SOLUTIONS

# 17

# Choice and Markets in the Presence of Risk

## Solutions for *Microeconomics: An Intuitive Approach with Calculus*



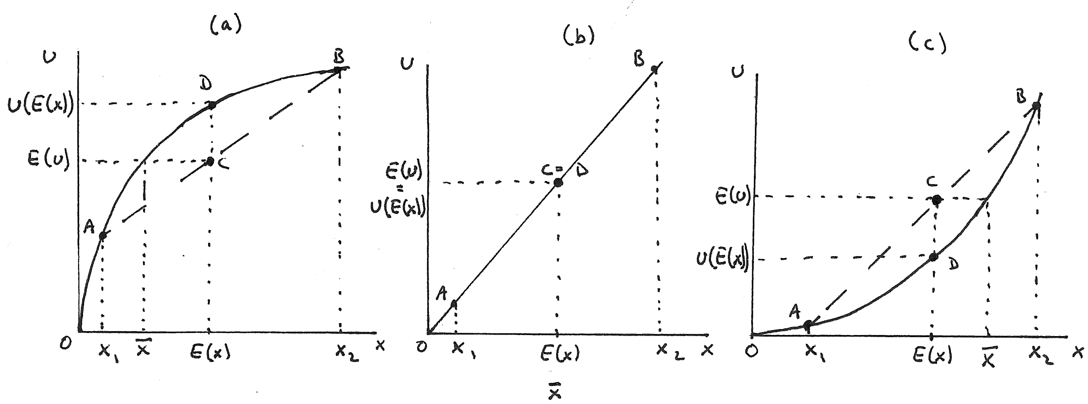
- Each end-of-chapter exercise begins on a new page. This is to facilitate maximum flexibility for instructors who may wish to share answers to some but not all exercises with their students.
- If you are assigning only the A-parts of exercises in *Microeconomics: An Intuitive Approach with Calculus*, you may wish to instead use the solution set created for the companion book *Microeconomics: An Intuitive Approach*.
- *Solutions to Within-Chapter Exercises are provided in the student* Study Guide.

**17.1** *In this exercise we review some basics of attitudes toward risk when tastes are state-independent and, in part B, we also verify some of the numbers that appear in the graphs of part A of the chapter.*

**A:** *Suppose that there are two possible outcomes of a gamble: Under outcome A, you get \$$x_1$ and under outcome B you get \$$x_2$ where $x_2 > x_1$. Outcome A happens with probability $\delta = 0.5$ and outcome B happens with probability $(1-\delta) = 0.5$.*

(a) *Illustrate three different consumption/utility relationships — one that can be used to model risk averse tastes over gambles, one for risk neutral tastes and one for risk loving tastes.*

Answer: This is done in panels (a) through (c) of Graph 17.1.




Graph 17.1: Different Attitudes about Risk

(b) *On each graph illustrate your expected consumption on the horizontal axis and your expected utility of facing the gamble on the vertical. Which of these — expected consumption or expected utility — does not depend on whether your degree of risk aversion*

Answer: The expected consumption level is simply $E(x) = 0.5x_1 + 0.5x_2$ and is illustrated in each panel as lying halfway between $x_1$ and $x_2$ on the horizontal axis. It does not depend on attitudes toward risk because it is simply a probability-weighted average of the two consumption levels that might happen. The expected utility $E(u)$ is the probability-weighted average of the utilities associated with each of the two possible outcomes — and is read off the line connecting $A$ and $B$ in each panel.

(c) *How does the expected utility of the gamble differ from the utility of the expected consumption level of the gamble in each graph?*

Answer: The expected consumption level of the gamble is $E(x)$. The utility associated with that level of consumption is read off the consumption/utility relationship itself — and is indicated as $u(E(x))$ in each panel of the graph. It is the utility the person gets from getting the expected consumption level without risk. It differs from the utility of the gamble because, although the gamble has the same expected consumption value, it involves risk. If you don't like risk, then the utility of the gamble will be less than the utility of the expected value of the gamble (as in panel (a)). But if you like risk, the utility of the gamble will be greater than the utility of the expected value of the gamble (as in panel (c)). The two will be the same in the case of risk neutrality (panel (b)) where the individual does not care one way or another about risk.

(d) *Suppose I offer you \$$\overline{x}$ to not face this gamble. Illustrate in each of your graphs where $\overline{x}$ would lie if it makes you just indifferent between taking $\overline{x}$ and staying to face the gamble.*

Answer: This is illustrated in each panel as the quantity that, if obtained without risk, will provide the same utility as the expected utility $E(u)$ of the gamble. It is what we called in the text the certainty equivalent.

(e) *Suppose I come to offer you some insurance — for every dollar you agree to give me if outcome B happens, I will agree to give you y dollars if outcome A happens. What's y if the deal I am offering you does not change the expected value of consumption for you?*

Answer: If the expected value of consumption is to remain unchanged, it must mean the expected value of what you are getting is the same as the expected value of what you are paying. When you agree to pay me $1 if $B$ happens, you agree to give me $1 with probability 0.5 (since $B$ happens with probability 0.5). Thus, the expected value of what you are giving me is 0.5. In return I give you $y if $A$ happens — which means the expected value of what I am giving you is 0.5y because $A$ happens with probability 0.5. For the expected value of consumption to remain the same, it must therefore be the case that 0.5y=0.5 — i.e. y=$1.

(f) *What changes in your 3 graphs if you buy insurance of this kind — and how does it impact your expected consumption level on the horizontal axis and the expected utility of the remaining gamble on the vertical?*

Answer: In each graph, $x_1$ increases by the same amount that $x_2$ decreases as I buy such insurance — thus reducing the risk of the gamble. However, the expected value $E(x)$ remains the same. In panel (a), however, the line on which expected utility is measured shifts up as a result of insurance — implying that $E(u)$ increases with insurance (as the expected value of the gamble remains unchanged but risk falls). But in panel (c), the line on which $E(u)$ is measured falls with insurance — implying the the expected utility of the gamble falls as risk is decreased by insurance (while the expected value of consumption remains unchanged). This should make sense: In panel (a), you dislike risk — while in panel (c) you like it. Insurance that keeps the expected value of the gamble unchanged will therefore makes you better off in panel (a) and worse off in panel (c) — because such insurance reduces risk. In panel (b), on the other hand, we don't care about risk one way or another — which implies insurance that lowers risk without changing the expected consumption value of the gamble leaves you indifferent.

**B:** *Suppose we can use the function $u(x) = x^\alpha$ for the consumption/utility relationship that allows us to represent your indifference curves over risky outcomes using an expected utility function. Assume the rest of the set-up as described in A.*

(a) *What value can $\alpha$ take if you are risk averse? What if you are risk neutral? What if you are risk loving?*

Answer: When $0 < \alpha < 1$, we get the concave shape required for risk aversion; when $\alpha = 1$, we simply get the equation of a line $u(x) = x$ and thus get the shape required for risk neutrality; and if $\alpha > 1$, we get the convex shape required for risk loving. These correspond to the cases graphed in panels (a) through (c) of Graph 17.1.

(b) *Write down the equations for the expected consumption level as well as the expected utility from the gamble. Which one depends on $\alpha$ and why?*

Answer: The expected consumption value of the gamble is given by

$$E(x) = \delta x_1 + (1-\delta)x_2 = 0.5x_1 + 0.5x_2 \tag{17.1}$$

which does not depend on $\alpha$ because it has nothing to do with tastes. The expected utility is given by

$$U = E(u) = \delta u(x_1) + (1-\delta)u(x_2) = 0.5x_1^\alpha + 0.5x_2^\alpha. \tag{17.2}$$

(c) *What's the equation for the utility of the expected consumption level?*

Answer: This is

$$u(0.5x_1 + 0.5x_2) = (0.5x_1 + 0.5x_2)^\alpha. \tag{17.3}$$

(d) *Consider $\overline{x}$ as defined in A(d). What equation would you have to solve to find $\overline{x}$?*

Answer: It has to be the case that $u(\overline{x}) = E(u)$; i.e.

$$\overline{x}^{\alpha} = 0.5x_1^{\alpha} + 0.5x_2^{\alpha}. \tag{17.4}$$

(e) *Suppose $\alpha = 1$. Solve for $\overline{x}$ and explain your result intuitively.*

Answer: In this case, equation (17.4) simply becomes

$$\overline{x} = 0.5x_1 + 0.5x_2 \tag{17.5}$$

where the right hand side is simply $E(x)$. This is reflected in panel (b) of Graph 17.1 where tastes are risk neutral and the certainty equivalent of a gamble is simply equal to the expected consumption value of the gamble (since risk neutral individuals don't care one way or another about the risk of the gamble).

(f) *Suppose that, instead of 2 outcomes, there are actually 3 possible outcomes: A, B and C, with associated consumption levels $x_1$, $x_2$ and $x_3$ occurring with probabilities $\delta_1$, $\delta_2$ and $(1-\delta_1-\delta_2)$. How would you write the expected utility of this gamble?*

Answer: You would then simply write it as

$$\begin{aligned} U = E(u) &= \delta_1 u(x_1) + \delta_2 u(x_2) + (1-\delta_1-\delta_2)u(x_3) = \\ &= \delta_1 x_1^{\alpha} + \delta_2 x_2^{\alpha} + (1-\delta_1-\delta_2)x_3^{\alpha}. \end{aligned} \tag{17.6}$$

(g) *Suppose that u took the form*

$$u(x) = 0.1x^{0.5} - \left(\frac{x}{100{,}000}\right)^{2.5} \tag{17.7}$$

*This is the equation that was used to arrive most of the graphs in part A of the chapter, where x is expressed in thousands but plugged into the equation as its full value; i.e. consumption of 200 in a graph represents $x = 200{,}000$. Verify the numbers in Graphs 17.1 and 17.3. (Note that the numbers in the graphs are rounded.)*

Answer: For Graph 17.1, the utility levels associated with points $A$, $B$, $C$ and $D$ are

$$\text{For } A\text{: } u(250{,}000) = 0.1(250{,}000)^{0.5} - \left(\frac{250{,}000}{100{,}000}\right) = 40.1179 \approx 40 \tag{17.8}$$

$$\text{For } B\text{: } u(10{,}000) = 0.1(10{,}000)^{0.5} - \left(\frac{10{,}000}{100{,}000}\right) = 9.9968 \approx 10 \tag{17.9}$$

$$\text{For } C\text{: } E(u) = 0.25(10) + 0.75(40) = 32.5 \tag{17.10}$$

$$\text{For } D\text{: } u(190{,}000) = 0.1(190{,}000)^{0.5} - \left(\frac{190{,}000}{100{,}000}\right) = 38.613 \approx 38.5 \tag{17.11}$$

In Graph 17.3 of the text, we also calculated the certainty equivalent. Since the expected utility of the gamble is 32.5, the certainty equivalent $\overline{x}$ must satisfy $u(\overline{x}) = 32.5$. Plugging 115,000 (which appears in the graph as 115 on the horizontal axis) into the consumption/utility relationship $u(x)$, we get

$$u(115{,}000) = 0.1(115{,}000)^{0.5} - \left(\frac{115{,}000}{100{,}000}\right) = 32.4934 \approx 32.5 \tag{17.12}$$

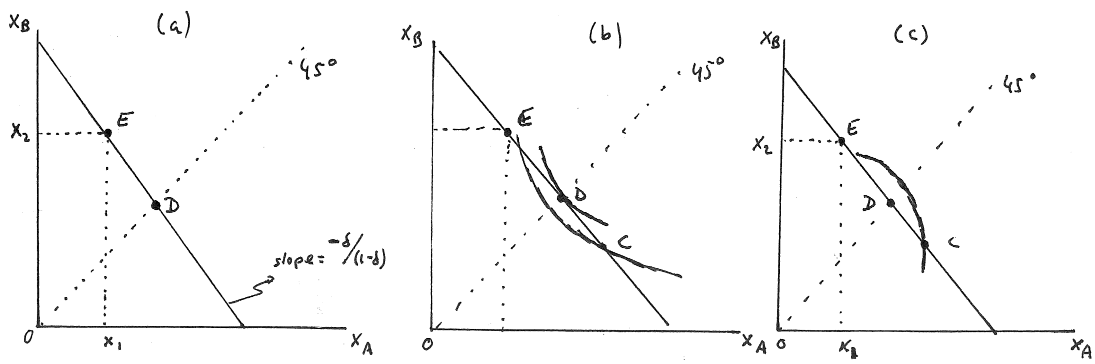
which verifies that indeed this consumer is indifferent between the gamble and getting \$115,000 for sure.

**17.2** *In our development of consumer theory, we made a big point about the fact that neoclassical economics does not put much stock in the idea of "cardinally" measuring utility (in terms of units of happiness or "utils"). Rather, our theory of consumer behavior is based only on the assumption that individuals can simply "order" pairs of bundles in terms of which they prefer or whether they are indifferent between the two. In this sense, we said* neoclassical consumer theory was ordinal and not cardinal. *We now ask whether the same continues to hold for our theory of choice in the presence of risk.*

**A:** *Return to the example from exercise 17.1 — where consumption levels differ depending on whether outcome A or outcome B occurs (where A occurs with probability $\delta$ and B with probability $(1-\delta)$). In the absence of insurance, these outcomes are $x_1$ and $x_2$ respectively (with $x_1 < x_2$).*

(a) *Draw a graph with consumption $x_A$ in outcome A on the horizontal and consumption $x_B$ in outcome B on the vertical axis. Then locate $(x_1, x_2)$ — the consumption levels you will enjoy in the absence of insurance depending on which outcome occurs.*

Answer: This is done in panel (a) of Graph 17.2.




Graph 17.2: Choice with Risk: Cardinal or Ordinal?

(b) *Calculate, as you did in part A(e) of exercise 17.1, how much I would have to give you in outcome A if you agree to give me \$1 in outcome B assuming we want your expected consumption level to remain unchanged.*

Answer: If the expected value of consumption is to remain unchanged, it must mean the expected value of what you are getting is the same as the expected value of what you are paying. When you agree to pay me \$1 if *B* happens, you agree to give me \$1 with probability $(1-\delta)$ (since *B* happens with probability $(1-\delta)$). Thus, the expected value of what you are giving me is $(1-\delta)$. In return I give you some amount \$$y$ if *A* happens — which means the expected value of what I am giving you is $\delta y$ because *A* happens with probability $\delta$. For the expected value of consumption to remain the same, it must therefore be the case that $\delta y = (1-\delta)$ — i.e. $y = (1-\delta)/\delta$.

(c) *Now identify all bundles in your graph that become available if we assume that you and I are willing to make trades of this kind on these terms — i.e. on terms that keep your expected consumption unchanged. Indicate the slope (in terms of $\delta$) of the line you have just drawn.*

Answer: This is illustrated in panel (a) of Graph 17.2 by the line through *E*. The slope of this line is the opportunity cost of one more dollar of consumption in outcome *A* (in terms of decreased consumption in outcome *B*). In the previous part we calculated that we can trade \$1 in *B* for $(1-\delta)/\delta$ in *A* — which is the opportunity cost of consumption in *B* (in terms of decreased consumption in *A*). The opportunity cost of increased consumption in *A* is therefor the inverse — giving us a slope $-\delta/(1-\delta)$ for the line of outcome bundles that keeps expected consumption constant. (This is the inverse of the slopes in the graphs in the text — because in the text the probability associated with the outcome on the horizontal axis was $(1-\delta)$ while here it is $\delta$.)

(d) *If you are risk neutral, should you care which bundle you face on the line you just drew?*

Answer: If you are risk neutral, all you care about is the expected value of consumption — you simply do not care whether or not there is risk or how much risk there is. All the bundles on the line we have drawn represent outcome bundles with the same expected consumption value but different levels of risk. A risk neutral individual should be indifferent between all of them.

(e) *We can define someone as risk averse if, when faced with two gambles that give rise to the same expected consumption level, she prefers the one that has less risk. Using this definition, which bundle on our line should a risk averse individual prefer? Could the same bundle be optimal for someone that loves risk?*

Answer: Since all the outcome bundles on our line have the same expected consumption value, the risk averse person would want to choose the outcome bundle with the least risk — i.e. the outcome bundle that has $x_A = x_B$ where all risk is eliminated. This outcome bundle lies at the intersection of the line with the 45 degree line and is labeled $D$ in panel (a) of Graph 17.2. This bundle could never be optimal for someone who is risk loving and can choose from our "budget line" — because all the bundles on the line have the same expected consumption value and therefore only differ in terms of risk. If you love risk, you won't choose the bundle that minimizes risk.

(f) *Now suppose that we assume individuals can make ordinal comparisons between bundles — i.e. when faced by two bundles in your graph, they can tell us which they prefer or whether they are indifferent. Suppose these rankings are "rational", that "more is better" and that there are "no sudden jumps" as we defined these in our development of consumer theory earlier in the text. Is this sufficient to allow us to assume there exist downward-sloping indifference curves which describe an individual's tastes over the risky gambles we are graphing?*

Answer: Yes. On a merely technical level, there is no difference between what we are doing here and what we did in Chapter 4. The difference is that in Chapter 4 we interpreted a bundle $(x_1, x_2)$ as a basket of goods, both of which we consume — whereas now we interpret the same bundle as a consumption level $x_1$ that occurs if outcome $A$ happens and a consumption level $x_2$ that occurs if outcome $B$ happens. Thus, we now do not assume that you consume both $x_1$ and $x_2$ — you'll only consume one of these depending on which outcome happens. But, as a technical matter, the preferences over pairs $(x_1.x_2)$ are just rankings that satisfy some properties.

(g) *What does your answer to (d) further imply about these indifference curves when tastes are risk neutral?*

Answer: Given that we concluded that someone with risk neutral tastes would be indifferent between all the outcome bundles on the line we have drawn through $E$, we can conclude that indifference curves for risk neutral individuals will have to be downward sloping lines (with constant $MRS$). In our treatment of consumer theory in the absence of risk, we would have called these goods perfect substitutes for such an individuals. Now, consumption under outcome $A$ and outcome $B$ are, in some sense, again perfect substitutes — because a risk neutral individual only cares about the probability-weighted average of these — and not exactly how that average comes about.

(h) *Now consider the case of risk aversion. Pick a bundle C that lies off the 45-degree line on the "budget line" you have drawn in your graph. In light of your answer to (e), is the point D that lies at the intersection of your "budget line" with the 45-degree line more or less preferred? What does this imply for the shape of the indifference curve that runs through C?*

Answer: This is illustrated in panel (b) of Graph 17.2 where $C$ is picked to just arbitrarily lie somewhere on our budget line but not on the 45 degree line. We already concluded that $D$ is the best point on the budget line for someone who is risk averse — so $D$ must be preferred to $C$ by someone who is risk averse. This then implies that the downward sloping line that passes through $C$ must lie "below" $D$ — or, put differently, it must mean that $D$ lies in the region to the northeast of the indifference curve through $C$. And this further implies that this indifference curve must have the usual convex shape as illustrated in panel (b) of the Graph.

(i) *What does your answer to (e) imply about the $MRS$ along the 45-degree line in your graph?*

Answer: The fact that $D$ is the best point from the budget line for a risk averse individual implies that her indifference curve is tangent to the budget line at that point. Given that the slope of the budget line is $-\delta/(1-\delta)$, it must therefore be the case that the $MRS$ along the 45 degree line is $-\delta/(1-\delta)$.

(j) True or False*: Risk aversion implies strict convexity of indifference curves over bundles of consumption for different outcomes, with all risk averse tastes sharing the same $MRS$ along the 45-degree line if tastes are state-independent.*

Answer: This is true, as we have just shown in the preceding parts of the question.

(k) True or False*: As the probability of each outcome changes, so do the indifference curves.*

Answer: This is also true. For the risk neutral case, for instance, we have concluded that the slope of the linear indifference curves is equal to $-\delta/(1-\delta)$ — which changes as $\delta$ changes. Similarly, we have concluded that this is the slope for risk averse tastes along the 45 degree line — and thus indifference curves for risk averse individuals similarly change in slope as $\delta$ changes.

(l) *Have we needed to make any appeal to being able to measure utility in "cardinal" terms?* True or False*: Although risk aversion appears to arise from how we measure utility in our graphs of consumption/utility relationships (such as those in exercise 17.1), the underlying theory of tastes over risky gambles does not in fact require any such cardinal measurements.*

Answer: No, we have not needed to appeal to cardinal measurements of utility in describing the underlying indifference curves over risky outcome bundles. We have shown that risk aversion has implications for how indifference curves look — but not for how they are numbered. Thus, the statement is true.

(m) *Repeat (h) for the case of someone who is risk loving.*

Answer: This is illustrated in panel (c) of Graph 17.2. The risky $C$ must now be preferred to safe $D$ (given that expected consumption levels are the same) — which means that the indifference curve that passes through $C$ must lie "above" $D$. Put differently, $D$ must lie in the region that is less preferred to the bundles on the indifference curve that passes through $C$ — i.e. in the region to the southwest of the indifference curve through $C$. This implies that the indifference curve must bend outward — i.e. it is no longer convex.

**B:** *Consider again the case of the consumption/utility relationship described by $u(x) = x^\alpha$ with $\alpha > 0$. In exercise 17.1B(a), you should have concluded that $\alpha < 1$ implies risk aversion, $\alpha = 1$ implies risk neutrality and $\alpha > 1$ implies risk loving — because the first results in a concave relationship, the second in an upward sloping line and the third in a convex relationship.*

(a) *Let $x_A$ represent consumption under outcome A (which occurs with probability $\delta$) and $x_B$ consumption under outcome B (which occurs with probability $(1-\delta)$.) Suppose we can in fact use $u(x)$ to express tastes over risky gambles as expected utilities. Define the expected utility function $U(x_A, x_B)$.*

Answer: This is

$$U(x_A, x_B) = \delta u(x_A) + (1-\delta)u(x_B) = \delta x_A^\alpha + (1-\delta)x_B^\alpha. \quad (17.13)$$

(b) *Next, consider the shape of the indifference curves that are represented by the expected utility function $U$. First, derive the $MRS$ of $U(x_A, x_B)$.*

Answer: The $MRS$ is

$$MRS = -\frac{\partial U/\partial x_A}{\partial U/\partial x_B} = -\frac{\alpha\delta x_A^{(\alpha-1)}}{\alpha(1-\delta)x_B^{(\alpha-1)}} = -\left(\frac{\delta}{(1-\delta)}\right)\left(\frac{x_B}{x_A}\right)^{(1-\alpha)} \quad (17.14)$$

(c) *What is the $MRS$ when $\alpha = 1$? How does this compare to your answer to A(g)?*

Answer: Consistent with our answer to A(g), $MRS = -\delta/(1-\delta)$ when $\alpha = 1$ — i.e. the indifference curves have constant negative slope and are thus just downward sloping lines with slope equal to the terms of trade that imply constant expected consumption.

(d) *Regardless of the size of $\alpha$, what is the $MRS$ along the 45-degree line? How does this compare to your answer to A(i) for risk averse tastes?*

Answer: Consistent with our answer to A(i), the $MRS$ when $x_A = x_B$ is $-\delta/(1-\delta)$.

(e) *Is the $MRS$ diminishing — giving rise to convex tastes? Does your answer depend on what value $\alpha$ takes? How does your answer compare to your answer to A(h)?*

Answer: We move downward along an indifference curve as $(x_B/x_A)$ falls. When $0 < \alpha < 1$, equation (17.14) then implies that the $MRS$ gets smaller in absolute value as we move down along an indifference curve — implying diminishing $MRS$ and the usual convex shape. This is consistent with our conclusion in A(h) that risk aversion implies convex indifference curves. When $\alpha = 1$, equation (17.14) implies that the $MRS$ does not change as we move down an indifference curve — implying linear indifference curves consistent with our conclusions about the implications of risk neutrality. Finally, when $\alpha > 1$, the exponent on the ratio $(x_B/x_A)$ in equation (17.14) becomes negative — implying that the $MRS$ *increases* in absolute value as we move down an indifference curve — giving us the non-convex shape we concluded must hold for risk loving tastes in A(m).

(f) *What do indifference curves look like when $\alpha > 1$?*

Answer: As we just concluded, this implies outward bending (or non-convex) indifference curves. However, the $MRS$ along the 45 degree line is still $-\delta/(1-\delta)$ as we concluded in (d) irrespective of the value of $\alpha$.

(g) *Do the slopes of indifference curves change with $\delta$? How does your answer compare to your answer to A(k)?*

Answer: It is immediately clear from equation (17.14) that the $MRS$ depends on $\delta/(1-\delta)$ which changes as $\delta$ changes. Thus, the slopes of indifference curves change as $\delta$ changes — consistent with our answer to A(k).

(h) *Suppose we used $u(x) = \beta x^{\alpha}$ (instead of $u(x) = x^{\alpha}$) to calculate expected utilities. Would the indifference map that arises from the expected utility function change?*

Answer: The expected utility function would then be

$$U = \delta\beta x_A^{\alpha} + (1-\delta)\beta x_B^{\alpha}, \tag{17.15}$$

with $MRS$ equal to

$$MRS = -\frac{\partial U/\partial x_A}{\partial U/\partial x_B} = -\frac{\alpha\delta\beta x_A^{(\alpha-1)}}{\alpha(1-\delta)\beta x_B^{(\alpha-1)}} = -\left(\frac{\delta}{(1-\delta)}\right)\left(\frac{x_B}{x_A}\right)^{(1-\alpha)}. \tag{17.16}$$

Since the $\beta$ cancels out, we get the same $MRS$ as in equation (17.14) — thus the indifference map is unaffected.

(i) *Suppose we used $u(x) = x^{\alpha\beta}$ (instead of $u(x) = x^{\alpha}$) to calculate expected utilities. Would the indifference map that arises from the expected utility function change?*

Answer: The expected utility function would then be

$$U = \delta x_A^{\alpha\beta} + (1-\delta)x_B^{\alpha\beta}, \tag{17.17}$$

with $MRS$ equal to

$$MRS = -\frac{\partial U/\partial x_A}{\partial U/\partial x_B} = -\frac{\alpha\beta\delta x_A^{(\alpha\beta-1)}}{\alpha\beta(1-\delta)x_B^{(\alpha\beta-1)}} = -\left(\frac{\delta}{(1-\delta)}\right)\left(\frac{x_B}{x_A}\right)^{(1-\alpha\beta)} \tag{17.18}$$

This is not the same as the original $MRS$ from equation (17.14). By changing the exponent on the last term, we have changed the curvature — and may in fact have turned risk averse tastes into risk loving ones or the other way around.

(j) True or False*: The tastes represented by expected utility functions are immune to* linear *transformations of the consumption/utility relationship $u(x)$ that is used to calculate expected utility — but are not immune to all types of positive transformations.*

Answer: This is true, as we have just shown in the two previous parts.

(k) *Consider the expected utility function $U(x_A, x_B)$ that uses $u(x) = x^{\alpha}$. Will the underlying indifference curves change under any order-preserving transformation?*

Answer: Mathematically, the expected utility function $U(x_A, x_B)$ is no different than utility functions (that take such a form) over bundles of goods. We know from our work on consumer theory that we can undertake order-preserving transformations of such functions without altering the shapes of indifference curves (and just changing labels) — so there is no reason that could not also be done to the expected utility function.

(l) True or False*: Expected utility functions can be transformed like all utility functions without changing the underlying indifference curves, but such transformations can then no longer be written as if they were the expected value of two different utility values emerging from an underlying function u*.

Answer: This is also true. We argued in the preceding part that we can in fact subject expected utility functions — like all utility functions over bundles of goods — to order preserving transformations without altering the shapes of indifference curves. Suppose, for instance, we square the function $U(x_A, x_B) = \delta u(x_A) + (1-\delta)u(x_B)$. You can check for yourself that this does not alter the $MRS$ — but the resulting function

$$\left[U(x_A, x_B)\right]^2 = \left[\delta u(x_A) + (1-\delta)u(x_B)\right]^2 = \tag{17.19}$$

$$= \delta^2 \left(u(x_A)\right)^2 + \delta(1-\delta)u(x_A)u(x_B) + (1-\delta)^2 \left(u(x_B)\right)^2 \tag{17.20}$$

is no longer the simple probability-weighted average of two utility values associated with the two outcomes.

(m) *In light of all this, can you reconcile the assertion that expected utility theory is not a theory that relies on cardinal interpretations of utility?*

Answer: The impression that expected utility theory is based on cardinal interpretations of utility comes from the fact that the shape of the $u(x)$ function is related to attitudes about risk — and this shape is not immune to transformations that we can usually make with respect to utility functions. However, the role of $u(x)$ in expected utility theory is *not* as a real "utility function". Rather, von-Neumann and Morgenstern showed that (assuming preference relations over gambles satisfy the independence axiom) we can always find a function $u$ over consumption that will allow us to represent *ordinal* tastes (i.e. indifference curves) over risky gambles with an expected utility function that assigns values to outcome bundles as a simply probability-weighted average of the values assigned to individual outcomes by the $u$ function. Put differently, we take as given the indifference curves and simply know that there is a way to write a utility function that represents these and takes the expected utility form using some underlying $u$ function. The shape of these indifference curves will indeed determine what kind of $u$ function will work for us — but the right $u$ function emerges from these indifference curves and is thus simply a tool for us to be able to work with expected utility functions.

**17.3** *We have illustrated in several settings the role of actuarily fair insurance contracts* $(b, p)$ *(where b is the insurance benefit in the "bad state" and p is the insurance premium that has to be paid in either state). In this problem we will discuss it in a slightly different way that we will later use in Chapter 22.*

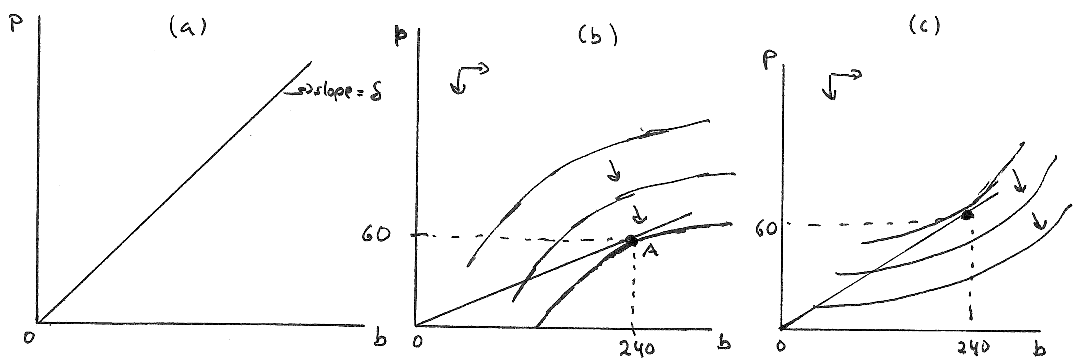
**A:** *Consider again the example, covered extensively in the chapter, of my wife and life insurance on me. The probability of me not making it is* $\delta$, *and my wife's consumption if I don't make it will be 10 and her consumption if I do make it will be 250 in the absence of any life insurance.*

(a) *Now suppose that my wife is offered a full set of actuarily fair insurance contracts. What does this imply for how p is related to* $\delta$ *and b?*

Answer: Actuarial fairness implies that what my wife pays is equal to what she receives *in expectation*. She will receive $(b-p)$ with probability $\delta$, and she will pay $p$ with probability $(1-\delta)$. Thus, actuarial fairness implies that $\delta(b-p) = (1-\delta)p$ or simply $p = \delta b$.

(b) *On a graph with b on the horizontal axis and p on the vertical, illustrate the set of all actuarily fair insurance contracts.*

Answer: This is illustrated in panel (a) of Graph 17.3.




Graph 17.3: Tastes over premiums $p$ and benefits $b$

(c) *Now think of what indifference curves in this picture must look like. First, which way must they slope (given that my wife does not like to pay premiums but she does like benefits)?*

Answer: Indifference curves must slope up. Consider any initial bundle $(b, p)$. We know that an increase in $b$ to $b'$ will make my wife unambiguously better off — which means that the bundle containing $b'$ that is indifferent to $(b, p)$ must have an offsetting increase in $p$ which, by itself, would make my wife unambiguously worse off. You can thus think of this as indifference curves over two goods where one of the goods, namely the premium $p$, is really a "bad".

(d) *In which direction within the graph does my wife have to move in order to become unambiguously better off?*

Answer: She becomes unambiguously better off as $p$ falls and $b$ increases — thus, she becomes better off moving to the southeast in the graph.

(e) *We know my wife will fully insure if she is risk averse (and her tastes are state-independent). What policy does that imply she will buy if* $\delta = 0.25$?

Answer: As was shown in the text, this would imply buying a policy $(b, p) = (240, 60)$ which satisfies the actuarily fair relationship derived in (a). Under this policy, she would have consumption of only 190 in the "good" state (where she has income of 250 but needs to pay the premium of 60) but she also has consumption of 190 in the "bad" state (where she has income of 10, has to pay the premium of 60 but also gets a benefit of 240).

(f) *Putting indifference curves into your graph from (b), what must they look like in order for my wife to choose the policy that you derived in (e).*

Answer: This is illustrated in panel (b) of Graph 17.3.

(g) *What would her indifference map look like if she were risk neutral? What if she were risk-loving?*

Answer: If her tastes were risk neutral, she should be indifferent between all the actuarily fair insurance policies along the budget line $p = \delta b$. Thus, her indifference curves must be straight lines with slope $\delta$. If she were risk loving, then she would still become better off moving to the southeast in the graph, but her indifference curves would bow in the opposite direction from those involving risk aversion. This is pictured in panel (c) of Graph 17.3.

**B:** *Suppose $u(x) = \ln(x)$ allows us to write my wife's tastes over gambles using the expected utility function. Suppose again that my wife's income is 10 if I am not around and 250 if I am — and that the probability of me not being around is $\delta$.*

(a) *Given her incomes in the good and bad state in the absence of insurance, can you use the expected utility function to arrive at her utility function over insurance policies $(b, p)$?*

Answer: Her expected utility is

$$U(x_B, x_G) - \delta u(x_B) + (1-\delta)u(x_G) = \delta \ln x_B + (1-\delta)\ln x_G \tag{17.21}$$

where $x_B$ is her consumption in the event that I am not around and $x_G$ is her consumption in the event that I am around. For any insurance policy $(b, p)$, $x_B = (10 + b - p)$ and $x_G = (250 - p)$. We can therefore write her expected utility of the policy $(b, p)$ as

$$U(b, p) = \delta \ln(10 + b - p) + (1-\delta)\ln(250 - p). \tag{17.22}$$

(b) *Derive the expression for the slope of an indifference curve in a graph with b on the horizontal and p on the vertical axis.*

Answer: This is just the $MRS$ which is

$$\begin{aligned} MRS = -\frac{\partial U(b,p)/\partial b}{\partial U(b,p)/\partial p} &= -\frac{\delta/(10+b-p)}{\left(-\delta/(10+b-p)\right) - \left((1-\delta)/(250-p)\right)} \\ &= \frac{\delta(250-p)}{\delta(250-p) + (1-\delta)(10+b-p)}. \end{aligned} \tag{17.23}$$

(c) *Suppose $\delta = 0.25$ and my wife has fully insured under policy $(b, p) = (240, 60)$. What is her $MRS$ now?*

Answer: Plugging $\delta = 0.25$, $b = 240$ and $p = 60$ into equation (17.23) gives us

$$MRS = \frac{0.25(190)}{0.25(190) + 0.75(190)} = 0.25. \tag{17.24}$$

(d) *How does your answer to (c) compare to the the slope of the budget formed by mapping out all actuarily fair insurance policies (as in A(b))? Explain in terms of a graph.*

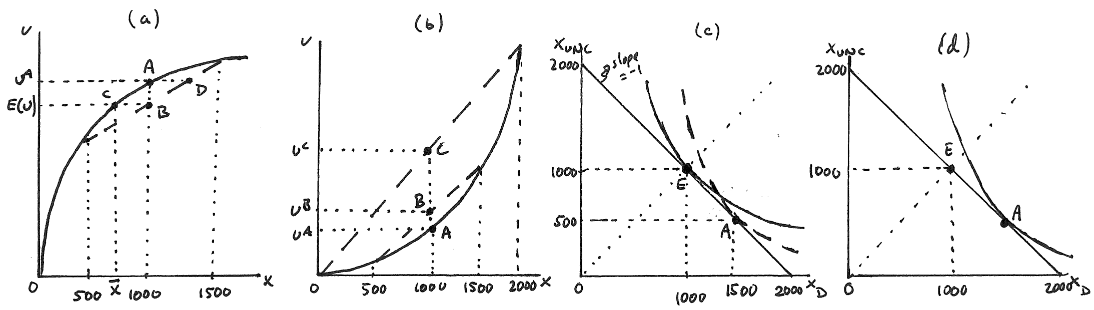
Answer: We concluded in A(b) that the slope of the budget line is $\delta$ which is equal to 0.25 in our case. Now we concluded that, at the actuarily fair full insurance policy, the $MRS$ of our indifference curve is also 0.25. Thus, the indifference curve is tangent to the budget line at the full insurance policy — implying that my wife is optimizing by fully insuring in the acuarily fair insurance market. We depicted this already in panel (b) of Graph 17.3 where tastes were assumed to be risk averse (as they are when we can use the concave function $u(x) = \ln x$ to represent tastes over gambles using the expected utility function.)

**17.4** Everyday Application: *Gambling on Sporting Events: Some people gamble on sporting events strictly to make money while others care directly about which teams win quite apart from whether or not they gambled on the game.*

**A:** *Consider your consumption level this weekend and suppose that you have \$1,000 available. On Friday night, Duke is playing UNC in an NCAA basketball tournament, and you have the opportunity to bet an amount* $\$X \leq \$1,000$ *on the game. If you bet \$X, you will only have (\$1,000-\$X) if you lose the bet, but you will have (\$1,000+\$X) if you win. We would say in this case that you are being given even odds (since your winnings if you win are as big as your losses if you lose). Suppose that you believe that each team has a 50% chance of winning (and that, if a game is tied, it goes into overtime until the tie is broken.)*

(a) *First, suppose you don't care about sports and only care which team wins to the extent to which it increases your consumption. I offer you the opportunity to place a bet of* $X = \$500$ *on either team. Will you take the bet?*

Answer: No, you will not take the bet. This is illustrated in panel (a) of Graph 17.4 where you can consume at $A$ without betting — thus getting utility $u^A$ — while your expected utility from betting \$500 is read off at $B$ as $E(u)$. Because of the concave shape of the consumption/utility relationship (that incorporates risk aversion), $E(u) < u^A$ — which implies you are better off not betting.




Graph 17.4: Betting on Duke (or UNC)

(b) *Suppose you got a little inebriated and wake up in the middle of the game to find that you did place the \$500 bet on Duke. You notice the game is tied — and you ask me if you can get out of the bet. How much would you be willing to pay to get out?*

Answer: Since you placed a bet, your expected utility is now $E(u)$ read off $B$ in panel (a) of Graph 17.4. If you can get out of the bet, you will again be on your consumption/utility relationship — and the most you'd be willing to pay is an amount that leaves you with a consumption level $\overline{x}$ that leaves you with the same utility level as $E(u)$. This occurs at $C$ in the graph — which implies you'd be willing to pay $(\$1,000 - \overline{x})$ to get out of the bet. (This is your risk premium).

(c) *Suppose that, just as you come to realize that alcohol had made you place the bet, Duke scores a series of points and you now think that the probability of Duke winning is* $\delta > 0.5$. *Might you choose to stay in the bet even if I give you a chance to get out for free?*

Answer: You might, or you might not, depending on just how much you think $\delta$ has increased. In panel (a) of Graph 17.4, an increase in $\delta$ moves your expected utility up along the dashed line starting at $B$ when $\delta = 0.5$. If $\delta$ has gone up sufficiently for your expected utility to reach $D$, then you will stay in the bet (and would in fact choose to enter the bet if you had not already). But if $\delta$ has not gone up enough for your expected utility to reach $D$, you will get out of the bet if you can do so for free (and would be willing to pay something to get out if you had to.)

(d) *Suppose you were actually a risk-lover. If you could choose to place any bet (that you can afford), how much would you bet on the game (assuming you again think each team is equally likely to win)?*

Answer: This is illustrated in panel (b) of Graph 17.4 where the consumption/utility relationship is now convex in order to incorporate risk-loving. If you do not enter a bet, you will simply consume \$1,000 and get utility $u^A$. If you enter a \$500 bet, you will get expected utility $u^B$ and if you enter a \$1000 bet you will get expected utility of $u^C$. Of these three choices, $u^C$ is clearly the highest — and you should be able to see that it in fact is higher than the expected utility from any gamble of less than \$1,000 (which is the most you can bet). Thus, you will bet \$1,000 (on one of the teams).

(e) *Illustrate your answer to (a) and (c) again, but this time in a graph with $x_D$ on the horizontal and $x_{UNC}$ on the vertical (with the two axes representing consumption in the "state" where Duke wins (on the horizontal) and where UNC wins (on the vertical).) (*Hint*: The "budget constraint" in the picture does not change as you go from re-answering (a) to re-answering (c).)*

Answer: The "endowment" point in panel (c) of Graph 17.4 is indicated by $E$ — and it is simply the consumption level of \$1,000 the you will enjoy in each state if you do not bet. Risk averse tastes that are state-independent imply that your indifference curves take the usual convex shape and, furthermore, that the optimum involves no risk when your probability estimate of either team winning is the same as what is incorporated in the budget constraint. Thus, $E$ is the optimal outcome bundle as indicated by the tangency with the solid indifference curve. When $\delta$ goes above 0.5 in (c), the budget constraint in the picture does not change — because the bets are still incorporating even odds. But indifference curves — which arise from the probability weighted sum of outcome utilities — become steeper as you place greater probability on Duke winning. Taking the bet implies that you are accepting $A$ over $E$. The dashed indifference curve has the steeper slopes associated with the increased probability of Duke winning — and the indifference curve that passes through $A$ lies above $E$. In this case, you would therefore take the bet even though your indifference curves continue to be convex (and thus risk averse). But it must be that your indifference curves have become sufficiently steeper — which means that $\delta$ must have increased sufficiently. If $\delta$ only increases by a little bit, the dashed indifference curves would still be steeper, but the indifference curve passing through $A$ would lie *below* $E$ — implying that you would not take the bet. (While the dashed indifference curve is drawn tangent to the budget constraint, that is not required for the answer to this problem — all that is required for you to take the bet is for the indifference curve that passes through $A$ once it incorporates the higher probability of Duke winning to lie above $E$.)

(f) *Suppose that you love Duke and hate UNC. When Duke wins, everything tastes better — and if UNC wins, there is little you want to do other than lie in bed. Might you now enter my betting pool (prior to the start of the game) even if you are generally risk averse and not at all drunk? Illustrate your answer.*

Answer: Yes. This is illustrated in panel (d) of Graph 17.4 where we again have $x_D$ and $x_{UNC}$ indicating consumption levels in the two states. Tastes are now state-dependent — which means that you can in fact believe that the two teams are equally likely to win but you'd like to have more consumption if Duke wins (when consumption means a lot) than when UNC wins (when consumption means less). This gives indifference curves of the type graphed — with the optimal bet moving you to the tangency at $A$. If only one bet is offered — i.e. if you are only offered the opportunity to bet \$500 — then it is not clear whether you will take the bet. It will depend on how much more you actually enjoy consumption when Duke wins. But as long as you enjoy it somewhat more, you will bet at least a little bit if I accept all bets of any size.

(g) True or False*: Gambling by risk averse individuals can arise if the gambler has a different probability estimate of each outcome occurring than the "house". Alternatively, it can also arise from state-dependent tastes.*

Answer: This is essentially what we have shown. First, we showed that you will not gamble if your probability estimate of each outcome is the same as mine and you are risk averse. Second, we showed that you will enter a gamble if you believe the terms of the gamble are sufficiently favorable — i.e. if you and I differ sufficiently much on how likely each of the outcomes is. And third, we showed that introducing state-dependence of your tastes can get you to gamble even if we don't differ on the probabilities.

(h) True or False*: If you are offered a bet with even odds and you believe that the odds are different, you should take the bet.*

Answer: This is false because it depends on just how different you believe the odds to be (as we have shown in (c)). If you are risk averse, it has to be the case that your estimate of the odds differs sufficiently from those offered in order for the risk of taking the gamble to be worth the higher expected payoff.

**B:** *Consider again the types of bets described in part A, and suppose the function $u(x) = x^{\alpha}$ allows us to represent your indifference curves over gambles using an expected utility function.*

(a) *Suppose $\alpha = 0.5$. What is your expected utility of betting \$500 on one of the teams, and how does this compare to your utility of not gambling?*

Answer: Your utility of not gambling is $u(1000) = 1000^{0.5} \approx 31.623$. Your expected utility of betting \$500 is

$$0.5(1000-500)^{0.5} + 0.5(1000+500)^{0.5} \approx 22.361. \tag{17.25}$$

(b) *Consider the scenario in A(b). How much would you be willing to pay to get out of the bet?*

Answer: The certainty equivalent of staying in the bet is an amount $\overline{x}$ such that $u(\overline{x})$ is equal to the expected utility of the gamble; i.e. $\overline{x}^{0.5} = 22.361$. This solves to $\overline{x} = 500$ — which implies you are willing to pay $(1000-500) = \$500$ to get out of the gamble. (The certainty equivalent and the risk premium are the same in this case, but the risk premium is the amount you are willing to pay to get out of the gamble.)

(c) *Consider the scenario in A(c). For what values of $\delta$ will you choose to stay in the bet?*

Answer: Your expected utility from staying in the gamble is

$$E(u) = (1-\delta)(1000-500)^{0.5} + \delta(1000+500)^{0.5} \approx 22.361 + 16.369\delta. \tag{17.26}$$

If you get out, you get $u(1000) \approx 31.623$. Thus, we have to set the above expected utility equal to 31.623 and solve for $\delta$ to get $\delta \approx 0.566$. Thus, you will choose to stay in the bet for all $\delta \geq 0.566$.

(d) *Suppose $\alpha = 2$. How much will you bet?*

Answer: When $\alpha = 2$, tastes are risk-loving — which implies you will be the maximum amount of \$1,000.

(e) *Consider what you were asked to do in A(e). Can you show how the $MRS$ changes as $\delta$ changes? (*Hint*: Express the expected utility function in terms of $x_{UNC}$ and $x_D$ and derive the $MRS$.) For what value of $\delta$ is the \$500 bet on Duke the optimal bet to place?*

Answer: Your expected utility is

$$U(x_D, x_{UNC}) = (1-\delta)\left(x_{UNC}\right)^{0.5} + \delta\left(x_D\right)^{0.5}. \tag{17.27}$$

The $MRS$ of the utility function $U$ is then

$$MRS = -\frac{\partial U/\partial x_D}{\partial U/\partial x_{UNC}} = -\frac{0.5\delta x_D^{-0.5}}{0.5(1-\delta)x_{UNC}^{-0.5}} = -\left(\frac{\delta}{(1-\delta)}\right)\left(\frac{x_{UNC}}{x_D}\right)^{0.5}. \tag{17.28}$$

Thus, as the probability $\delta$ that Duke wins increases, $(1-\delta)$ decreases — implying that $(\delta/(1-\delta))$ increases. This implies that, for any outcome pair $(x_D, x_{UNC})$, the $MRS$ increases in absolute value as $\delta$ increases — i.e. indifference curves become steeper. The \$500 bet will be the optimal bet if the $MRS$ at $(x_D, x_{UNC}) = (1500, 500)$ is equal to the slope of the budget constraint formed by bets that give even odds; i.e. the slope of $-1$. This tangency is described by the equation

$$-\left(\frac{\delta}{(1-\delta)}\right)\left(\frac{500}{1500}\right)^{0.5} = -1. \tag{17.29}$$

Solving for $\delta$, we get $\delta \approx 0.634$.

(f) *Suppose that $u_D(x) = \alpha x^{0.5}$ and $u_{UNC}(x) = (1-\alpha)x^{0.5}$ are two functions that allow us to represent your tastes over bets like this using an expected utility function. What is the equation for the $MRS$ in your indifference map assuming that you think the probability of each team winning is in fact 0.5? For what value of $\alpha$ will the \$500 bet be the optimal bet.*

Answer: The expected utility function now is

$$U(x_D, x_{UNC}) = 0.5\alpha(x_D)^{0.5} + 0.5(1-\alpha)(x_{UNC})^{0.5}. \tag{17.30}$$

The $MRS$ of this utility function is

$$MRS = -\frac{\partial U/\partial x_D}{\partial U/\partial x_{UNC}} = -\frac{0.25\alpha x_D^{-0.5}}{0.25(1-\alpha)x_{UNC}^{-0.5}} = -\left(\frac{\alpha}{(1-\alpha)}\right)\left(\frac{x_{UNC}}{x_D}\right)^{0.5}. \tag{17.31}$$

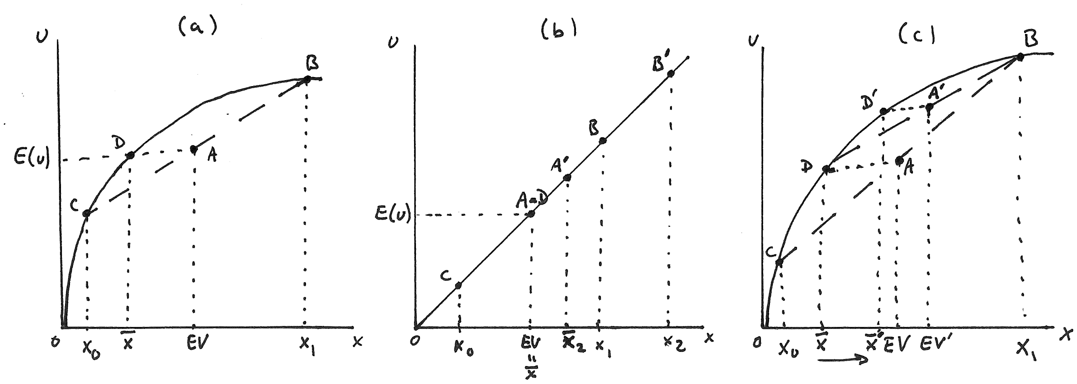
Thus, the $\alpha$ now plays the role that $\delta$ played in the previous part — which implies we will get the \$500 bet to be optimal when $\alpha = 0.634$ — i.e. you have to place more weight on consumption in the state where Duke won than in the state where UNC won. This illustrates what we concluded in A(g) — you can get risk averse individuals to engage in gambling either if they think the probabilities are sufficiently different than the terms of the bet — or you can get the same result if tastes are sufficiently state dependent.

**17.5** Everyday Application: *Teenage Sex and Birth Control*: *Consider a teenager who evaluates whether she should engage in sexual activity with her partner of the opposite sex. She thinks ahead and expects to have a present discounted level of life-time consumption of $x_1$ in the absence of a pregnancy interrupting her educational progress. If she gets pregnant, however, she will have to interrupt her education and expects the present discounted value of her life-time consumption to decline to $x_0$ — considerably below $x_1$.*

**A:** *Suppose that the probability of a pregnancy in the absence of birth control is 0.5 and assume that our teenager does not expect to evaluate consumption any differently in the presence of a child.*

(a) *Putting the present discounted value of lifetime consumption $x$ on the horizontal axis and utility on the vertical, illustrate the consumption/utility relationship assuming that she is risk averse. Indicate the expected utility of consumption if she chooses to have sex.*

Answer: This is done in panel (a) of Graph 17.5 where the concave shape of the relationship incorporates risk aversion. The expected utility of consumption if the teenager has sex is indicated as $E(u)$.




Graph 17.5: Sex, Education and Birth Control

(b) *How much must the immediate satisfaction of having sex be worth in terms of lifetime consumption in order for her to choose to have sex?*

Answer: The expected utility of consumption if she has sex is $E(u)$ — which has a certainty equivalent of $\overline{x}$. Thus, she is giving up $(x_1 - \overline{x})$ by choosing to have sex — which is the least that the experience must be worth to rationalize the action.

(c) *Now consider the role of birth control which reduces the probability of a pregnancy. How does this alter your answers?*

Answer: Birth control reduces the probability of a pregnancy — and thus shifts $EV$ in panel (a) up along the dotted line that connects $B$ and $C$. Perfectly reliable birth control would imply that $A$ shifts on top of $B$. As the probability of a pregnancy declines, $\overline{x}$ increases — implying that $(x_1 - \overline{x})$ falls. Thus, sex does not have to be as valued in order for the teenager to choose to engage in it.

(d) *Suppose her partner believes his future consumption paths will develop similarly to hers depending on whether or not there is a pregnancy — but he is risk neutral. For any particular birth control method (and associated probability of a pregnancy), who is more likely to want to have sex assuming no other differences in tastes?*

Answer: This is illustrated in panel (b) of Graph 17.5 where the consumption/utility relationship is graphed as linear to incorporate risk neutrality. The points $A$, $B$ and $C$ correspond to those in panel (a) — with $E(u)$ again representing the expected utility from consumption if sexual activity ensues. Note, however, that $\overline{x}$ — the certainty equivalent — is now equal to $EV$ — which implies that $(x_1 - \overline{x})$ is lower in panel (b) than in panel (a). The risk neutral partner therefore requires less immediate satisfaction from sexual activity to rationalize it than the risk averse partner.

(e) *As the payoff to education increases in the sense that $x_1$ increases, what does the model predict about the degree of teenage sexual activity assuming that the effectiveness and availability of birth control remains unchanged and assuming risk neutrality?*

Answer: Consider the case where the probability of a pregnancy is 0.5. We have already shown in panel (b) of Graph 17.5 that a risk neutral partner would need to place value of at least $(x_1 - \overline{x})$ on sex in order to engage in it under these assumptions. Now suppose $x_1$ increases to $x_2$. This implies the expected value as well as the certainty equivalent increase to $\overline{x}_2$ — and the increase from $\overline{x}$ to $\overline{x}_2$ is half as much as the increase from $x_1$ to $x_2$. The new minimum value that this person must place on sex in order to justify it rationally is $(x_2 - \overline{x}_2)$ — as compared to the previous $(x_1 - \overline{x}_1)$. But, since the distance from $x_1$ to $x_2$ is twice the distance from $\overline{x}$ to $\overline{x}_2$, $(x_2 - \overline{x}_2) > (x_1 - \overline{x}_1)$ — meaning the value one must place on sex to engage in it has increased. Thus, fewer people will do so.

(f) *Do you think your answer to (e) also holds under risk aversion?*

Answer: Yes. Under risk aversion, the certainty equivalent changes more slowly as $x_1$ increases — which implies that the value that one must place on sex in order to engage in it (holding birth control constant) would increase more than in the case of risk neutrality.

(g) *Suppose that a government program makes daycare more affordable — thus raising $x_0$. What happens to the number of risk averse teenagers having sex according to this model?*

Answer: This is illustrated in panel (c) of Graph 17.5 where the original certainty equivalent is $\overline{x}$ and the original minimum value one must place on sex in order to engage in it is $(x_1 - \overline{x})$. As $x_0$ increases, the certainty equivalent increases (to $\overline{x}'$) but $x_1$ remains unchanged — which implies that $(x_1 - \overline{x}')$, the new minimum value one must place on sex, is less than the original $(x_1 - \overline{x})$. Thus, more teenagers will have sex according to this model (assuming teenagers vary in the value they place on having sex).

**B:** *Now suppose that the function $u(x) = \ln(x)$ allows us to represent a teenager's tastes over gambles involving lifetime consumption using an expected utility function. Let $\delta$ represent the probability of a pregnancy occurring if the teenagers engage in sexual activity, and let $x_0$ and $x_1$ again represent the two lifetime consumption levels.*

(a) *Write down the expected utility function.*

Answer: The expected utility function is

$$U(x_0, x_1) = \delta \ln x_0 + (1-\delta) \ln x_1. \tag{17.32}$$

(b) *What equation defines the certainty equivalent? Using the mathematical fact that $\alpha \ln x + (1-\alpha)\ln y = \ln(x^{\alpha} y^{(1-\alpha)})$, can you express the certainty equivalent as a function $x_0$, $x_1$ and $\delta$?*

Answer: The certainty equivalent $\overline{x}$ is the level of consumption whose utility is equal to the expected utility of the gamble; i.e. $\overline{x}$ is such that

$$\ln \overline{x} = \delta \ln x_0 + (1-\delta) \ln x_1. \tag{17.33}$$

Using the mathematical fact pointed out in the question, this implies

$$\overline{x} = x_0^{\delta} x_1^{(1-\delta)}. \tag{17.34}$$

(c) *Now derive an equation $y(x_0, x_1, \delta)$ that tells us the least value (in terms of consumption) that this teenager must place on sex in order to engage in it.*

Answer: This is

$$y(x_0, x_1, \delta) = x_1 - \overline{x} = x_1 - x_0^{\delta} x_1^{(1-\delta)}. \tag{17.35}$$

(d) *What happens to y as the effectiveness of birth control increases? What does this imply about the fraction of teenagers having sex (as the effectiveness of birth control increases) assuming that all teenagers are identical except for the value they place on sex?*[1]

Answer: To see this, we can take the partial derivative of $y$ with respect to $\delta$. This gives us

$$\frac{\partial y(x_0,x_1,\delta)}{\partial \delta} = -(\ln x_0)x_0^{\delta}x_1^{(1-\delta)} + (\ln x_1)x_0^{\delta}x_1^{(1-\delta)} = (\ln x_1 - \ln x_0)x_0^{\delta}x_1^{(1-\delta)} > 0. \quad (17.36)$$

Thus, as $\delta$ increases, $y$ rises; and as $\delta$ decreases, $y$ falls. Birth control becoming more effective implies $\delta$ decreases — which therefore implies that the consumption value placed on sex in order for a teenager to engage in it decreases. Put differently, as birth control becomes more effective, some teenagers for whom sex was not sufficiently valuable before will now find it worth it — and thus the fraction of teenagers having sex increases.

(e) *What happens to y as the payoff from education increases in the sense that $x_1$ increases? What does this imply for the fraction of teenagers having sex (all else equal)?*

Answer: Again, we take a partial derivative to find

$$\frac{\partial y(x_0,x_1,\delta)}{\partial x_1} = 1-(1-\delta)x_0^{\delta}x_1^{-\delta} = 1-(1-\delta)\left(\frac{x_0}{x_1}\right)^{0.5} > 0. \quad (17.37)$$

The reason this expression is greater than zero is because $(x_0/x_1) < 1$ (since $x_0 < x_1$) and $(1-\delta) < 1$ — which implies the term that is being subtracted from 1 in the equation is the product of three numbers that are all below 1 (which must itself then be below 1). This then implies that an increase in $x_1$ results in an increase in $y$ — i.e. the greater payoffs to education imply that the payoff from sex must increase in order for teenagers to be willing to engage in it. As a result, all else being equal, fewer teenagers will have sex.

(f) *What happens to y as the government makes it easier to continue going to school — i.e. as it raises $x_0$? What does this imply for the fraction of teenagers having sex?*

Answer: Again, taking the right partial derivative, we get

$$\frac{\partial y(x_0,x_1,\delta)}{\partial x_0} = -\delta x_0^{(\delta-1)}x_1^{(1-\delta)} = -\delta\left(\frac{x_1}{x_0}\right)^{(1-\delta)} < 0. \quad (17.38)$$

Thus, as it gets easier to continue going to school despite a pregnancy, $y$ falls — i.e. the value a teenager must place on sex in order to engage in it falls. This implies that more teenagers will have sex.

(g) *How do your answers change for a teenager with risk neutral tastes over gambles involving lifetime consumption that can be expressed using an expected utility function involving the function $u(x) = x$?*

Answer. The expected utility function would then be $U = \delta x_0 + (1-\delta)x_1$, and the certainty equivalent would be $\overline{x} = \delta x_0 + (1-\delta)x_1$. This implies that $y$ is

$$y(x_0,x_1,\delta) = x_1 - (\delta x_0 + (1-\delta)x_1) = \delta(x_1 - x_0). \quad (17.39)$$

Taking the three partial derivatives, we then get

$$\frac{\partial y}{\partial \delta} = (x_1 - x_0) > 0\,,\ \frac{\partial y}{\partial x_1} = \delta > 0 \text{ and } \frac{\partial y}{\partial x_0} = -\delta < 0. \quad (17.40)$$

The signs of these derivatives are the same as before — implying changes in the same direction as $\delta$, $x_1$ and $x_0$ change.

(h) *How would your answers change if $u(x) = x^2$?*

Answer: The expected utility function would be $U = \delta x_0^2 + (1-\delta)x_1^2$, and the certainty equivalent $\overline{x}$ is defined by the equation $u(\overline{x}) = \overline{x}^2 = \delta x_0^2 + (1-\delta)x_1^2$ which solves to

[1] It will be helpful to recall the mathematical fact that the derivative of $x^{\alpha}$ *with respect to* $\alpha$ is equal to $x^{\alpha}\ln x$.

$$\overline{x} = \left(\delta x_0^2 + (1-\delta)x_1^2\right)^{0.5} \tag{17.41}$$

which implies

$$y(x_0, x_1, \delta) = x_1 - \left(\delta x_0^2 + (1-\delta)x_1^2\right)^{0.5}. \tag{17.42}$$

Taking the three partial derivatives, we then get

$$\frac{\partial y}{\partial \delta} = -\frac{1}{2}(x_0^2 - x_1^2)\left(\delta x_0^2 + (1-\delta)x_1^2\right)^{-0.5} = \frac{x_1^2 - x_0^2}{2\left(\delta x_0^2 + (1-\delta)x_1^2\right)^{0.5}} > 0 \tag{17.43}$$

$$\frac{\partial y}{\partial x_1} = 1 - \frac{1}{2}(2(1-\delta)x_1)\left(\delta x_0^2 + (1-\delta)x_1^2\right)^{-0.5} =$$

$$= \frac{\left(\delta x_0^2 + (1-\delta)x_1^2\right)^{0.5} - (1-\delta)x_1}{\left(\delta x_0^2 + (1-\delta)x_1^2\right)^{0.5}} > 0 \tag{17.44}$$

$$\frac{\partial y}{\partial x_0} = -2\delta x_0 \left(\delta x_0^2 + (1-\delta)x_1^2\right)^{-0.5} = \frac{-2\delta x_0}{\left(\delta x_0^2 + (1-\delta)x_1^2\right)^{0.5}} < 0. \tag{17.45}$$

Thus we again get the same inequalities — implying all the effects still operate in the same direction. The first inequality is straightforward to see since $x_1 > x_0$. The second inequality holds so long as

$$\left(\delta x_0^2 + (1-\delta)x_1^2\right)^{0.5} - (1-\delta)x_1 > 0. \tag{17.46}$$

Adding the second term to both sides and squaring, we get

$$\delta x_0^2 + (1-\delta)x_1^2 > (1-\delta)^2 x_1^2 \tag{17.47}$$

which can be re-written as

$$\delta x_0^2 > x_1^2 \left[(1-\delta)^2 - (1-\delta)\right] = -\delta(1-\delta)x_1^2. \tag{17.48}$$

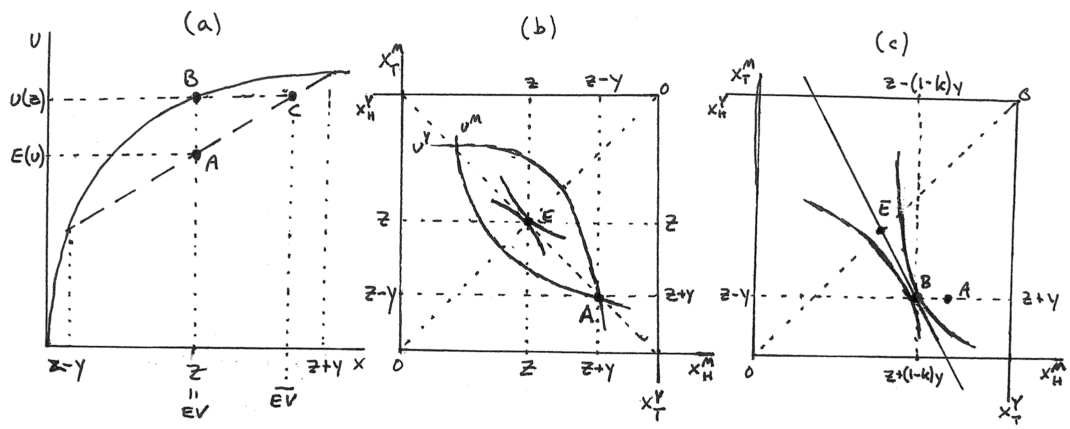
This always holds given that $\delta x_0^2 > 0$ and the right hand side is less than zero.

**17.6** Everyday Exercise: *Gambling with different Beliefs: Suppose you and I consider the following game: We both put \$y on the table, then flip a coin. If it comes up heads, I get everything on the table, and if it comes up tails, you get everything on the table.*

**A:** *Suppose we both have an amount $z > y$ available for consumption this week and both of us are risk averse.*

(a) *Draw my (weekly) consumption/utility relationship given that I am risk averse. On your graph, indicate the expected value of the gamble and the expected utility of the gamble assuming that we are playing with a fair coin — i.e. a coin that comes up heads half the time and tails the other half.*

Answer: This is done in panel (a) of Graph 17.6 where the concave shape arises from risk aversion. If the coin toss comes up heads, I win and will have $(z+y)$ in consumption available for the week, but if the coin toss comes up tails I will only have $(z-y)$. If I do not play the game, I simply have what I started with — which is $z$ — and this is also the expected value of the gamble $EV$. My expected utility $E(u)$ from the gamble is read off $A$ on the vertical axis.




Graph 17.6: Gambling with Fair and Unfair Coins

(b) *Will I agree to participate in the gamble if I think the coin is fair?*

Answer: No — the expected utility from the gamble is $E(u)$ while the utility from simply consuming $z$ is $u(z)$. Because I am risk averse, $E(u) < u(z)$ — which implies I will not gamble.

(c) *Now suppose that I exchanged the game coin for a weighted coin that comes up heads more often than tails. Illustrate in your graph how, if the coin is sufficiently unfair, I will now agree to participate in the gamble.*

Answer: The higher the probability of heads coming up, the more the point from which I read the expected utility from the gamble moves up the dashed line that contains $A$. If that probability is sufficiently high to cause the expected value of the gamble to rise to $\overline{EV}$, then we have a gamble whose expected utility is equal to $u(z)$ and I am not indifferent between playing and not playing. Any higher probability for heads will cause my expected utility from the game to be higher than the utility $u(z)$ of not playing.

(d) *Now consider both of us in the context of an Edgeworth Box, and suppose again that the coin is fair. Draw an Edgeworth box with consumption $x_H$ under "heads" on the horizontal and consumption $x_T$ under "tails" on the vertical axis. Illustrate our "endowment" bundle E before the gamble and the outcome bundle A if we do gamble.*

Answer: This is done in panel (b) of Graph 17.6 where $M$ superscripts consumption on axes that apply to me and $Y$ superscripts axes that apply to you. If we do not play the game, then we both have consumption level $z$ regardless of whether the coin comes up heads or tails (or regardless of whether the coin is tossed at all). Thus, the "endowment" $E$ lies in the center of the box that is a square with length $2z$. If we play and heads comes up, I get $(z+y)$ and you get $(z-y)$, and the reverse is true if tails comes up. This results in $A$ in the graph.

(e) *Illustrate the indifference curves through $E$ and $A$. Will we gamble? Is it efficient not to gamble?*

Answer: Since both of us are risk averse, we know that our indifference curves must be tangent to any actuarily fair budget along the 45 degree line. Given the equal probabilities of heads and tails occurring, such a budget would have to have slope of $-1$ — which implies our indifference curves both have slope of $-1$ on the 45 degree line. Since $E$ is on the 45-degree line, it must then be the case that our indifference curves are tangent to $-1$ and thus tangent to one another at $E$. Risk aversion implies our indifference curves have convex shapes — which then further implies that they must cross at $A$ as shown. It is therefore inefficient to engage in the gamble — because both of us would prefer outcome bundles inside the lens created by the indifference curves that pass through $A$. And it is efficient to not play the game — at $E$ we are at a point where there is no way to make one of us better off without making the other one worse off.

(f) *Suppose next that I have an unfair coin that is weighted toward coming up heads with probability $\delta > 0.5$. How do my indifference curves change as a result?*

Answer: If I know that heads comes up with probability $\delta$, then I know that the state "tails" comes up with probability $(1-\delta)$. If an insurance company were to give me \$1 if the state "heads" comes up — it would therefore give me $\delta$ in expectation. To make this actuarily fair, it should therefore demand that I pay $(1-\delta)y$ if the state "tails" comes up — where $(1-\delta)y=\delta$ or $y=\delta/(1-\delta)$. This is the opportunity cost of \$1 in $x_H$ consumption (in terms of $x_T$ consumption) if expected consumption remains unchanged by the terms of trade — i.e. if the terms are actuarily fair. And if I face actuarily fair trades, I will fully insure on the 45-degree line — implying that my $MRS$ on the 45 degree line is $-\delta/(1-\delta)$. Note that when $\delta = 0.5$, this implies an $MRS$ of $-0.5$ on the 45 degree line. But if $\delta > 0.5$, then $MRS$ along the 45 degree line is above 0.5 in absolute value — i.e. my indifference curves are becoming steeper. Intuitively, if I know that heads will come up more often than tails, then I would be willing to give up more consumption in the state "tails" to increase my consumption in the state "heads" than if I thought they happen with equal probability.

(g) *You do not know about the unfair coin, but you are delighted to hear that I have just sweetened the gamble for you: If the coin comes up heads, I agree to give you a fraction $k$ of my winnings. Draw a new Edgeworth box with the endowment bundle $E$ and the outcome bundle $B$ implied by the change I have made to the gamble.*

Answer: This is done in panel (c) of Graph 17.6 where the dimensions of the box stay the same (because there is still $2z$ available overall whether the coin comes up heads or tails.) The endowment point is also the same — if we do not play the gamble, we will both consume $z$ regardless of whether the coin comes up heads or tails (or whether it is flipped at all). But the outcome bundle that emerges from the gamble now changes from $A$ to $B$ — because, if heads comes up, I now get only $(1-k)y$ and you only lose $(1-k)y$ — whereas before I won $y$ and you lost $y$ if heads came up. Since nothing changes when tails comes up, $B$ is simply $A$ shifted to the left by a distance $ky$.

(h) *Can you illustrate how both of us engaging in the gamble might now be an efficient equilibrium?*

Answer: We can now see how $B$ might in fact be an efficient equilibrium. Your indifference curves are still the same (since you still believe the coin is a fair coin) — with slope $-0.5$ along the 45 degree line and steeper slope below the 45 degree line. My indifference curves have become steeper as argued in (f). Thus, if the coin is sufficiently unfair, I benefit sufficiently from playing the game to allow me to reduce your losses in the event of "heads" to induce you to play the game despite the fact that we are both risk averse.

(i) True or False*: If individuals have different beliefs about the underlying probabilities of different states occurring, then there may be gains from state-contingent consumption trades that would not arise if individuals agreed on the underlying probabilities.*

Answer: This is true, as we have just demonstrated.

**B:** *Suppose that the function $u(x) = \ln x$ allows us to represent both of our preferences over gambles using the expected utility function. Suppose further that $z$ and $y$ (as defined in part A) take on the values $z = 150$ and $y = 50$.*

(a) *Calculate the expected utility of entering this gamble (assuming a fair coin) and compare it to the utility of not entering. Will either us agree to play the game?*

Answer: The expected utility of entering the gamble is

$$E(u) = 0.5\ln(150-50) + 0.5\ln(150+50) \approx 4.9517. \tag{17.49}$$

The utility from not entering the game is $u(150) = \ln(150) \approx 5.0106$. Thus, neither one of us will play the game.

(b) *Suppose that I paid you a fraction $k$ of my winnings in the event that "heads" comes up. What is the minimum that $k$ has to be for you to agree to enter the game (assuming you think we are playing with a fair coin)?*

Answer: The least that $k$ has to be is an amount such that the utility for you of entering the gamble is equal to $u(150) = \ln(150)$ — the utility of not playing. Thus, we have to solve

$$0.5\ln(150+50) + 0.5\ln(150-(1-k)50) = \ln(150) \tag{17.50}$$

which can also be written as

$$0.5\ln(200) + 0.5\ln(100+50k) = \ln(150) \text{ or } \ln\left(200^{0.5}(100+50k)^{0.5}\right) = \ln(150). \tag{17.51}$$

This implies that

$$200^{0.5}(100+50k)^{0.5} = 150 \text{ or } 100+50k = \frac{150^2}{200} = \frac{225}{2} \tag{17.52}$$

which solves to give us $k = 0.25$. Thus, if I give 25% of my winnings to you in the event of heads, you will be indifferent between playing and not playing (as long as you think the coin is fair).

(c) *If I agreed to set $k$ to the minimum required to get you to enter the game, determine the lowest possible $\delta$ that an unbalanced coin must imply in order for* me *to want to enter the game.*

Answer: For me to be indifferent between entering and not entering the game given $k = 0.25$, it has to be the case that

$$\begin{aligned}&(1-\delta)\ln(150-50) + \delta\ln(150+(1-0.25)50) = \ln(150) \text{ or}\\&(1-\delta)\ln(100) + \delta\ln(187.5) = \ln(150).\end{aligned} \tag{17.53}$$

Evaluating the natural logs, this becomes approximately

$$4.6052(1-\delta) + 5.2338\delta = 5.0106 \tag{17.54}$$

which solves to $\delta \approx 0.645$.

(d) *Suppose my unbalanced coin comes up heads 75% of the time. Define the expected utility function for me and you as a function of $x_T$ and $x_H$ given that I know that the coin is unbalanced and you do not.*

Answer: For me, the expected utility function is then

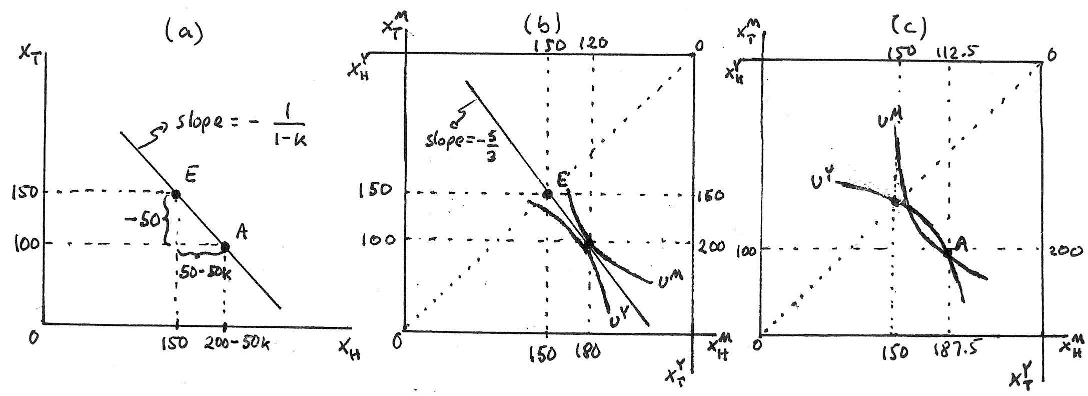
$$U^M(x_T, x_H) = 0.25\ln x_T + 0.75\ln x_H \tag{17.55}$$

and for you it is

$$U^Y(x_T, x_H) = 0.5\ln x_T + 0.5\ln x_H \tag{17.56}$$

(e) *Define p as the price for \$1 worth of $x_H$ consumption in terms of $x_T$ consumption. Suppose you wanted to construct a linear budget (with price p for $x_H$ and price of 1 for $x_T$) that contains our "endowment" bundle as well as the outcome bundle from the gamble (in which I return k of my winnings if the coin comes up heads). Derive p as a function of k.*

Answer: Panel (a) of Graph 17.7 illustrates the endowment bundle $E = (150, 150)$. My consumption will be $(150 - 50) = 100$ if I lose and the coin comes up tails; and it will be $150 + (1-k)50 = 200 - 50k$ if I win and the coin comes up heads. Thus, the outcome bundle from the gamble is $A = (200 - 50k, 100)$ for me. This implies that we go "down" by 50 and "over" by $(1-k)50$ when we move from $E$ to $A$ — giving us $p = 1/(1-k)$.




Graph 17.7: Gambling with Different Beliefs

(f) *Using our expected utility functions and the budget constraints (as a function of k), derive our demands for $x_H$ and $x_T$ as a function of k.*

Answer: Our budget constraints simply say that the value of our endowment $E = (150, 150)$ is equal to the value of our outcome bundle $(x_H, x_T)$ where the price of consumption under heads is $1/(1-k)$ and the price of our consumption under tails is 1; i.e.

$$\frac{150}{1-k} + 150 = \frac{x_H}{1-k} + x_T. \tag{17.57}$$

Rewriting the left-hand side by factoring out 150 and multiplying both sides by $(1-k)$, we can write this budget constraint as $150(2-k) = x_H + (1-k)x_T$. Using this, my optimization problem is then

$$\max_{x_H, x_T} 0.25 \ln x_T + 0.75 \ln x_H \text{ subject to } 150(2-k) = x_H + (1-k)x_T \tag{17.58}$$

and yours (given you have the same budget constraint) is

$$\max_{x_H, x_T} 0.5 \ln x_T + 0.5 \ln x_H \text{ subject to } 150(2-k) = x_H + (1-k)x_T. \tag{17.59}$$

Solving these in the usual way, we get

$$x_T^M = \frac{75(2-k)}{2(1-k)} \text{ and } x_H^M = \frac{225(2-k)}{2} \tag{17.60}$$

for me and

$$x_T^Y = \frac{75(2-k)}{(1-k)} \text{ and } x_H^Y = 75(2-k) \tag{17.61}$$

for you.

(g) *Determine the level of k that results in an equilibrium price and then verify that the resulting equilibrium output bundle is the one associated with the gamble we have been analyzing. Call this $k^*$ and illustrate what you have done in an Edgeworth Box.*

Answer: We can find the equilibrium by setting demand equal to supply in one of the two markets. For instance, we can use our demand equations for $x_H$ and know that, in equilibrium, they must sum to the total supply of consumption (which is 300); i.e. $x_H^M + x_H^Y = 300$ or, plugging in for our demands,

$$\frac{225(2-k)}{2} + 75(2-k) = 300. \tag{17.62}$$

Solving for $k$, we get $k^* = 2/5 = 0.4$ — with the associated price $p = 1/(1-0.4) = 5/3$. Plugging $k^*$ into our demand equations, we then get

$$x_T^M = \frac{75(2-0.4)}{2(1-0.4)} = 100 \text{ and } x_H^M = \frac{225(2-0.4)}{2} = 180 \text{ for me} \tag{17.63}$$

and

$$x_T^Y = \frac{75(2-0.4)}{(1-0.4)} = 200 \text{ and } x_H^Y = 75(2-0.4) = 120 \text{ for you.} \tag{17.64}$$

This is precisely what the gamble with $k = 0.4$ gives us: If the coin comes up heads, I win your 50 but give back 40% — or 20 — to wind up with $150+30 = 180$, leaving you with the remaining 120. When the coin comes up tails, on the other hand, I lose 50 and you win 50 — leaving you with 200 and me with 100. All this is illustrated in panel (b) of Graph 17.7.

(h) *Is the allocation chosen through the gamble efficient when $k = k^*$?*

Answer: The first welfare theorem tells us that any equilibrium allocation is efficient — and, since the gamble is an equilibrium, it is efficient. In the context of our Edgeworth box, this implies that the marginal rates of substitution for you and me must be equal at this allocation. We can check this by deriving the $MRS$ for me and you from our expected utility functions and plugging in the outcome pair from the gamble:

$$MRS^M = -\frac{\partial u^M(x_H^M, x_T^M)/\partial x_H}{\partial u^M(x_H^M, x_T^M)/\partial x_T} = -\frac{3x_T^M}{x_H^M} = -\frac{3(100)}{180} = -\frac{5}{3} \text{ and} \tag{17.65}$$

$$MRS^Y = -\frac{\partial u^Y(x_H^Y, x_T^Y)/\partial x_H}{\partial u^Y(x_H^Y, x_T^Y)/\partial x_T} = -\frac{x_T^Y}{x_H^Y} = -\frac{200}{120} = -\frac{5}{3}. \tag{17.66}$$

(i) *Suppose I had offered the lowest possible k that would induce you to enter the game instead — i.e. the one you derived in (b). Would the allocation chosen through the gamble be efficient in that case? Could it be supported as an equilibrium outcome with some equilibrium price?*

Answer: By offering $k = 0.25$, the gamble would have resulted in

$$x_H^M = 150 + (1-0.25)50 = 187.50 \text{ and } x_T^M = 150 - 50 = 100 \text{ for me} \tag{17.67}$$

and

$$x_H^Y = 150 - (1-0.25)50 - 112.50 \text{ and } x_T^Y = 150 + 50 = 200 \text{ for you.} \tag{17.68}$$

We can then check whether our marginal rates of substitution are equal at these allocations. They are

$$MRS^M = -\frac{3x_T^M}{x_H^M} = -\frac{3(100)}{187.50} = -1.6 \text{ and} \tag{17.69}$$

$$MRS^Y = -\frac{x_T^Y}{x_H^Y} = -\frac{200}{112.50} = -1.78. \tag{17.70}$$

Thus, $MRS^M \neq MRS^Y$ — which implies we are not at an efficient allocation. Since the first welfare theorem tells us that every equilibrium allocation is efficient, we then know immediately that we cannot support this outcome with an equilibrium price.

(j) *Illustrate what's different in an Edgeworth Box in part (i) than in part (g).*

Answer: This is done in panel (c) of Graph 17.7. Since we set $k$ at the minimum level to get you into the gamble, you are indifferent between the gamble and the endowment — which implies that your indifference curve through $E$ also passes through the gamble outcome pair $A$. We calculated above that your $MRS$ at $A$ is greater in absolute value than mine — implying that, at $A$, your indifference curve is steeper than mine. Finally, my indifference curve through $A$ lies above $E$ because I would prefer to gamble rather than not gamble given I have the unbalanced coin and I only offered you $k = 0.25$. The lens shape between our indifference curve illustrates that there are outcome pairs which both of us would prefer — implying $A$ is not efficient. You can also see immediately that there is no way to draw a budget line through $A$ and $E$ such that both of us will optimize at $A$.

**17.7** Everyday Application: *Venice and Regret*: *Suppose that you can choose to participate in one of two gambles: In Gamble 1 you have a 99% chance of winning a trip to Venice and a 1% chance of winning tickets to a movie about Venice; and in Gamble 2, you have a 99% of winning the same trip to Venice and a 1% chance of not winning anything.*

**A:** *Suppose you very much like Venice, and, were you to be asked to rank the three possible outcomes, you would rank the trip to Venice first, the tickets to the movie about Venice second, and having nothing third.*

(a) *Assume that you can create a consumption index such that getting nothing is denoted as 0 consumption, getting the tickets to the movie is $x_1 > 0$ and getting the trip is $x_2 > x_1$. Denote the expected value of Gamble 1 by $E(G_1)$ and the expected value of Gamble 2 by $E(G_2)$. Which is higher?*

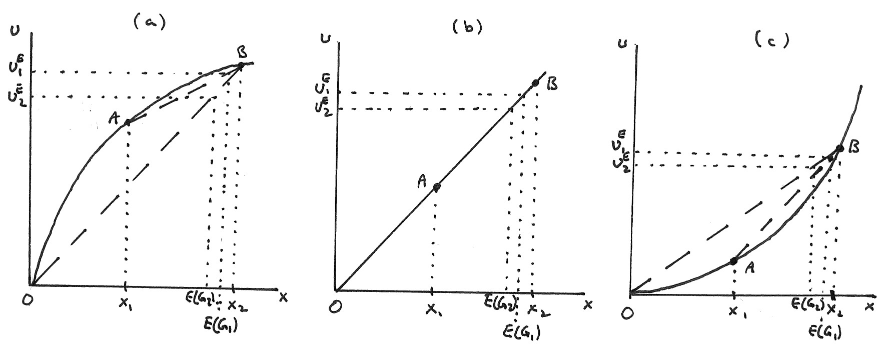
Answer: The expected value of Gamble 1, $E(G_1)$, is higher than the expected value of Gamble 2, $E(G_2)$ — because we take the same weighted average between $x_2$ and $x_1$ to get to $E(G_1)$ as we are between $x_2$ and 0 to get to $E(G_2)$. Thus

$$E(G_1) = 0.99x_2 + 0.01x_1 \text{ and } E(G_2) = 0.99x_2 + 0.01(0) = 0.99x_2. \tag{17.71}$$

Since $x_1 > 0$, it must therefore be the case that $E(G_1) > E(G_2)$.

(b) *On a graph with $x$ on the horizontal axis and utility on the vertical, illustrate a consumption/utility relationship that exhibits risk aversion.*

Answer: This is illustrated in panel (a) of Graph 17.8 where the relationship takes on the concave shape necessary for risk aversion.




Graph 17.8: Trips to and Movies about Venice

(c) *In your graph, illustrate the expected utility you receive from Gamble 1 and from Gamble 2. Which gamble will you choose to participate in?*

Answer: This is also illustrated in panel (a) of Graph 17.8. The expected utility of Gamble 1 is read off on the line connecting points $A$ and $B$, and the expected utility of Gamble 2 is read off the line connecting (the origin) 0 to $B$. Thus, $u_1^E$ is the expected utility of Gamble 1 and $u_2^E$ is the expected utility of Gamble 2. We can see immediately that $u_1^E > u_2^E$ — thus you would choose to participate in Gamble 1 over Gamble 2.

(d) *Next, suppose tastes are risk neutral instead. Re-draw your graph and illustrate again which gamble you would choose. (*Hint*: Be careful to accurately differentiate between the expected values of the two gambles.)*

Answer: This is illustrated in panel (b) of Graph 17.8 where the shape of the consumption/utility relationship is now linear (as is required for risk neutrality). The expected utility of the gambles is again read off the lines that connect $A$ and $B$ (for Gamble 1) and 0 and $B$ (for Gamble 2) — but these now lie on the consumption/utility relationship. Since $E(G_1) > E(G_2)$, we see that the expected utility of Gamble 1, $u_1^E$, is greater than the expected utility of Gamble 2, $u_2^E$. Again, you will choose Gamble 1 over Gamble 2.

(e) *It turns out (for reasons that become clearer in part B) that, risk aversion (or neutrality) is irrelevant for how individuals whose behavior is explained by expected utility theory will choose among these gambles. In a separate graph, illustrate the consumption/utility relationship again, but this time assume risk loving. Illustrate in the graph how your choice over the two gambles might still be the same as in parts (c) and (d). Can you think of why it in fact* has to be *the same?*

Answer: This is illustrated in panel (c) of Graph 17.8. Although the line connecting $A$ and $B$ now lies above the line connecting 0 and $B$, it is still the case that $E(G_1) > E(G_2)$. Thus, the graph can easily be drawn with the expected utility of Gamble 1 ($u_1^E$) greater than the expected utility of Gamble 2 ($u_2^E$). To see why this in fact *has to* be the case, denote the consumption/utility relationship $u(x)$. Thus, the utility of $x_2$ is given by $u(x_2)$, the utility of $x_1$ is given by $u(x_1)$ and the utility of 0 is simply $u(0) = 0$. The expected utility levels $u_1^E$ and $u_2^E$ (that lie on the lines connecting the outcomes) associated with Gambles 1 and 2 are then given by

$$u_1^E = 0.99u(x_2) + 0.01u(x_1) \text{ and } u_2^E = 0.99u(x_2) + 0.01u(0) = 0.99u(x_2). \tag{17.72}$$

Since $u(x_1) > 0$, it must then be that $u_1^E > u_2^E$, and it is irrelevant whether the $u$ function is concave or convex — so long as it slopes up and thus $u(x_1) > 0$. More intuitively, Gambles 1 and 2 place the same probability on winning the trip, but Gamble 1 places the remaining probability on winning a movie ticket while Gamble 2 does not. Thus, because Gamble 1 contains something "extra", it must be preferred to Gamble 2 under expected utility theory.

(f) *It turns out that many people, when faced with a choice of these two gambles, end up choosing Gamble 2. Assuming that such people would indeed rank the three outcomes the way we have, is there any way that such a choice can be explained using expected utility theory (taking as given that the choice implied by expected utility theory does not depend on risk aversion?)*

Answer: No, it cannot given the answer to (e). Put simply, the person gets the trip with probability 0.99 in both Gambles, but he gets something additional in Gamble 1 but not in Gamble 2. If that something additional — the movie ticket — is valuable, then Gamble 1 has to be better than Gamble 2 according to expected utility theory.

(g) *This example is known as* Machina's Paradox.[2] *One explanation for it (i.e. for the fact that many people choose Gamble 2 over Gamble 1) is that expected utility theory does not take into account* regret. *Can you think of how this might explain people's paradoxical choice of Gamble 2 over Gamble 1?*

Answer: Having had such a high chance of actually winning the trip, not getting it might cause regret — and then watching a movie about Venice might make it worse. Thus, it is not that the person does not, all else equal, prefer the movie ticket to nothing. But the movie ticket — after coming so close to being able to get to Venice in person — might actually be worse than nothing because of the fact that the person is reminded of what he has lost. None of this fits into expected utility theory.

**B:** *Assume again, as in part A, that individuals prefer a trip to Venice to the movie ticket, and they prefer the movie ticket to getting nothing. Furthermore, suppose there exists a function u that assigns $u_2$ as the utility of getting the trip, $u_1$ as the utility of getting the movie ticket and $u_0$ as the utility of getting nothing, and suppose that this function u allows us to represent tastes over risky pairs of outcomes using an expected utility function.*

(a) *What inequality defines the relationship between $u_1$ and $u_0$?*

Answer: It must be that $u_1 > u_0$.

(b) *Now multiply both sides of your inequality from (a) by 0.01, and then add* $0.99u_2$ *to both sides. What inequality do you now have?*

[2] The paradox is named after Mark Machina (1954-) who first identified it.

Answer: Multiplying the two inequalities as instructed, we get $0.01u_1 > 0.01u_0$, and adding $0.99u_2$ to both sides, we get

$$0.99u_2 + 0.01u_1 > 0.99u_2 + 0.01u_0. \tag{17.73}$$

(c) *Relate the inequality you derived in (b) to the expected utility of the two gambles in this example. What gamble does expected utility theory predict a person will choose (assuming the outcomes are ranked as we have ranked them)?*

Answer: The left hand side is the expected utility of Gamble 1 and the right hand side is the expected utility of Gamble 2. Since the left hand side is greater than the right hand side, expected utility theory implies that this person will choose Gamble 1 over Gamble 2.

(d) *When we typically think of a "gamble", we are thinking of different outcomes that will happen with different probabilities. But we can also think of "degenerate" gambles — i.e. gambles where one outcome happens with certainty. Define the following three such "gambles": Gamble A results in the trip to Venice with probability of 100%; Gamble B results in the movie ticket with probability of 100%; and Gamble C results in nothing with probability of 100%. How are these degenerate "gambles" ranked by someone who prefers the trip to the ticket to nothing?*

Answer: It must then be the case that $G_A \succ G_B \succ G_C$.

(e) *Using the notion of* mixed gambles *introduced in Appendix 1, define Gambles 1 and 2 as mixed gambles over the degenerate "gambles" we have just defined in (d). Explain how the* Independence Axiom *from Appendix 1 implies that Gamble 1 must be preferred to Gamble 2.*

Answer: Gamble 1 is simply Gamble $A$ (which is equivalent to getting the trip) and Gamble $B$ (which is equivalent to getting the movie ticket) mixed with weight 0.99 on $G_A$ and 0.01 weight on $G_B$. Similarly, Gamble 2 is equivalent to mixing Gamble $A$ with weight 0.99 and Gamble $C$ with weight 0.01. We can thus write that

$$G_1 = 0.99G_A + 0.01G_B \text{ and } G_2 = 0.99G_A + 0.01G_C. \tag{17.74}$$

(f) True or False*: When individuals who rank the outcomes the way we have assumed choose Gamble 2 over Gamble 1, expected utility theory fails because the independence axiom is violated.*

Answer: This is true. The independence axiom says that, if a Gamble $B$ is preferred to a Gamble $C$, then the mixture of Gamble $B$ with a third Gamble $A$ must be preferred to the mixture of Gamble $C$ with Gamble $A$ so long as they are mixed with equal weights; i.e.

$$G_B \succ G_C \text{ implies } \left(\delta G_B + (1-\delta)G_A\right) \succ \left(\delta G_C + (1-\delta)G_A\right) \text{ for all } 0 < \delta < 1. \tag{17.75}$$

When $\delta = 0.01$, the left hand side of this implication becomes $G_1$ and the right hand side becomes $G_2$. Thus, the independence axiom implies that, if the movie ticket is worth more than nothing to the individual, then $G_1 \succ G_2$. Expected utility theory cannot predict that someone like this will choose Gamble 2 over Gamble 1 because such a prediction would imply a violation of the independence axiom on which expected utility theory is built.

(g) *Would the paradox disappear if we assumed state-dependent tastes? (*Hint*: As with the Allais paradox in Appendix 2, the answer is no.)*

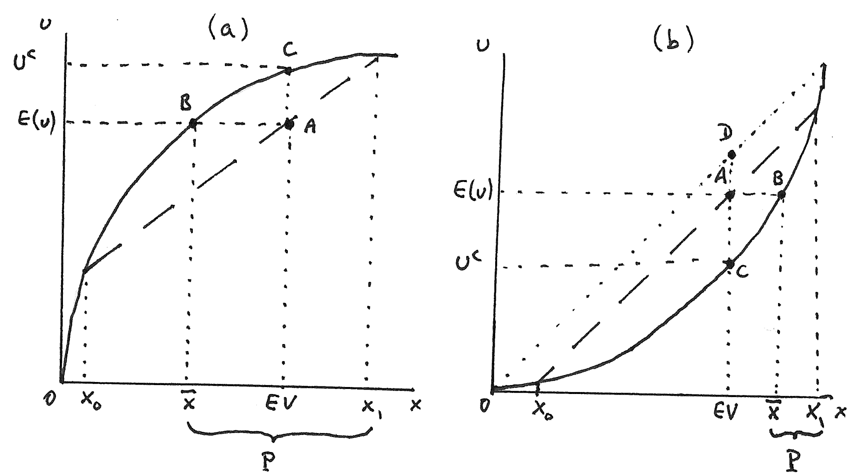
Answer: The reason that assuming state-dependent tastes does not resolve the Machina Paradox is because it does not matter whether we choose one function $u$ to assign utility to each outcome (as we would if tastes are state-independent) or we choose three separate functions $u_2, u_1$ and $u_0$ to assign utility to the outcomes. The question is whether we can assign utility values at all such that the expected utility of each gamble is a probability weighted average of the utilities associated with each outcome in the gamble. If we can find a way to assign such utility values, then expected utility theory can be applied, and it implies (as we have shown) that Gamble 1 is preferred to Gamble 2. Choosing Gamble 2 over Gamble 1 is then inconsistent with expected utility theory regardless of whether tastes are state dependent.

**17.8** Business Application: *Choosing a Mafia for Insurance: Consider Sunny who is committed to a life of crime. Sunny is risk averse, knows that he will enjoy consumption level $x_1$ if he does not get caught and consumption level of $x_0$ (very much below $x_1$) if he gets caught and goes to jail. He estimates the probability of getting caught as $\delta$.*

**A:** *Suppose there are various mafia organizations that have connections in the District Attorney's office and can affect the outcomes of court cases. Suppose initially that Sunny's tastes are state-independent.*

(a) *First, consider a really powerful mafia that can insure that any of its members who is caught is immediately released. Can you illustrate how much such a mafia would be able to charge Sunny if Sunny is risk averse? What about if Sunny is risk loving?*

Answer: The case where Sunny is risk averse is illustrated in panel (a) of Graph 17.9 and the risk loving case is illustrated in panel (b). In each panel, $EV$ represents the expected consumption value of a life of crime, and $E(u)$ (read off $A$) represents the expected utility of a life of crime. We can then identify the certainty equivalent $\overline{x}$ that results in the same utility as $E(u)$. If Sunny joins the mafia, he will always get $x_1$ minus the payment $P$ that he pays to get the mafia's protection — thus ending up with $(x_1 - P)$. When $P = (x_1 - \overline{x})$ (as illustrated in the two panels), then those in the mafia get $x_1 - P = x_1 - (x_1 - \overline{x}) = \overline{x}$ — and thus get the same utility as they would from a life of crime without insurance. The most Sunny is willing to pay is therefore the amount $P$ indicated in each panel, with $P$ being lower when he is risk averse.




Graph 17.9: Paying to Join Powerful Mafia

(b) *Next, suppose that the local mafia is not quite as powerful and can only get jail sentences reduced — thus in effect raising $x_0$. It approaches Sunny to offer him a deal: Pay us p when you don't get caught, and we'll raise your consumption level if you do get caught by b. If the local mafia insurance business is perfectly competitive (and faces no costs other than paying for increased consumption in jail), what is the relationship between b and p? (*Hint*: Note that this is different than the insurance example in the text where my wife had to pay p* regardless *of whether she was in the good or bad outcome.)*

Answer: If the mafia insurance market is perfectly competitive, it must be that the mafia is making zero profit (in expectation). For every member, it faces a probability $\delta$ of having to pay $b$ and a probability $(1-\delta)$ of receiving $p$. Expected zero profit therefore implies

$$\delta b = (1-\delta)p \quad \text{or, written differently,} \quad b = \left(\frac{(1-\delta)}{\delta}\right)p. \tag{17.76}$$

(c) *Suppose that Sunny can choose any combination of b and p that satisfies the relationship you derived in (b). What would he choose if he is risk averse? What if he is risk loving?*

Answer: This is actuarily fair insurance — which implies Sunny (whose tastes at this point are still assumed to be state-independent) will fully insure if risk averse and not insure at all (or, if possible, buy "negative insurance") if risk loving. You can see this in panels (a) and (b) of Graph 17.9. In (a), where Sunny is risk averse, any actuarily fair insurance (with $b = (1-\delta)p/\delta$) keeps $EV$ unchanged but reduces risk — with full insurance eliminating risk and moving us to point $C$ in the graph — giving utility $u^C$. In panel (b), where Sunny is risk loving, full insurance would also get us to $C$ — but now $u^C < E(u)$, implying that full insurance makes Sunny worse off. Instead, if given the option, he would prefer to increase risk — lowering $x_0$ and raising $x_1$ until $x_0 = 0$. If he can do this on actuarily fair terms, it will get him to $D$ in the graph — with the expected utility of this riskier life higher than the initial expected utility.

To be more precise, if he is risk averse, he will insure until his consumption is the same whether he gets caught or not; i.e. he will pick $(b, p)$ such that $(x_0 + b) = (x_1 - p)$ or, replacing $b$ with our result from part (a),

$$x_0 + \left(\frac{(1-\delta)}{\delta}\right)p = x_1 - p, \tag{17.77}$$

which solves to $p = \delta(x_1 - x_0)$. Plugging back into the relationship between $b$ and $p$ from part (a), we also get that $b = (1-\delta)(x_1 - x_0)$.

If Sunny is risk loving and negative insurance is possible, he would insure to the point where consumption in jail falls to zero; i.e. until $x_0 + b = 0$ which implies $b = -x_0$. Plugging this into the actuarily fair relationship between $p$ and $b$, we then also get $p = -\delta x_0/(1-\delta)$ — implying consumption of $((1-\delta)x_1 + \delta x_0)/(1-\delta)$ when he does not get caught. If negative insurance is not possible, he will simply choose $(b, p) = (0, 0)$.

(d) *Why does Sunny join the mafia in (a) but not in (c) if he is risk loving (and if "negative" insurance is not possible)?*

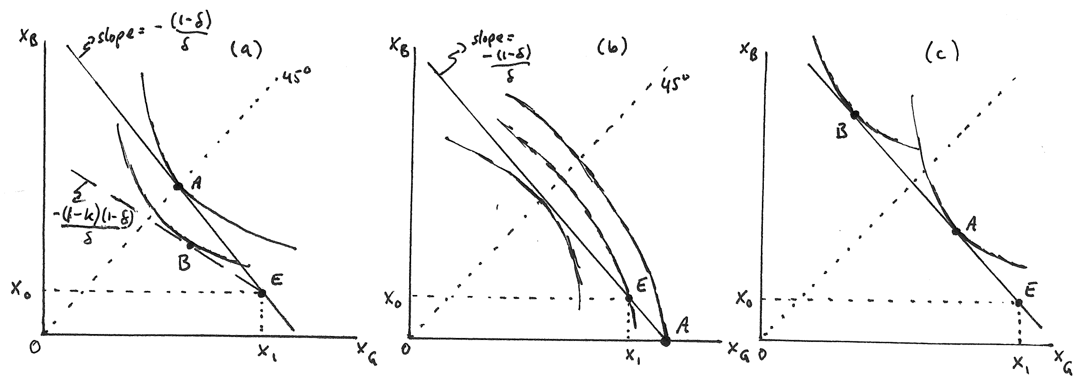
Answer: The difference between (a) and (c) is that in (a), the consumption level rises *above* the expected consumption level for an uninsured life of crime as risk is reduced — while in (c) the expected consumption value remains unchanged under insurance and only the risk falls. For someone who is risk loving, reducing risk without increasing expected consumption levels is not a good deal — but reducing risk while increasing expected consumption can be a good deal (and is in (a) if $P$ is not too high).)

(e) *How much consumer surplus does Sunny get for buying his preferred $(b, p)$ package when he is risk averse; i.e. how much more would Sunny be willing to pay to eliminate risk than he has to pay?*

Answer: Without insurance, he would have had utility of $E(u)$ in panel (a) of Graph 17.9. Eliminating risk without increasing utility would put Sunny at $B$ in the graph — with consumption $\overline{x}$ — but instead, Sunny gets to be at $C$ with utility $u^C$. He would therefore have been willing to pay $(EV - \overline{x})$ more — i.e. he would have been willing to pay more by an amount equal to what we have called the risk premium.

(f) *Construct a graph with $x_G$ — defined as consumption when not caught — on the horizontal and $x_B$ — defined as consumption when caught — on the vertical axis. Illustrate, in the form of a budget line, all the combinations of insurance contracts that Sunny is offered by the local mafia.*

Answer: This is illustrated in panel (a) of Graph 17.10 (on the next page) where $E = (x_1, x_0)$ is the outcome "endowment point" where Sunny starts in the absence of getting mafia-insured. Since we determined that $b = (1-\delta)p/\delta$, we know that for every dollar he gives up when he does not get caught he gets $(1-\delta)/\delta$ more when he is in jail — thus, the slope of the budget line that represents all the possible (actuarily fair) insurance contracts has slope $-(1-\delta)/\delta$.




Graph 17.10: Buying Mafia Protection (cont'd.)

(g) *Illustrate his optimal choice when he is risk averse and his tastes are still state-independent. How does this change if the corrupt jailer takes a fraction k of every dollar that the mafia makes available to Sunny in jail?*

Answer: When Sunny is risk averse and his tastes are state-independent, we have concluded that he will fully insure — which, in the picture we are graphing, implies his consumption in jail should be equal to his consumption outside after he insured. Thus, his optimal insurance contract gets him to $A$ in panel (a) of Graph 17.10. If, however, the jailer takes a fraction $k$ of every dollar that comes in, it means that, for every dollar Sunny gives up in the good state, he now gets

$$\frac{(1-k)(1-\delta)}{\delta} \tag{17.78}$$

when he is in jail. This implies that the slope of his "budget" becomes $-(1-k)(1-\delta)/\delta$ — shallower than the original slope. As a result, the optimal contract is found at $B$ — and at a shallower portion of the indifference map where Sunny no longer fully insures. This is because insurance is now no longer actuarily fair.

(h) *Can you show in this type of graph where Sunny would optimize if he is risk-loving?*

Answer: If Sunny is risk loving, his indifference map is no longer convex — i.e. indifference curves bow out rather than in. This is pictured in panel (b) of Graph 17.10 where the slope of the indifference curves is still the same on the 45 degree line as it was in panel (a), but the tangency formed along the budget line is no longer an optimum. This is because now the optimum is a corner solution at $A$ (or, depending on $\delta$, on the other end of the budget constraint). Sunny would therefore buy "negative insurance" if possible — i.e. insurance that increases the variance in his consumption across the good and bad states (while keeping the expected value constant). If "negative insurance" is not possible, his budget ends at $E$ — making $E$ (with no insurance) his optimum.

(i) *Finally, suppose Sunny's utility from consumption is different when he is forced to consume in jail than when he consumes on the outside. Can you tell an intuitive story for how this might cause Sunny to pick a $(b,p)$ combination that either over- or under-insures him?*

Answer: If consumption is now simply not as meaningful in jail as it is outside, then Sunny will not fully insure and instead make sure he consumes more outside jail than in jail. If, on the other hand, consumption is more meaningful in jail, he would want to make sure that he has more of it there than on the outside. (Perhaps in jail, having money buys respect that is necessary to enjoy life in jail.) The first case is illustrated with the indifference curve that is tangent at $A$ in panel (c) of Graph 17.10, and the second case is illustrated with the indifference curve that is tangent at $B$.

**B:** *Suppose we express consumption in thousands of dollars per year and that* $x_0 = 20$ *and* $x_1 = 80$. *Suppose further that* $\delta = 0.25$ *and that the function* $u(x) = x^\alpha$ *is the utility function over consumption that allows us to express tastes over gambles through an expected utility function.*

(a) *Consider first the powerful mafia (from part A(a)) that can eliminate any penalties from getting caught. How much would Sunny be willing to pay to join this mafia if* $\alpha = 0.5$*? What if* $\alpha = 2$*?*

Answer: As we showed in A(a), the most Sunny is willing to pay is the difference between $x_1$ and the certainty equivalent $\overline{x}$. To determine the certainty equivalent, we have to set $u(\overline{x}) = \overline{x}^\alpha$ equal to the expected utility $(0.25(20)^\alpha + 0.75(80)^\alpha)$. When $\alpha = 0.5$, this gives us $\overline{x} = 61.25$, implying that the most Sunny is willing to pay for the mafia eliminating the bad outcome is $(80 - 61.25) = 18.75$. When $\alpha = 2$, on the other hand, $\overline{x} = 70$ — implying that Sunny would only be willing to pay 70 to join the mafia.

(b) *One of these cases represents risk averse tastes, the other risk loving. In light of this, can you explain your answer intuitively?*

Answer: Note that the expected value of crime is $EV = 0.25(20) + 0.75(80) = 65$. The case of $\alpha = 0.5$ represents risk averse tastes while $\alpha = 2$ creates the convex function that implies risk loving. This is consistent with the certainty equivalent $\overline{x}$ being below the expected consumption value from crime (i.e. $\overline{x} = 61.25 < 65 = EV$) when $\alpha = 0.5$ and with the certainty equivalent being above the expected consumption value of crime (i.e. $\overline{x} = 70 > 65 = EV$) when $\alpha = 2$. Under risk aversion, Sunny is thus willing to pay an amount that reduces his overall consumption level below the expected value of crime because he is willing to pay to eliminate risk; under risk loving behavior, on the other hand, he is willing to pay to raise his consumption level above the expected value but not to reduce risk.

(c) *Next, consider the weaker mafia that can raise consumption in jail. Suppose this mafia asks Sunny to pay* $p$ *during times when he is not caught in exchange for getting an increase of* $b$ *in consumption when he finds himself in jail. If you have not already done so in part A of the question, derive the relationship between* $p$ *and* $b$ *if the mafia insurance market is perfectly competitive (and faces no costs other than paying* $b$ *to members who are in jail).*

Answer: Repeating from part A: If the mafia insurance market is perfectly competitive, it must be that the mafia is making zero profit (in expectation). For every member, it faces a probability $\delta$ of having to pay $b$ and a probability $(1-\delta)$ of receiving $p$. Expected zero profit therefore implies

$$\delta b = (1-\delta)p \quad \text{or, written differently,} \quad b = \left(\frac{(1-\delta)}{\delta}\right)p. \tag{17.79}$$

Given that $\delta = 0.25$, this implies $b = 3p$.

(d) *Using the function* $u(x) = x^\alpha$*, set up the optimization problem for Sunny who is considering which combination of* $b$ *and* $p$ *he should choose from all possible combinations that satisfy the relationship you derived in (c). Then derive his optimal insurance contract with the mafia.*

Answer: We would like to maximize the expected utility of consuming $(20 + b)$ when caught and $(80 - p)$ when not caught subject to the constraint that $b = 3p$ (as derived in the previous part); i.e.

$$\max_{b,p} \; 0.25(20+b)^\alpha + 0.75\left(80-p\right)^\alpha \quad \text{subject to} \quad b = 3p, \tag{17.80}$$

or, substituting the constraint into the objective,

$$\max_{p} \; 0.25\left(20+3p\right)^\alpha + 0.75\left(80-p\right)^\alpha. \tag{17.81}$$

Taking the first derivative and setting it to zero, we get

$$3\alpha(0.25)\left(20+3p\right)^{(\alpha-1)} - 0.75\alpha\left(80-p\right)^{(\alpha-1)} = 0 \tag{17.82}$$

which solves to give us $p = 15$. Substituting this back into our equation $b = 3p$, we also get $b = 45$. Thus, the optimal deal for Sunny to accept is $(b, p) = (45, 15)$ — which results in consumption of 65 regardless of whether Sunny is caught or not. (This is, however, the right solution only so long as $\alpha < 1$.)

(e) *If $\alpha = 0.5$, what is Sunny's consumer surplus from participating in the mafia.*

Answer: Had he not joined the mafia, Sunny's expected utility would have been

$$E(u) = 0.25(20)^{0.5} + 0.75(80)^{0.5} \approx 7.826 \tag{17.83}$$

which is the same utility he gets at the certainty equivalent $\overline{x} = 61.25$ (with $u(61.25) = (61.25)^{0.5} \approx 7.826$). By joining the mafia, Sunny gets 65 no matter what — giving him a higher level of utility. He would thus have been willing to pay $(65-61.25) = 3.75$ more to eliminate the risk, implying a consumer surplus of 3.75 for joining the mafia and fully insuring at the actuarily fair rate.

(f) *Why does your solution to (d) give the wrong answer when $\alpha > 1$? Explain using the example of $\alpha = 2$.*

Answer: When $\alpha = 2$, Sunny is risk loving — which implies he would prefer to end up at a corner solution with risk as large as possible (while keeping the expected value the same). The solution given by the calculus in (d) assumes an interior solution. The solution given by the calculus method we used becomes a (local) minimum rather than maximum when $\alpha > 1$. For instance, when $\alpha = 2$, the utility under full insurance is $u(65) = 65^2 = 4{,}225$ but the expected utility under no insurance is $E(u) = 0.25(20)^2 + 0.75(80)^2 = 4{,}900$. Full insurance that keeps the expected value unchanged therefore makes Sunny worse off when he is risk loving. In fact, Sunny could become even better off than he is without insurance if he can buy "negative" insurance at the actuarily fair rate $b = 3p$. Consider the package $(b, p) = (-20, -20/3)$ — which implies consumption of zero in jail and consumption of 86.67 outside jail — giving expected utility $(0.25(0)^2 + 0.75(86.67)^2 = 5{,}633.33$ — a further increase in utility from the no insurance case that gives utility of 4,900.

(g) *Suppose again that $\alpha = 0.5$. What changes when the jailer takes a fraction $k = 0.25$ of every dollar that is smuggled into the jail?*

Answer: We would now write the optimization problem as

$$\max_{b,p}\ 0.25\,(20+b)^{0.5} + 0.75\left(80-p\right)^{0.5} \ \text{ subject to }\ b = 3p(1-0.25) = 2.25p, \tag{17.84}$$

or, substituting the constraint into the objective,

$$\max_{p}\ 0.25\left(20+2.25p\right)^{0.5} + 0.75\left(80-p\right)^{0.5}. \tag{17.85}$$

Taking the first derivative and setting it to zero, we get

$$0.5(0.25)(2.25)(20+2.25p)^{-0.5} - 0.75(80-p)^{-0.5} = 0, \tag{17.86}$$

which solves to $p = 8.89$ and, given that $b = 2.25p$, $b = 20$. Thus, when the jailer takes a portion of what the mafia ships in, Sunny will insure less than he did under actuarily fair terms.

(h) *Finally, suppose that tastes are state dependent and that the functions $u_B(x) = 0.47x^{0.5}$ and $u_G(x) = 0.53x^{0.5}$ (where $u_B$ applies in jail and $u_G$ applies outside) allow us to represent Sunny's tastes over gambles using an expected utility function. Assuming that Sunny still chooses from the insurance contracts that satisfy the relationship between b and p you derived in (c), what contract will he pick? What if $u_B(x) = 0.53x^{0.5}$ and $u_G(x) = 0.47x^{0.5}$ instead? Can you make intuitive sense of your answers?*

Answer: Sunny would again maximize his expected utility subject to the constraint that $b = 3p$ — i.e. he would solve

$$\max_{b,p}\ 0.25(0.47)(20+b)^{0.5} + 0.75(0.53)(80-p)^{0.5} \ \text{ subject to }\ b = 3p \tag{17.87}$$

or, substituting the constraint into the objective,

$$\max_{p}\ 0.1175(20+3p)^{0.5}+0.3975(80-p)^{0.5}. \tag{17.88}$$

Solving this as before, we get $p = 11.33$ and $b = 34$ — implying consumption of 54 in jail and 68.67 outside. Thus, Sunny is underinsuring because he is placing relatively more weight on consumption outside than on consumption inside.

If $u_B(x) = 0.53x^{0.5}$ and $u_G(x) = 0.47x^{0.5}$ instead, then, using analogous steps, we get $p \approx 19.13$ and $b = 57.40$ — giving us consumption of 77.40 in jail and approximately 60.87 outside. Thus, because he is placing more weight on consumption in jail than outside, he is now over-insuring at the actuarily fair terms.

**17.9** Business Application: *Diversifying Risk along the Business Cycle: Suppose you own a business that does well during economic expansions but not so well during recessions which happen with probability $\delta$. Let $x_E$ denote your consumption level during expansions and let $x_R$ denote your consumption level during recessions. Unless you do something to diversify risk, these consumption levels are $E = (e_E, e_R)$ where $e_E$ is your income during expansions and $e_R$ your income during recessions (with $e_E > e_R$). Your tastes over consumption are the same during recessions as during expansions and you are risk averse. For any asset purchases described below, assume that you pay for these assets from whatever income you have depending on whether the economy is in recessions or expansion.*

**A:** *Suppose I own a financial firm that manages asset portfolios. All I care about as I manage my business is expected returns, and any asset I sell is characterized by $(p, b_R, b_E)$ where $p$ is how much I charge for 1 unit of the asset, $b_R$ is how much the asset will pay you (as, say, dividends) during recessions and $b_E$ is how much the asset will pay you during expansions.*

(a) *Is someone like me — who only cares about expected returns — risk averse, risk loving or risk neutral?*

Answer: Someone who only cares about expected returns (but not risk) is risk neutral.

(b) *Suppose that all the assets I offer have the feature that those who buy these assets experience no change in their expected consumption levels as a result of buying my assets. Derive an equation that expresses the price $p$ of my assets in terms of $\delta$, $b_R$ and $b_E$.*

Answer: In order for your expected consumption to remain unchanged, it must be that the expected change in consumption during recessions is exactly offset by the expected change in consumption during expansions — i.e.

$$\delta(-p + b_R) = -(1-\delta)(-p + b_E) \tag{17.89}$$

which solves to give us

$$p = \delta b_R + (1-\delta) b_E. \tag{17.90}$$

(c) *What happens to my expected returns when I sell more or fewer of such assets?*

Answer: Just as your expected consumption is unchanged when you buy these assets, my expected returns are unchanged.

(d) *Suppose you buy 1 asset $(p, b_R, b_E)$ that satisfies our equation from (b). How does your consumption during expansions and recessions change as a result?*

Answer: Your consumption during recessions will be

$$x_R = e_R - p + b_R = e_R - \left(\delta b_R + (1-\delta) b_E\right) + b_R = e_R + (1-\delta)(b_R - b_E), \tag{17.91}$$

and your consumption during expansion would be

$$x_E = e_E - p + b_E = e_E - \left(\delta b_R + (1-\delta) b_E\right) + b_E = e_E + \delta(b_E - b_R). \tag{17.92}$$

(e) *At what rate do assets of the kind I am offering allow you to transfer consumption opportunities from expansions to recessions? On a graph with $x_E$ on the horizontal and $x_R$ on the vertical axis, illustrate the "budget line" that the availability of such assets creates for you.*

Answer: In order to transfer consumption from expansions to recessions, you need to pick assets with $b_E < b_R$. Suppose you pick an asset with $b_R - b_E = 1$. Then, when you buy one unit of such an asset, your consumption in the two states becomes

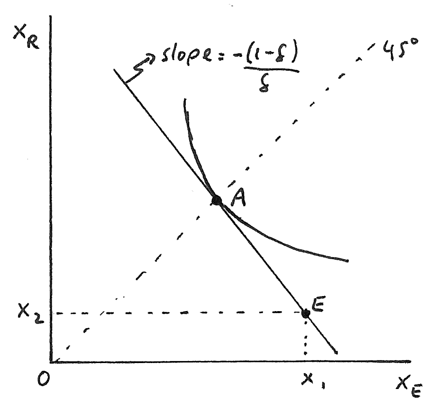
$$x_R = e_R + (1-\delta) \text{ and } x_E = e_E - \delta. \tag{17.93}$$

Thus, you are trading $\delta$ in consumption during expansions for $(1-\delta)$ in consumption during recessions — or, put differently, for every \$1 in consumption you give up during expansions, you get $\$(1-\delta)/\delta$ during recessions.

This is then illustrated in a graph in Graph 17.11 (on the next page).

(f) *Illustrate in your graph your optimal choice of assets.*

Answer: This is also illustrated in Graph 17.11 (on the next page) where risk aversion and state-independence of tastes implies that you will "fully insure" because the terms of trade are "actuarially fair" in the sense that your expected consumption level does not change. As a result, you optimize at point $A$ in the graph.




Graph 17.11: Reducing Risk across Business Cycles

(g) *Overall output during recessions is smaller than during expansions. Suppose everyone is risk averse. Is it possible for us to all end up doing what you concluded you would do in (f)? (We will explore this further in exercise 17.10.)*

Answer: No — if the economy shrinks, it is not possible for everyone to fully insure in the sense of maintaining the same level of consumption regardless of the state of the economy.

**B:** *Suppose that the function $u(x) = x^{\alpha}$ is such that we can express your tastes over gambles using expected utility functions.*

(a) *If you have not already done so in part A, derive the expression $p(\delta, b_R, b_E)$ that relates the price of an asset to the probability of a recession $\delta$, the dividend payment $b_R$ during recessions and the dividend payment $b_E$ during economic expansions assuming that purchase of such assets keeps expected consumption levels unchanged.*

Answer: Repeating our derivation from above: In order for your expected consumption to remain unchanged, it must be that the expected change in consumption during recessions is exactly offset by the expected change in consumption during expansions — i.e.

$$\delta(-p + b_R) = -(1-\delta)(-p + b_E) \tag{17.94}$$

which solves to give us

$$p = \delta b_R + (1-\delta) b_E. \tag{17.95}$$

(b) *Suppose you purchase k units of the same asset $(b_E, b_R)$ which is priced as you derived in part (a) and for which $(b_R - b_E) = y > 0$. Derive an expression for $x_R$ defined as your consumption level during recessions (given your recession income level of $e_R$) assuming you purchase these assets. Derive similarly an expression for your consumption level $x_E$ during economic expansions.*

Answer: Your consumption during recessions will be equal to your recession income plus the dividends from your assets minus the price of the assets:

$$\begin{aligned} x_R &= e_R + kb_R - kp = e_R + kb_R - k(\delta b_R + (1-\delta) b_E) = \\ &= e_R + (1-\delta)k(b_R - b_E) = e_R + (1-\delta)ky. \end{aligned} \tag{17.96}$$

Similarly,

$$x_E = e_E + kb_E - kp = e_E + kb_E - k(\delta b_R + (1-\delta)b_E) = \\ = e_E + \delta k(b_E - b_R) = e_E - \delta ky. \quad (17.97)$$

(c) *Set up an expected utility maximization problem where you choose $k$ — the number of such assets that you purchase. Then solve for $k$.*

Answer: We have already determined the consumption levels $x_R$ and $x_B$ conditional on how many assets you buy subject to the pricing constraints. Thus, all we have to solve is the unconstrained optimization problem

$$\max_k \delta u(x_R) + (1-\delta)u(x_E) = \delta x_R^\alpha + (1-\delta)x_E^\alpha = \\ = \delta[e_R + (1-\delta)ky]^\alpha + (1-\delta)[e_E - \delta ky]^\alpha. \quad (17.98)$$

Taking the first derivative of the right-hand side and setting it to zero, we get

$$\alpha\delta(1-\delta)y[e_R + (1-\delta)ky]^{(\alpha-1)} = \alpha(1-\delta)\delta y[e_E - \delta ky]^{(\alpha-1)} \quad (17.99)$$

which simplifies to

$$e_R + (1-\delta)ky = e_E - \delta ky \quad (17.100)$$

which we can solve for

$$k = \frac{(e_E - e_R)}{y} = \frac{(e_E - e_R)}{(b_R - b_E)}. \quad (17.101)$$

(For the last equality, we simply substituted back in for $y = (b_R - b_E)$.)

(d) *How much will you consume during recessions and expansions?*

Answer: Substituting (17.101) into (17.96) and (17.97), we get

$$x_R = e_R + (1-\delta)\left(\frac{(e_E - e_R)}{y}\right)y = \delta e_R + (1-\delta)e_E \quad (17.102)$$

$$x_E = e_E + \delta\left(\frac{(e_E - e_R)}{y}\right)y = \delta e_R + (1-\delta)e_E. \quad (17.103)$$

Thus, you will buy sufficient numbers of assets such that consumption in recessions and expansions is equalized.

(e) *For what values of $\alpha$ is your answer correct?*

Answer: The answer is correct for $\alpha < 1$ when tastes over gambles are risk averse. It is not correct for $\alpha > 1$ when tastes are risk-loving. The calculus still produces the same answer, but the indifference curves now bow out and, while they are tangent to the budget at the "full insurance" bundle, they are tangent from below and therefore a local minimum rather than a maximum. When tastes are risk loving, the true solution is a corner solution. And when $\alpha = 1$, all outcome bundles with the same expected consumption value are optimal — including the one derived.

(f) True or False*: So long as assets that pay more dividends during recessions than expansions are available at "actuarily fair" prices, you will be able to fully insure against consumption shocks from business cycles.*

Answer: This is true, as we have just shown. We assumed $(b_R > b_E)$ for the assets that we are buying — and equation (17.101) shows that the smaller the difference between the recession and expansion dividends, the more assets we will buy — always with the ultimate goal of equalizing consumption across the business cycle.

(g) *Could you accomplish the same outcome by instead creating and selling assets with* $(b_E > b_R)$*?*

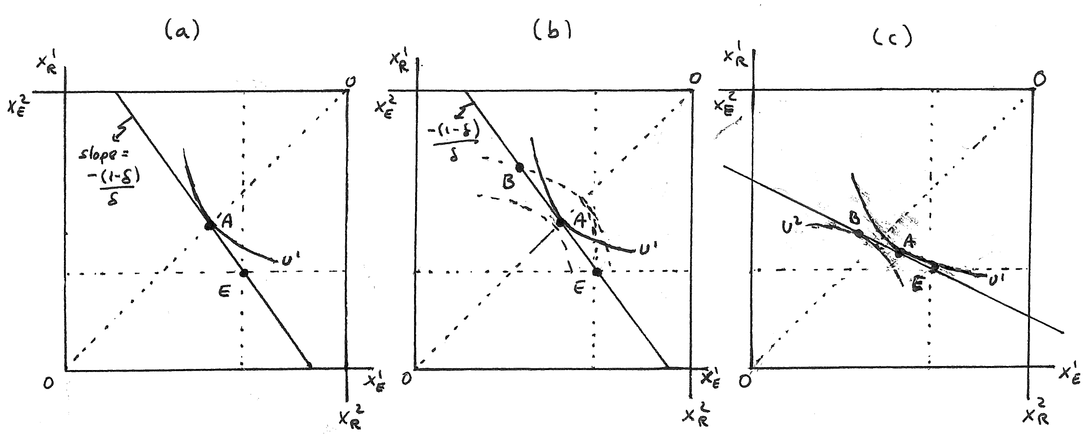
Answer: Yes — you can do exactly the same thing if you price such assets according to our pricing formula (equation (17.95)) that keeps expected consumption constant. In this case, you would be paying someone else $(p - b_E)$ during expansions, but you would receive $(p - b_R) > 0$ during recessions.

**17.10** Business Application: *Diversifying Risk along the Business Cycle in General Equilibrium*: *In exercise 17.9, we considered the case of me trading assets that allow you to transfer consumption from good times to bad times. Suppose again that your income during economic expansions is $e_E$ and your income during recessions is $e_R$, and that the probability of a recession is $\delta < 0.5$.*

**A:** *Also, suppose again that my tastes are risk neutral while yours are risk averse and that $e_E > e_R$. My consumption opportunity endowment, however, is the reverse of yours — with $e_R$ equal to my income during economic expansions and $e_E$ equal to my income during recessions.*

(a) *Draw an Edgeworth box representing the economy of you and me.*

Answer: This is done in panel (a) of Graph 17.12 where the Edgeworth Box is drawn as a square because the aggregate endowment of the economy is the same during both states (since our endowments are symmetric). Superscripts of 1 refer to you and superscripts of 2 refer to me.




Graph 17.12: Protecting against Risk in General Equilibrium

(b) *Illustrate the equilibrium in this economy. Will you do in equilibrium what we concluded you would do in exercise 17.9?*

Answer: Any equilibrium must be such that your indifference curve as well as mine is tangent to a line passing through our endowment point $E$. Since my tastes are risk neutral, the indifference curves are straight lines with slope $-(1-\delta)/\delta$. This is because I am indifferent between trades that leave my expected value of consumption unchanged — and a trade that takes $(1-\delta)/\delta$ during "recessions" that happen with probability $\delta$ and gives me \$1 during "expansions" that happen with probability $(1-\delta)$ changes my expected consumption by

$$-\delta\left(\frac{(1-\delta)}{\delta}\right)+(1-\delta)(1)=0. \tag{17.104}$$

Our equilibrium budgets will then be formed by a line with slope $-(1-\delta)/\delta$ passing through $E$ — and all points on it will be optimal for me since I have an indifference curve that lies on top of it. For you, however, only $A$ is optimal — because you are risk averse and, given that these are actuarily fair terms of trade, you will "fully insure". Thus, $A$ is the equilibrium allocation — with both of us equalizing consumption across the two states.

(c) *Next, suppose that there was a third person in our economy — your identical twin who shares your tastes and endowments. Suppose the terms of trade for transferring consumption in one state to the other remain unchanged, and suppose an equilibrium exists in which everyone ends up at an interior solution. Illustrate what this would look like — given that there are* *now 2 of you and only one of me. (*Hint*: It should no longer be the case that our indifference curves within the box are tangent to one another — because equilibrium now implies that two of your trades have to be exactly offset by one of mine.)*

Answer: This is illustrated in panel (b) of Graph 17.12. For the same reasons as in the previous part, if there is an equilibrium with an interior solution for everyone, then the slope of the budgets must be equal to the slope of my linear indifference curve. (Ignore the dashed indifference curves in panel (b) — which are not relevant until part (e).) Thus, one of my linear indifference curves (with slope $-(1-\delta)/\delta$) lies right on top of the budget line through our endowment $E$. As a result, any outcome bundle on the budget line is optimal for me. For you (and your twin brother), though, the optimum will be just as it was in panel (a) since neither your tastes nor the terms of trade have changed. The outcome bundle $A$ is therefore the optimum for you and your twin. This requires me to provide to you the difference between the outcome bundle at $E$ and the outcome bundle at $A$ — but now I have to provide it to your twin as well. Thus, I have to end up at a point on the budget that is twice as far from $E$ as $A$ is — which puts me at the outcome bundle $B$ (which, like all outcome bundles on the budget line, is (expected) utility maximizing for me). The equilibrium therefore has the same terms of trade as when there was only one of you, with the same outcome bundle for you (and your twin) but a different outcome bundle for me.

(d) *Is anyone fully insured against consumption swings in the business cylce? Is everyone?*

Answer: The terms of trade are still "actuarily fair" — which is why you and your twin are still fully insuring on the 45-degree line (where consumption in both the expansion and the recession end up being the same). I, on the other hand, can no longer fully insure in equilibrium (with $B$ not on the 45 degree line) — but since I am risk neutral, I am just as well off at $B$ as I would be at $A$ because the expected value of consumption is the same in both (and my indifference curve therefore lies right on top of the budget line.) Notice that I was fully insuring in panel (a) when it was just the two of us — but now am not because there is two of you that are trying to fully insure and I am the only one on the other side of the market.

(e) *Now continue with the example but suppose that my tastes, instead of being risk neutral, were also risk averse. Would the same terms of trade still produce an equilibrium?*

Answer: If I am also risk averse, then my indifference curves have the usual strictly convex shape with $MRS$ equal to $(1-\delta)/\delta)$ along the 45 degree line. Under the same terms of trade, I would therefore want to maximize my expected utility at $A$ as indicated by the dashed indifference curves in panel (b) of Graph 17.12. But this cannot be an equilibrium — because there are two of you and only one of me — and $A$ would only work out as an equilibrium if there was an equal number of us on both sides of the market.

(f) *How do the terms of trade now have to change to support an equilibrium when all of us are risk averse?*

Answer: Given that everyone maximizing at $A$ (in panel (b) of Graph 17.12) when the terms of trade are $(1-\delta)/\delta$ does not work when there are two of you and one of me, it must be that the budget line gets shallower so that it will be tangent closer to $E$ for you (and your twin) and farther from $E$ for me. This is illustrated in panel (c) of the graph where you (and your twin) optimize at $A$ and I optimize at $B$. The shallower slope means one has to give up more consumption during expansions to increase consumption during recessions or, alternatively, one has to give up less in recessions to increase consumption during expansions. These terms of trade are more favorable to me (who wants to trade from recessions to expansions) and less favorable to you (who wants to trade from expansions to recessions) — and they insure that demand is equal to supply (given that there are two of you and only one of me).

(g) *Will anyone be fully insured — i.e. will anyone enjoy the same level of consumption during recessions as during expansions?*

Answer: Neither $A$ nor $B$ fall on the 45-degree line in panel (c) of Graph 17.12 — which means no one chooses to equalize consumption across recessions and expansions under the equilibrium terms of trade. In fact all of us choose to consume less during recessions than during expansions.

(h) *Relate your conclusion to the existence of aggregate risk in economies that experience expansions and recessions. Who would you rather be — me or you?*

Answer: When there are two of you and only one of me, the economy faces aggregate risk (which it did not face when there was only one of each of us). This is because overall consumption in the economy now has to be less during recessions than during expansions — and it is for this reason that it is not possible for anyone to fully insure. At the terms of trade $(1-\delta)/\delta$ at which we would fully insure (given that we are risk averse and our tastes are not state dependent), there can be no equilibrium because demand is not equal to supply; and at the equilibrium prices no one wants to fully insure (because the terms of trade are not actuarly fair). Risk is reduced in the economy and expected utility increases for everyone in the equilibrium (relative to the endowments) — but the constraints of recessions and expansion imply that all of us consume less during recessions than during expansions even when our incomes are higher during recessions.

Whether or not you would rather be me depends on how frequently recessions happen. If they are relatively frequent, then having high income during recessions is better because one can command the more favorable terms of trade when one has the scarcer good to trade.

**B:** *Suppose that the function $u(x) = \ln x$ allows us to express your tastes over gambles as expected utilities. Also, suppose again that your income during expansions is $e_E$ and your income during recessions is $e_R$, with $e_E > e_R$.*

(a) *Let $p_R$ be defined as the price of \$1 of consumption in the event that a recession occurs and let $p_E$ be the price of \$1 of consumption in the event that an economic expansion occurs. Explain why we can simply normalize $p_R = 1$ and then denote the price of \$1 of consumption in the event of expansions as $p_E = p$.*

Answer: In general equilibrium models of this kind we can determine relative prices only. Thus, we can determine the price ration $p_E/p_R$, and any individual price combinations $(p_E, p_R)$ will support an equilibrium with identical allocations. (This is because only the slope of the budget line matters, with the relevant budget line passing through the endowment point of the economy.)

(b) *Using these normalized prices, write down your budget constraint and your expected utility optimization problem.*

Answer: Your budget constraint then simply says that the value of your endowment opportunities must be equal to the value of your consumption opportunity bundle; i.e.

$$pe_E + e_R = px_E + x_R. \tag{17.105}$$

Your expected utility maximization problem can then be written as

$$\max_{x_R, x_E} \delta \ln x_R + (1-\delta)\ln x_E \text{ subject to } pe_E + e_R = px_E + x_R. \tag{17.106}$$

(c) *Solve for your demand for $x_R$ and $x_E$.*

Answer: First, we set up the Lagrangian

$$\mathcal{L} = \delta \ln x_R + (1-\delta)\ln x_E + \lambda(pe_E + e_R - px_E - x_R) \tag{17.107}$$

that gives us first order conditions

$$\frac{\partial \mathcal{L}}{\partial x_R} = \frac{\delta}{x_R} - \lambda = 0 \text{ and } \frac{\partial \mathcal{L}}{\partial x_E} = \frac{(1-\delta)}{x_E} - \lambda p = 0. \tag{17.108}$$

These solve to give us $x_E = (1-\delta)x_R/(\delta p)$. Plugging this into the budget constraint, we can then solve for $x_R$ — and plugging that back into $x_E = (1-\delta)x_R/(\delta p)$, we can solve for $x_E$. Using these steps, we get

$$x_R = \delta(pe_E + e_R) \text{ and } x_E = \frac{(1-\delta)(pe_E + e_R)}{p}. \tag{17.109}$$

(d) *Repeat parts (b) and (c) for me — assuming I share your tastes but my income during* recessions *is $e_E$ and my income during* expansions *is $e_R$ — exactly the mirror image of your incomes over the business cycle.*

Answer: The only difference is that the role of $e_R$ and $e_E$ are reversed — giving us an optimization problem

$$\max_{x_R,x_E} \delta \ln x_R + (1-\delta)\ln x_E \text{ subject to } pe_R + e_E = px_E + x_R. \tag{17.110}$$

Solving this, we get my demands as

$$x_R = \delta(pe_R + e_E) \text{ and } x_E = \frac{(1-\delta)(pe_R + e_E)}{p}. \tag{17.111}$$

(e) *Assuming we are the only ones in this economy, derive the equilibrium price, or terms of trade across the two states.*

Answer: In equilibrium, the total demand for $x_R$ must be equal to the economy's supply of goods during recessions. The total demand is simply the sum of our two demands derived above; the total supply of goods during the recession is your income during recessions ($e_R$) plus my income during recessions ($e_E$). Thus, in equilibrium, the following must hold:

$$\delta(pe_E + e_R) + \delta(pe_R + e_E) = e_E + e_R. \tag{17.112}$$

Solving this for $p$, we get

$$p = \frac{(1-\delta)}{\delta}, \tag{17.113}$$

a familiar equation from our graphs where this is the slope of the budget constraint. (You could have also derived this by setting the demand and supply during economic expansions equal to another.)

(f) *How much do each of us consume during expansions and recessions at this equilibrium price?*

Answer: Plugging $p = (1-\delta)/\delta$ into your demand equations (17.109), we get

$$x_R = \delta\left(\frac{(1-\delta)e_E}{\delta} + e_R\right) = (1-\delta)e_E + \delta e_R \tag{17.114}$$

$$x_E = \frac{(1-\delta)\left(\frac{(1-\delta)e_E}{\delta} + e_R\right)}{\frac{(1-\delta)}{\delta}} = (1-\delta)e_E + \delta e_R. \tag{17.115}$$

Similarly, plugging $p = (1-\delta)/\delta$ into my demand equations (17.111), we get

$$x_R = \delta\left(\frac{(1-\delta)e_R}{\delta} + e_E\right) = (1-\delta)e_R + \delta e_E \tag{17.116}$$

$$x_E = \frac{(1-\delta)\left(\frac{(1-\delta)e_R}{\delta} + e_E\right)}{\frac{(1-\delta)}{\delta}} = (1-\delta)e_R + \delta e_E. \tag{17.117}$$

Thus, both of us end up equalizing consumption across the business cycle.

(g) *Now suppose that there are two of you and only one of me in this economy. What happens to the equilibrium price?*

Answer: We now have to multiply your demand for $x_R$ by two and add it to my demand for $x_R$ to get the total demand during recessions — and we have to sum twice your recession endowment $e_R$ with my recession endowment $e_E$. This gives us the equilibrium equation

$$2\delta(pe_E + e_R) + \delta(pe_R + e_E) = 2e_R + e_E. \tag{17.118}$$

Solving for $p$, we get the equilibrium price

$$p = \frac{(1-\delta)}{\delta}\left[\frac{2e_R + e_E}{2e_E + e_R}\right] < \frac{(1-\delta)}{\delta}. \tag{17.119}$$

The inequality emerges from the fact that, since $e_E > e_R$, the denominator of the bracketed term is larger than the numerator. This is what we also concluded in part A and panel (c) of Graph 17.12.

(h) *Do you now consume less during recessions than during expansions? Do I?*

Answer: To cut down on notation, define $c$ as

$$c = \left[\frac{2e_R + e_E}{2e_E + e_R}\right] \tag{17.120}$$

which allows us to write the equilibrium price from equation (17.119) as

$$p = \frac{c(1-\delta)}{\delta} \text{ where } c < 1. \tag{17.121}$$

Plugging this into your demand equations (17.109), we get your equilibrium consumption levels as

$$x_R = c(1-\delta)e_E + e_R \tag{17.122}$$

$$x_E = (1-\delta)e_E + \frac{\delta e_R}{c}. \tag{17.123}$$

Comparing these to what we derived for you in part (e) (where $x_R = x_E = (1-\delta)e_E + \delta e_R$), the fact that $c < 1$ implies immediately that $x_R$ has fallen and $x_E$ has increased — implying you now consume more during economic expansions than during recessions. Similarly, we can plug $p = c(1-\delta)/\delta$ into my demand equations from (17.111) to get my equilibrium consumption levels

$$x_R = c(1-\delta)e_R + e_E \tag{17.124}$$

$$x_E = (1-\delta)e_R + \frac{\delta e_E}{c}. \tag{17.125}$$

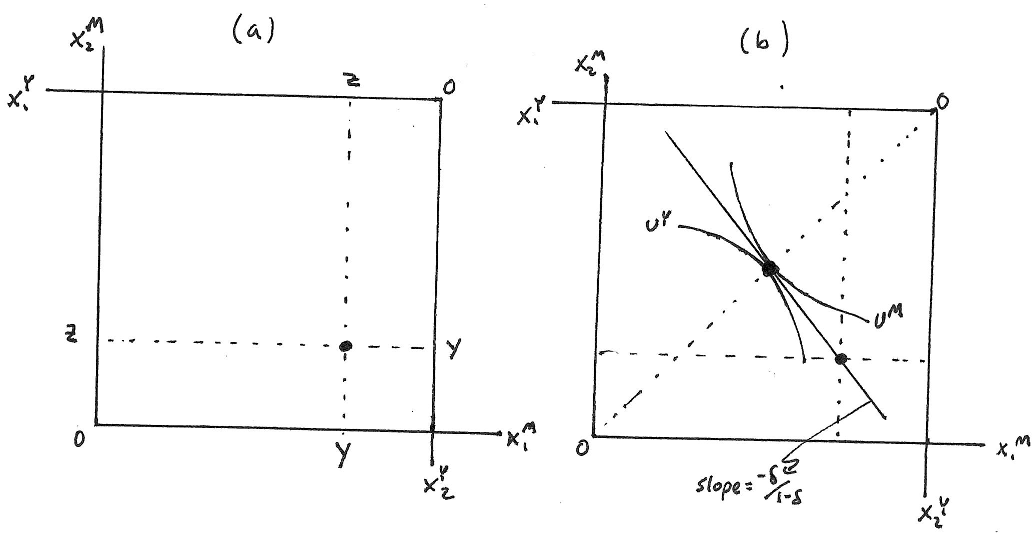
Again comparing this to what we derived for me earlier (where $x_R = x_E = (1-\delta)e_R + \delta e_E$), we again see that, since $c < 1$, $x_R$ falls and $x_E$ rises. Thus, I also now consume more during economic expansions than during recessions (even though my income is higher during recessions than expansions).

**17.11** Business Application: *Local versus National Insurance: Natural disasters are local phenomena — impacting a city or a part of a state but rarely impacting the whole country, at least if the country is geographically large. To simplify the analysis, suppose there are two distinct regions that might experience local disasters.*

**A:** *Define "state 1" as region 1 experiencing a natural disaster, and define "state 2" as region 2 having a natural disaster. I live in region 2 while you live in region 1. Both of us have the same risk averse and state-independent tastes, and our consumption level falls from $y$ to $z$ when a natural disaster strikes. The probability of state 1 is $\delta$ and the probability of state 2 is $(1-\delta)$.*

(a) *Putting consumption $x_1$ in state 1 on the horizontal axis and consumption $x_2$ in state 2 on the vertical, illustrate an Edgeworth box assuming you and I are the only ones living in our respective regions. Illustrate our "endowment" bundle in this box.*

Answer: This is illustrated in panel (a) where my consumption is superscripted by $M$ and yours by $Y$. Since I live in region 2, my consumption is high in state 1 when the disaster strikes in region 1; and since you live in region 1, your consumption is high in state 2 when the disaster strikes in region 2. The box is a square because, no matter where the disaster strikes, the overall level of consumption in the economy is $(y+z)$ — i.e. there is no aggregate risk.




Graph 17.13: Disaster Insurance

(b) *Suppose an insurance company wanted to insure us against the risks of natural disasters. Under actuarily fair insurance, what is the opportunity cost of state 2 consumption in terms of state 1 consumption? What is the opportunity cost of state 1 consumption in terms of state 2 consumption? Which of these is the slope of the acuarily fair budget in your Edgeworth Box?*

Answer: Actuarily fair insurance means that our expected consumption remains unchanged from being insured. Thus, if I want insurance that gives me \$1 in state 2 that happens with probability $(1-\delta)$, I am asking for an expected net benefit of $(1-\delta)$. This needs to be offset by an expected net payment of $\delta p$ that will be made to the insurance company in state 1 — and actuarial fairness implies $\delta p = (1-\delta)$ or $p = (1-\delta)/\delta$. Thus, the opportunity cost of \$1 in state 2 is $(1-\delta)/\delta$ in state 1. Alternatively, the opportunity cost of \$1 in state 1 is $\delta/(1-\delta)$ in state 2. The latter is the slope of the actuarily fair budget constraint when state 1 is on the horizontal axis.

(c) *Illustrate the budget line that arises from the set of all actuarily fair insurance contracts within the Edgeworth Box. Where would you and I choose to consume assuming we are risk averse?*

Answer: This is done in panel (b) of Graph 17.13. Under actuarily fair insurance terms, we would both choose to fully insure along the 45 degree line that connects the lower left to the upper right corners of the box.

(d) *How does this outcome compare to the equilibrium outcome if you and I were simply to trade state-contingent consumption across the two states?*

Answer: It is identical as can quickly be seen in panel (b) of Graph 17.13 where the budget line formed by the actuarily fair insurance terms causes us to optimize at the same point in the box.

(e) *Suppose there were two of me and two of you in this world. Would anything change?*

Answer: No, nothing would change as the same prices would still get us to optimize at the same point and, because there is no aggregate risk, there are enough resources in both states for the relevant trades to take place.

(f) *Now suppose that the two of me living in region 2 go to a local insurance company that operates only in region 2. Why might this company not offer us actuarily fair insurance policies?*

Answer: This insurance company may find it difficult to insure us because of the aggregate risk that the local economy faces. The insurance company needs to be able to write enough policies with risks that are offsetting so that it can in expectation meet the costs of all the benefits it has to pay with the premiums it is collecting in all those places where disaster does not strike. But if a local insurance company only sells local policies, it does not have people in other places where disaster won't strike to write offsetting policies.

(g) *Instead of insurance against the consequences of natural disasters, suppose we instead considered insurance against non-communicable illness. Would a local insurance company face the same kind of problem offering actuarily fair insurance in this case?*

Answer: No, the same problem would not arise for a local insurance company — because the "disasters" are not striking randomly without being clustered in geographic areas.

(h) *How is the case of local insurance companies insuring against local natural disasters similar to the case of national insurance companies insuring against business cycle impacts on consumption? How might international credit markets that allow insurance companies to borrow and lend help resolve this?*

Answer: In both cases, the problem is aggregate risk that does not make sufficient resources available in one state to make it possible to pay the necessary obligations. If insurance companies have access to full international credit markets, though, they can resolve this problem. They would do so by borrowing in such markets during times when bad times hit all at once (due to aggregate risk) and lend in good times.

**B:** *Suppose that, as in exercise 17.10 the function $u(x)=\ln x$ allows us to represent our tastes over gambles as expected utilities. Assume the same set-up as the one described in A.*

(a) *Let $p_1$ be defined as the price of \$1 of consumption if state 1 occurs and let $p_2$ be the price of \$1 of consumption in the event that state 2 occurs. Set $p_2=1$ and then denote the price of \$1 of consumption in the event of state 1 occurring as $p_1=p$ and write down your budget constraint.*

Answer. Your budget constraint is then $pz+y=px_1+x_2$, where the left hand side is the value of your endowment and the right hand side is the value of your consumption opportunity bundle to which you trade.

(b) *Solve the expected utility maximization problem given this budget constraint to get your demand $x_1$ for state 1 consumption as well as your demand $x_2$ for state 2 consumption.*

Answer: Your expected utility function is $U(x_1,x_2)=\delta\ln x_1+(1-\delta)\ln x_2$, and your expected utility maximization problem is

$$\max_{x_1,x_2} \delta \ln x_1 + (1-\delta)\ln x_2 \text{ subject to } pz + y = px_1 + x_2. \quad (17.126)$$

Solving this in the usual way, we get your demands

$$x_1 = \frac{\delta(pz+y)}{p} \text{ and } x_2 = (1-\delta)(pz+y). \quad (17.127)$$

(c) *Repeat (a) and (b) for me.*

Answer: For me, the budget constraint is $py + z = px_1 + x_2$ (because my endowment is the symmetric opposite of yours.) Otherwise everything is the same — giving us the following demands for me:

$$x_1 = \frac{\delta(py+z)}{p} \text{ and } x_2 = (1-\delta)(py+z). \quad (17.128)$$

(d) *Derive the equilibrium price. Is this acutarily fair?*

Answer: In equilibrium, the demand for $x_1$ must be equal to the economy's endowment $(y+z)$ (as must the demand for $x_2$ since the economy's endowment is the same in both states.) Equilibrium in state 1 therefore implies

$$x_1^M + x_1^Y = y + z, \quad (17.129)$$

where $M$ superscripts my demand and $Y$ superscripts yours. Plugging in the demands we calculated above, this can be written as

$$\frac{\delta(pz+y)}{p} + \frac{\delta(py+z)}{p} = y + z. \quad (17.130)$$

Solving this, we get

$$p = \frac{\delta}{(1-\delta)}. \quad (17.131)$$

(Note that this is the inverse of what we have often derived under similar conditions because the probability of the state 1 rather than the probability of state 2 is $\delta$ here.)

(e) *How much do we consume in each state?*

Answer: Plugging the equilibrium price back into our demands from above, we get

$$x_1^Y = \delta y + (1-\delta)z = x_2^Y \text{ and } x_1^M = \delta z + (1-\delta)y = x_2^M \quad (17.132)$$

where $Y$ again superscripts you and $M$ superscripts me. Thus, we both fully insure.

(f) *Does the equlibrium price change if there are 2 of you and 2 of me?*

Answer: We would still need that the demand is equal to the available endowment in each state. For state 1, this implies

$$2x_1^M + 2x_1^Y = 2(y+z), \quad (17.133)$$

which, once we cancel the 2's, is identical to the previous equilibrium equation (17.129). Thus, the equilibrium does not change as we increase the number of parties on each side of the market.

(g) *Finally, suppose that the two of me attempt to trade state-contingent consumption just between us. What will be the equilibrium price?*

Answer: The equilibrium in state 1 now requires twice my demand to sum to twice my endowment — i.e.

$$2\left(\frac{\delta(py+z)}{p}\right) = 2y \quad (17.134)$$

which we can solve for

$$p = \frac{\delta z}{(1-\delta)y}. \quad (17.135)$$

(h) *Will we manage to trade at all?*

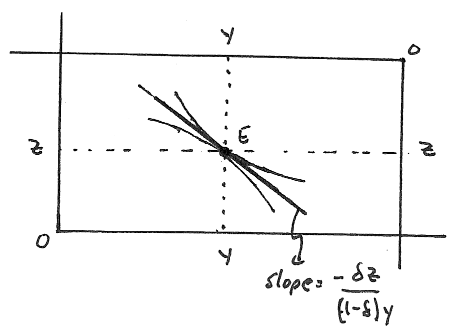
Answer: Plugging this back into my demand for $x_1$, we get

$$x_1 = \frac{\delta\left(\left(\frac{\delta z}{(1-\delta)y}y + z\right)\right)}{\frac{\delta z}{(1-\delta)y}} = y. \tag{17.136}$$

Thus, at the equilibrium price, each of us simply consumes our endowment and no trade occurs.

(i) *Can you illustrate this in an Edgeworth Box? Is the equilibrium efficient?*

Answer: This is illustrated in Graph 17.14 where the Edgeworth Box is no longer a square since the economy's endowment in state 1 is now $2y$ and the endowment in state 1 is $2z$ (where $y > z$). Our individual endowment bundle is now in the center of the box, and the equilibrium price keeps both of us optimizing at that bundle. Since our indifference curves are tangent to one another, the equilibrium is efficient even though we both continue to face risk. The risk cannot be reduced because of the presence of aggregate risk.



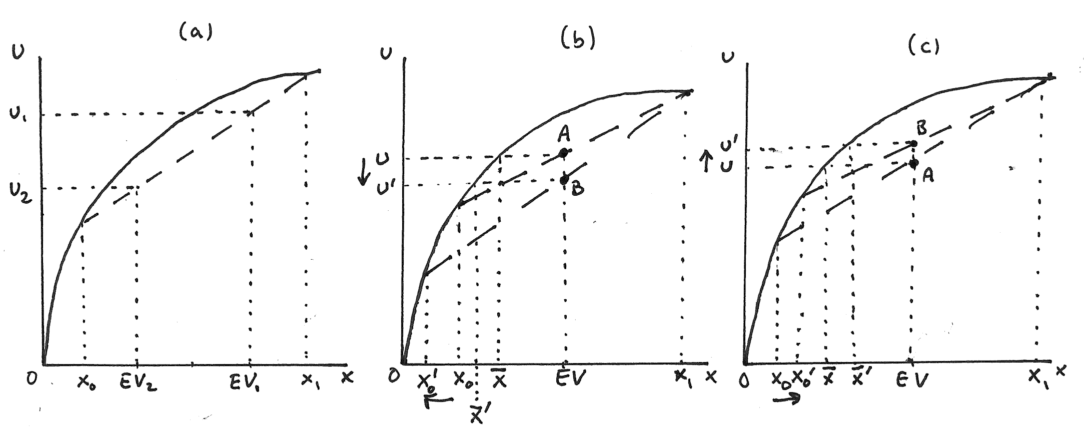

Graph 17.14: No Trade Equilibrium

**17.12** Policy Application: *More Police or More Jails? Enforcement versus Deterrence: Consider a person who is thinking about whether to engage in a life of crime. He knows that, if he gets caught, he will be in jail and will sustain a consumption level of $x_0$ but if he does not get caught, he will be able to consume $x_1$ considerably above $x_0$.*

**A:** *Suppose that this person cares only about his consumption level (i.e. he has state-independent tastes).*

(a) *On a graph with consumption x on the horizontal axis and utility on the vertical, illustrate this person's consumption/utility relationship assuming he is risk averse.*

Answer: This is illustrated in panel (a) of Graph 17.15— where the concave shape of the relationship is due to the assumption of risk aversion.




Graph 17.15: More Police or Jails?

(b) *Suppose the probability $\delta$ of getting caught is 0.25. Illustrate the expected utility of choosing a life of crime. What if $\delta = 0.75$?*

Answer: This is also shown in panel (a) of Graph 17.15 where $EV_1$ is the expected consumption value from a life of crime when $\delta = 0.25$ — i.e. when the probability of getting caught is low. This then results in an expected utility from a life of crime equal to $u_1$. When $\delta = 0.75$ and the probability of getting caught is therefore high, on the other hand, the expected consumption value from a life of crime falls to $EV_2$ with expected utility of $u_2$.

(c) *Redraw the consumption/utility graph and suppose $\delta = 0.5$. Let $\overline{x}$ indicate the income this person would need to be able to make honestly in order for him to be indifferent between an honest living and a life of crime.*

Answer: This is done in panel (b) of Graph 17.15. The expected value from a life of crime is now equal to $EV$ (halfway between $x_0$ and $x_1$ because $\delta = 0.5$). This gives an expected utility (read off point $A$) of $u$ — which is the same utility the person would get if he received $\overline{x}$ in the graph for sure. This consumption value $\overline{x}$ is in fact the certainty equivalent of the crime gamble. If the person can make more than $\overline{x}$ honestly, he will therefore not engage in a life of crime, and if he can only make less than $\overline{x}$ honestly, he will engage in a life of crime.

(d) *Senator C believes the criminal justice system spends too much effort on identifying criminals but not enough effort on punishing them harshly. He proposes an* increased deterrence policy *under which penalties for committing crimes are raised while less is spent on law enforcement. This implies a drop in both $x_0$ as well as $\delta$. Suppose the expected consumption level for a person engaged in a life of crime remains unchanged under this policy. Will the* *person who was previously indifferent between an honest living and a life of crime still be indifferent?*

Answer: In panel (b) of Graph 17.15, we illustrate this policy by reducing $x_0$ to $x_0'$. This gives us the steeper dashed line connecting the utility of the good and bad outcomes — and if the expected consumption level from a life of crime remains unchanged, we now read the expected utility from such a life off point $B$ — giving us $u'$ that must lie below $u$. Thus, by making the consequence of getting caught harsher but enforcing less (and thus keeping the expected value from crime unchanged), fewer people will commit crimes. (You can also see in the graph that the certainty equivalent has dropped to $\overline{x}'$ — which means people with honest incomes between $\overline{x}'$ and $\overline{x}$ will switch from crime to honest living as a result of this policy.)

(e) *Senator L believes we are treating criminals too harshly. He proposes an* increased enforcement policy *that devotes more resources toward catching criminals but then lowers the penalties that criminals face if caught. The policy thus increases $x_0$ as well as $\delta$. Suppose that the expected consumption level of a person engaged in a life of crime is again unchanged under this policy. Will the person who was previously indifferent between an honest living and a life of crime still be indifferent?*

Answer: This is illustrated in panel (c) of Graph 17.15 where we again begin with expected utility $u$. By increasing the consumption level when getting caught from $x_0$ to $x_0'$ and keeping the expected consumption value of a life of crime unchanged, we again read the new expected utility off from point $B$ — but now point $B$ lies above point $A$ giving us an expected utility $u'$ above the original expected utility $u$. Thus, more people will commit crimes when we lessen the penalty and increase enforcement to keep the expected consumption value of crime constant. (You can also again see that the certainty equivalent now goes up from $\overline{x}$ to $\overline{x}'$ — implying that people who made honest livings between those two values before will now switch to lives of crime under the new policy.)

(f) True or False*: If criminals are risk averse, the increased deterrence policy is more effective at reducing crime than the increased enforcement policy.*

Answer: This is true as illustrated above. In fact, the increased enforcement policy (with more lenient sentences to keep the expected value of a life of crime unchanged) will increase crime.

(g) *How would your answers change if criminals were risk loving?*

Answer: The results are now the reverse of what we concluded before — with crime increasing under the deterrence policy and decreasing under the increased enforcement policy. This is illustrated in panels (a) and (b) of Graph 17.16 (on the next page). In panel (a), the increase deterrence policy is modeled precisely as before, with $x_0$ falling to $x_0'$. As a result, expected utility now increases from $u$ to $u'$ (and the consumption level from an honest living required to prevent someone from committing crimes rising from $\overline{x}$ to $\overline{x}'$. The reverse happens in panel (b) where we illustrate the increased enforcement policy with more lenient penalties.

**B:** *Suppose that $x_0 = 20$ and $x_1 = 80$ (where we can think of these values as being expressed in terms of thousands of dollars), and suppose the probability of getting caught is $\delta = 0.5$.*

(a) *What is the expected consumption level if the life of crime is chosen?*

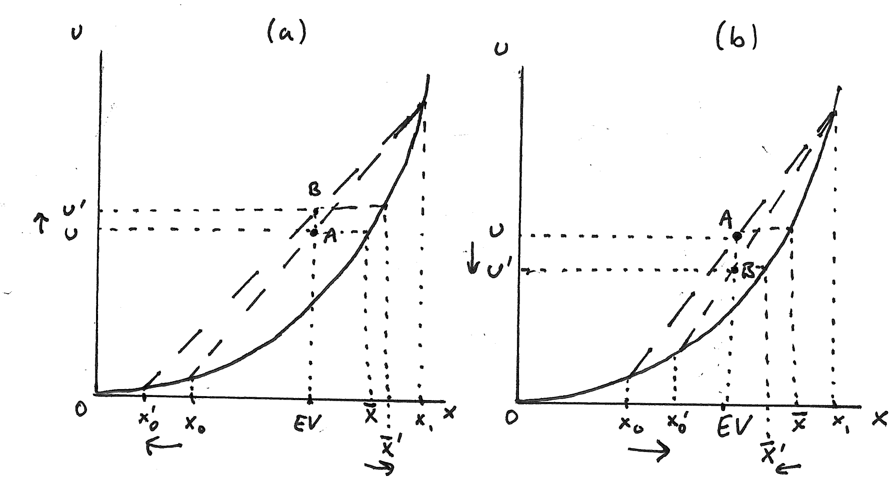
Answer: The expected consumption level is

$$E(x) = 0.5(20) + 0.5(80) = 50. \tag{17.137}$$

(b) *Suppose the potential criminal's tastes over gambles can be expressed using an expected utility function that evaluates the utility of consumption as $u(x) = \ln(x)$. What is the person's expected utility from a life of crime?*

Answer: The expected utility is then

$$U(20,80) = 0.5\ln(20) + 0.5\ln(80) \approx 3.689. \tag{17.138}$$



Graph 17.16: More Police or Jails? The case of Risk-Loving Criminals

(c) *How does the expected utility compare to the utility of the expected value of consumption? Can you tell from this whether the criminal is risk averse?*

Answer: The utility of the expected value of crime is $u(50) = \ln(50) \approx 3.912$. Since this is greater than the expected utility calculated in (b), the individual is risk averse. (This is of course due to the fact that the $u(x)$ function we have chosen is concave.)

(d) *Consider the level of consumption this person could attain by not engaging in a life of crime. What level of consumption from an honest living would make the person be indifferent between a life of crime and an honest living? Denote this consumption level $\overline{x}$.*

Answer: This would be the level of consumption that, if available with certainty, will result in utility equal to the expected utility of the life of crime — i.e. equal to 3.689 we calculated in (b). We therefore need to solve the equation

$$u(\overline{x}) = \ln \overline{x} = 3.689 \quad (17.139)$$

which gives us $\overline{x} = e^{3.689} = 40$. Thus individuals who can make more than 40 honestly will not engage in crime while individuals who can only make less than 40 honest will commit crimes.

(e) *Now consider the* increased deterrence policy *described in A(d). In particular, suppose that the policy increases penalties to the point where $x_0$ falls to 5. How much can $\delta$ drop if the expected consumption level in a life of crime is to remain unchanged?*

Answer: We need to solve the equation

$$\delta(5) + (1-\delta)(80) = 50 \quad (17.140)$$

which gives $\delta = 0.4$. Thus, by lowering the consumption level if caught to 5, we can also lower the probability of getting caught by 0.1 and keep the expected consumption level of a life of crime unchanged.

(f) *What happens to the $\overline{x}$ as a result of this increased deterrence policy?*

Answer: First, we need to calculate the expected utility of a life of crime under the new policy. This would be

$$0.4\ln(5) + 0.6\ln(80) \approx 3.273. \quad (17.141)$$

The certainty equivalent is then determined by solving $\ln(\overline{x}') = 3.273$ — which solves to $\overline{x}' \approx 26.39$; i.e. the minimum honest income level required to keep someone from entering a life of crime falls from the initial 40 to 26.39. Thus, anyone whose honest living falls into that range will switch from a life of crime to an honest living as a result of this policy change.

(g) *Now consider the* increased enforcement policy *described in A(e). In particular, suppose that $\delta$ is increased to 0.6. How much can $x_0$ increase in order for the expected consumption in a life of crime to remain unchanged?*

Answer: In order for the expected consumption value of crime to remain unchanged, it must be that

$$0.6x_0 + 0.4(80) = 50. \quad (17.142)$$

Solving this, we get $x_0 = 30$; i.e. if we increase the rate at which we catch people to 0.6, we can increase their consumption level if caught from 20 to 30 and keep the expected consumption value of crime unchanged.

(h) *What happens to $\overline{x}$ as a result of this increased enforcement policy?*

Answer: The expected utility of crime now increases from the initial 3.689 to

$$0.6\ln(30) + 0.4\ln(80) \approx 3.794. \quad (17.143)$$

The new certainty equivalent $\overline{x}'$ then needs to satisfy $\ln(\overline{x}') = 3.794$ which solves to $\overline{x}' \approx 44.41$. Thus, the minimum (honest) income level required for someone not to enter a life of crime rises from the initial 40 to 44.41, and anyone whose honest income falls between these two levels will switch from an honest living to a life of crime under this policy.

(i) *Which policy is more effective at reducing crime assuming potential criminals are risk averse?*

Answer: The deterrence policy is more effective — and in fact the enforcement policy (that is more lenient on those who are caught) increases crime.

(j) *Suppose that the function $u(x)$ that allows us to represent this individual's tastes over gambles with an expected utility function is $u(x) = x^2$. How do your answers change?*

Answer: The initial expected utility is

$$0.5(20)^2 + 0.5(80)^2 = 3,400 \quad (17.144)$$

compared to the utility of the expected consumption value of crime which is $u(50) = 50^2 = 2,500$. People are therefore now risk loving (as we could have guessed from the fact that $u(x) = x^2$ is a convex function.) The certainty equivalent $\overline{x}$ solves the equation

$$\overline{x}^2 = 3400 \quad (17.145)$$

which gives us $\overline{x} \approx 58.31$. Thus, anyone whose honest work yields less than 58.31 engages in a life of crime.

The deterrence policy lowers $x_0$ from 20 to 5 while simultaneously lowering $\delta$ from 0.5 to 0.4 to keep the expected consumption value of crime unchanged. This gives expected utility of

$$0.4(5)^2 + 0.6(80)^2 = 3,850; \quad (17.146)$$

i.e. the policy has increased the expected utility from crime. The certainty equivalent similarly increases — from the initial 58.31 to 62.05 — implying more people will now engage in crime. This is the opposite of what we found in the risk averse case.

The enforcement policy, on the other hand, results in $x_0 = 30$ and $\delta = 0.6$. This gives us expected utility

$$0.6(30)^2 + 0.4(80)^2 = 3,100; \quad (17.147)$$

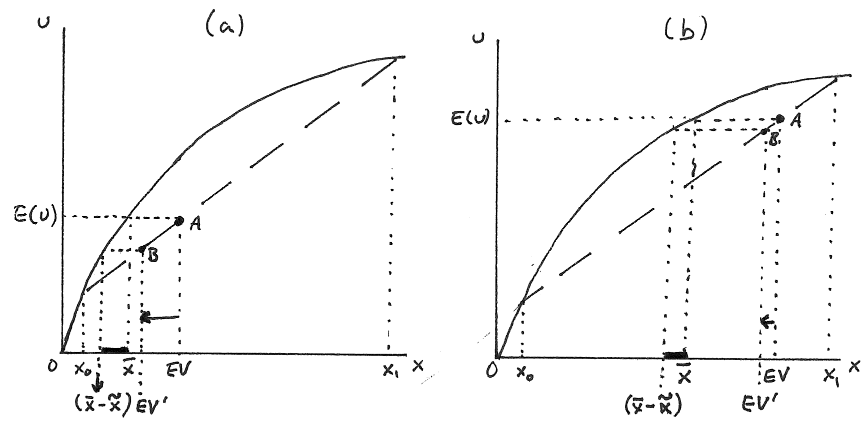
i.e. the expected utility from crime falls as a result of this policy, as does the certainty equivalent which drops to 55.68. Crime therefore falls under the increased enforcement policy — again opposite from what we derived in the risk averse case.

**17.13** Policy Application: *More Police or More Teachers? Enforcement versus Education: Suppose again (as in exercise 17.12) that the payoff from engaging in a life of crime is $x_1$ if you don't get caught and $x_0$ (significantly below $x_1$) if you end up in jail, with $\delta$ representing the probability of getting caught. Suppose everyone has identical tastes but we differ in terms of the amount of income we can earn in the (legal) labor market — with (legal) incomes distributed uniformly (i.e. evenly) between $x_0$ and $x_1$.*

**A:** *Suppose there are two ways to lower crime rates: Spend more money on police officers so that we can make it more likely that those who commit crimes get caught, or spend more money on teachers so that we increase the honest income that potential criminals could make. The first policy raises $\delta$; the second raises individual incomes through better education.*

(a) *Begin by drawing a risk averse individual's consumption/utility relationship and assume a high $\delta$. Indicate the corresponding $\overline{x}$ that represents the (honest) income level at which a person is indifferent between an honest life and a life of crime.*

Answer: This is done in panel (a) of Graph 17.17 where the consumption/utility relationship is concave due to the assumption of risk aversion. If $\delta$ is high, it means the probability of getting caught is high — which means the expected consumption value $EV$ of a life of crime is relatively low. Thus, $EV$ is drawn at a relatively low level — with the expected utility of a life of crime read off point $A$. The certainty equivalent is the level of consumption $\overline{x}$ that yields the same level of utility as the expected utility of the life of crime — and it is equal to the honest income level at which a person is indifferent between an honest life and a life of crime.




Graph 17.17: More Police or More Teachers?

(b) *Consider a policy that invests in education and results in a uniform increase in all incomes by an amount $\tilde{x}$. On the horizontal axis of your graph, indicate which types of individuals (identified by their pre-policy income levels) will now switch from a life of crime to an honest life.*

Answer: An individual who previously could make an honest living of $(\overline{x}-\tilde{x})$ will now end up earning $\overline{x}$ in the (legal) labor market. Thus, individuals whose pre-policy (honest) income falls in the darkened interval from $(\overline{x}-\tilde{x})$ to $\overline{x}$ will switch from a life of crime to an honest life as a result of the investments in education.

(c) *Next, consider the alternative policy of investing in more enforcement — thus increasing the probability of getting caught $\delta$. Indicate in your graph how much the expected consumption* *level of a life of crime must be shifted in order for the policy to achieve the same reduction in crime as the policy in part (b).*

Answer: This is also illustrated in panel (a) of Graph 17.17. In order for a policy focused on raising $\delta$ to reduce crime by the same amount (without legal incomes rising), it must be that the expected consumption value of a life of crime falls sufficiently to make the expected *utility* of crime equal to the utility of an individual with (legal) income $(\overline{x}-\tilde{x})$. This would imply an increase in $\delta$ sufficiently high to move us to $B$ — which requires a shift of the expected consumption value of crime from $EV$ to $EV'$. Because the consumption/utility relationship is steep, $(EV-EV')$ is greater than $\overline{x}$.

(d) *If it costs the same to achieve a \$1 increase in everyone's income through education investments as it costs to achieve a \$1 reduction in the expected consumption level of a life of crime, which policy is more cost effective at reducing crime given we started with an already high $\delta$.*

Answer: Since $(EV-EV')$ is greater than $\overline{x}$, it is more cost effective to reduce crime through investments in education in this case.

(e) *How does your answer change if $\delta$ is very low to begin with?*

Answer: This is pictured in panel (b) of Graph 17.17. Following the same steps as before, we now find that $(EV-EV')$ is less than $\overline{x}$ — implying it is more cost effective to reduce crime through increases in policing rather than investments in education.

(f) True or False*: Assuming people are risk averse, the following is an accurate policy conclusion from our model of expected utility: The higher current levels of law enforcement, the more likely it is that investments in education will cause greater reductions in crime than equivalent investments in additional law enforcement.*

Answer: This is true based on our analysis above. When $\delta$ is high, law enforcement levels are already high — in which case we found it is more likely to be cost effective to invest in education rather than additional law enforcement than when $\delta$ is low.

**B:** *Now suppose that, as in exercise 17.12,* $x_0=20$ *and* $x_1=80$ *(where we can think of these values as being expressed in terms of thousands of dollars).*

(a) *Suppose, again as in exercise 17.12, that expressing utility over consumption by* $u(x)=\ln x$ *allows us to express tastes over gambles using the expected utility function. If* $\delta=0.75$*, what is the income level* $\overline{x}$ *at which an individual is indifferent between a life of crime and an honest life?*

Answer: When $\delta=0.75$, the expected utility from a life of crime is given by

$$0.75\ln(20)+0.25\ln(80)\approx 3.342. \quad (17.148)$$

The certainty equivalent $\overline{x}$ is the obtained by setting $\ln(\overline{x})=3.342$ which solves to $\overline{x}=e^{3.342}\approx 28.28$ — the value of an honest income that makes individuals indifferent between an honest life and a life of crime.

(b) *If an investment in eduction results in a uniform increase of income of 5, what are the pre-policy incomes of people who will now switch from a life of crime to an honest life?*

Answer: Since we concluded above that 28.28 is the cut-off (honest) income level above which the expected utility from a life of crime is below the utility of an honest life, it is those with pre-policy incomes between 23.28 and 28.28 that will switch from lives of crime to honest lives.

(c) *How much would $\delta$ have to increase in order to achieve an equivalent reduction in crime? How much would this change the expected consumption level under a life of crime?*

Answer: In order for an increase in $\delta$ to accomplish the same thing, it must be that $\delta$ is sufficiently high for the expected utility of crime to lie below the utility from consumption of 23.28; i.e.

$$\delta\ln(20)+(1-\delta)\ln(80)=\ln(23.28). \quad (17.149)$$

Solving for $\delta$, we get $\delta\approx 0.89$ — i.e. we have to increase enforcement from 0.75 to 0.89 in order to achieve the same reduction in crime as the education policy analyzed above.

(d) *If it is equally costly to raise incomes by $1 through education investments as it is to reduce the expected value of consumption in a life of crime through an increase in* $\delta$*, which policy is the more cost effective way to reduce crime?*

Answer: The education investment policy raises incomes for everyone by 5. The increased enforcement policy that raises $\delta$ to 0.89 results in a reduction in the expected consumption value of crime to

$$EV' = 0.89(20) + 0.11(80) = 26.60, \tag{17.150}$$

down 8.4 from the initial expected consumption value of crime $EV = 35$. If it is equally costly to raise everyone's income as it is to lower $EV$, the increased enforcement policy is therefore a significantly more costly way of reducing crime than the education investment policy.

(e) *How do your answers change if* $\delta = 0.25$ *to begin with?*

Answer: When $\delta = 0.25$, we have an initial expected consumption level of crime equal to $EV = 65$ and an expected utility of crime of $E(u) \approx 4.035$ — with certainty equivalent of $\overline{x} \approx 56.57$. Thus, initially everyone whose honest income falls below 56.57 lives a life of crime. Under the education investment policy, all incomes rise by 5 — which implies that those who previously could earn between 51.57 and 56.57 in the (legal) labor market would switch from a life of crime to an honest life under this policy. In order to achieve an equivalent reduction in crime through lowering $\delta$ (using an increased enforcement policy), we need to find the $\delta$ for which the expected utility from a life of crime is equal to the utility of consuming 51.57. This gives us $\delta \approx 0.317$ — up from the initial 0.25. And, $\delta = 0.317$ implies an expected consumption value of crime equal to approximately 61 — down by only 4 from the initial 65. If it is equally costly to fund education investments that raise everyone's income by a dollar as it is to reduce the expected consumption value of crime by one dollar through increased enforcement, it now costs less to reduce crime through increased enforcement rather than investments in education.