient — which means that the allocation will not lie on the contract curve. The core is the portion of the contract curve that lies between the indifference curves which contain our endowment bundles — and since the tax allocation is not on the contract curve, it is not in the core. It is, however, in the mutually beneficial region — i.e. so long as there is still trade, individuals are better off than they would be by simply consuming their endowments.

(g) *If you have done exercise 16.10, can you tell whether the same inverse U-shaped Laffer curve also arises when tastes are quasilinear?*

Answer: Yes, it emerges there as well — because there, too, the amount that is traded falls as t increases — with trade ceasing at sufficiently high levels of t. The quasilinear case is a little less complicated than the case analyzed here because there are no wealth effects when tastes are quasilinear — thus we only needed to realize that the substitution effect pushes me in the direction of selling less and you in the direction of buying less as t increases.

B: *Assume, as in exercise 16.11, that our tastes can be represented by the utility function $u(x_1, x_2) = x_1 x_2$ and that our endowments are as specified at the beginning of the problem.*

(a) *If you did not already do so in exercise 16.11, derive the equilibrium pre- and post-tax prices as a function of t.*

Answer: Letting p equal the post-tax (or buyer) price and letting $(p - t)$ equal the pre-tax (or seller) price, we derived in the last exercise that

$$x_1^M = \frac{100(p-t)+50}{(p-t)} \text{ and } x_2^M = 100(p-t)+50 \text{ for me, and} \tag{16.97}$$

$$x_1^Y = \frac{50p+100}{p} \text{ and } x_2^Y = 50p+100 \text{ for you.} \tag{16.98}$$

Adding the demands for x_1 and setting them equal to the economy endowment of x_1 (equal to 300), we then solved for p and $(p - t)$ as

Tax Rates and Tax Revenues							
	$t = 0.00$	$t = 0.25$	$t = 0.50$	$t = 0.75$	$t = 1.00$	$t = 1.25$	$t = 1.50$
Tax Rev	0.00	10.26	15.69	16.44	13.40	7.63	0.00
p	1.00	1.10	1.23	1.39	1.58	1.78	2.00
$(p-t)$	1.00	0.85	0.73	0.64	0.58	0.53	0.50
x_1^M	150.00	158.95	168.61	178.08	186.60	193.90	200.00
x_1^Y	150.00	141.05	131.39	121.92	113.40	106.10	100.00

Table 16.2: Laffer Curve Relationship

$$p = \frac{(t+1)+\sqrt{t^2 - \frac{2}{3}t + 1}}{2} \tag{16.99}$$

which is the post-tax equilibrium price that buyers pay. The pre-tax price that sellers receive is then simply t less; i.e.

$$(p-t) = \frac{(1-t)+\sqrt{t^2 - \frac{2}{3}t + 1}}{2}. \tag{16.100}$$

(b) *Construct a table relating t to tax revenues, buyer price p, seller price $(p-t)$, my consumption level of x_1 and your consumption level of x_1 in 0.25 increments. (This is easiest done by putting the relevant equations into an excel spreadsheet and changing t.)*

Answer: This is done in Table 16.2.

(c) *Can you see the Laffer curve for this example within your table?*

Answer: The Laffer curve is visible in the first row where tax revenue initially increases and eventually decreases as t increases.

(d) *Does the inverse U-shaped Laffer curve also emerge in the case where we assumed quasilinear tastes such as those in exercise 16.10?*

Answer: Yes — you can see tax revenues initially rising and then falling as t increases in Table 16.1 which is analogous to Table 16.2 except that we used the quasilinear tastes specified in exercise 16.10.