

## S O L U T I O N S

# 7

## Income and Substitution Effects in Consumer Goods Market

### **Solutions for *Microeconomics: An Intuitive Approach with Calculus***

**Apart from end-of-chapter exercises provided in the student *Study Guide*, these solutions are provided for use by instructors.** (End-of-Chapter exercises with solutions in the student *Study Guide* are so marked in the textbook.)

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- Each end-of-chapter exercise begins on a new page. This is to facilitate maximum flexibility for instructors who may wish to share answers to some but not all exercises with their students.
- If you are assigning only the A-parts of exercises in *Microeconomics: An Intuitive Approach with Calculus*, you may wish to instead use the solution set created for the companion book *Microeconomics: An Intuitive Approach*.
- *Solutions to Within-Chapter Exercises are provided in the student Study Guide.*

**7.1** Below we consider some logical relationships between preferences and types of goods.

**A:** Suppose you consider all the goods that you might potentially want to consume.

- (a) Is it possible for all these goods to be luxury goods at every consumption bundle? Is it possible for all of them to be necessities?

Answer: Neither is possible. If they were all luxuries, then, as income increases by some percentage, consumption of each good would increase by a greater percentage. This is logically impossible. If they were all necessities, then, as income increases by some percentage, consumption of each good would increase by a lesser percentage. This implies that some income would remain unspent, which is inconsistent with optimization.

- (b) Is it possible for all goods to be inferior goods at every consumption bundle? Is it possible for all of them to be normal goods?

Answer: The first is not possible but the second is. If all goods are inferior, then, as income falls, the consumer would increase her consumption of all goods. But that is logically impossible since income is declining. If all goods are normal goods, then consumption of all increases with increases in income and decreases with decreases in income — which is logically possible.

- (c) True or False: When tastes are homothetic, all goods are normal goods.

Answer: True. Homothetic tastes are defined by the fact that the *MRS* remains constant along any ray from the origin. Thus, if we find a tangency of an indifference curve with a budget line, we know that, as income changes, indifference curves will always be tangent to the new budget along the ray that connects the original tangency to the origin. Thus, as income increases, consumption of all goods increases, and when income decreases, consumption of all goods decreases.

- (d) True or False: When tastes are homothetic, some goods could be luxuries while others could be necessities.

Answer: False. We just explained that for homothetic tastes, the optimal bundles (for a given set of prices) lie on rays from the origin as income changes. Thus, as income increases by some percentage, consumption of all goods increases by the same percentage. Thus, all goods are borderline between luxuries and necessities.

- (e) True or False: When tastes are quasilinear, one of the goods is a necessity.

Answer: True. As income changes, consumption of one of the goods does not change. Thus, as income increases, the percentage of income spent on that good decreases — making that good a necessity.

- (f) True or False: In a two good model, if the two goods are perfect complements, they must both be normal goods.

Answer: True — since the goods are always consumed as pairs, consumption of both increases as income increases.

- (g) True or False: In a 3-good model, if two of the goods are perfect complements, they must both be normal goods.

Answer: False. Since there is a third good, it may be that this third good is a normal good while the perfectly complementary goods are (jointly) inferior. Suppose, for instance, that rum and coke are perfect complements for someone, but that the person also has a taste for really good single malt scotch. As income goes up, he increases his consumption of single malt scotch and lowers his consumption of rum and cokes. Rum and coke would be perfect complements, but as income goes up, less of both would be consumed.

**B:** In each of the following cases, suppose that a person whose tastes can be characterized by the given utility function has income  $I$  and faces prices that are all equal to 1. Illustrate mathematically how his consumption of each good changes with income and use your answer to determine whether the goods are normal or inferior, luxuries or necessities.

- (a)  $u(x_1, x_2) = x_1 x_2$

Answer: In each case, we can set up the optimization problem

$$\max_{x_1, x_2} u(x_1, x_2) \text{ subject to } x_1 + x_2 = I \quad (7.1)$$

and solve it for  $x_1$  and  $x_2$  as a function of  $I$ . For the function  $u(x_1, x_2) = x_1 x_2$ , this gives us  $x_1(I) = x_2(I) = I/2$ . Thus, half of all income is spent on  $x_1$  and half on  $x_2$ , which implies that, when income doubles, so does consumption of each of the two goods. Thus, the goods are borderline between luxuries and necessities — and they are both normal.

(b)  $u(x_1, x_2) = x_1 + \ln x_2$

Answer: Solving this optimization problem again with the new utility function, we get  $x_1(I) = I - 1$  and  $x_2(I) = 1$ . Consumption of  $x_2$  is therefore independent of income — which means the good is borderline between normal and inferior. The fraction of income spent on  $x_2$  declines with income — which means the good is a necessity. Good  $x_1$ , on the other hand, is a normal good — and a luxury.

(c)  $u(x_1, x_2) = \ln x_1 + \ln x_2$

Answer: For this utility function, we again get  $x_1(I) = x_2(I) = I/2$  as in (a). (This makes sense since the utility function here is a monotone transformation of the utility function in (a).) So the same answer as in (a) applies.

(d)  $u(x_1, x_2, x_3) = 2 \ln x_1 + \ln x_2 + 4 \ln x_3$

Answer: We can again solve the same optimization problem, except that we now have 3 choice variables. We would write the Lagrange function as

$$\mathcal{L}(x_1, x_2, x_3, \lambda) = 2 \ln x_1 + \ln x_2 + 4 \ln x_3 + \lambda(I - x_1 - x_2 - x_3) \quad (7.2)$$

and the first three first order conditions as

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= \frac{2}{x_1} - \lambda = 0, \\ \frac{\partial \mathcal{L}}{\partial x_2} &= \frac{1}{x_2} - \lambda = 0, \\ \frac{\partial \mathcal{L}}{\partial x_3} &= \frac{4}{x_3} - \lambda = 0. \end{aligned} \quad (7.3)$$

The first and second can be used to write  $x_2 = x_1/2$ , and the first and third can be combined to give us  $x_3 = 2x_1$ . Substituting these into the budget constraint  $x_1 + x_2 + x_3 = I$  gives us  $x_1 + x_1/2 + 2x_1 = I$  which solves to  $x_1(I) = 2I/7$ . Substituting this back into  $x_2 = x_1/2$  and  $x_3 = 2x_1$  then gives us  $x_2(I) = I/7$  and  $x_3(I) = 4I/7$ . The consumption of each of the three goods is therefore a constant fraction of income — which implies all three goods are normal and borderline between luxuries and necessities.

(e)  $u(x_1, x_2) = 2x_1^{0.5} + \ln x_2$

Answer: Following the same set-up, we get<sup>1</sup>

$$x_1(I) = \left( \frac{-1 + (1 + 4I)^{1/2}}{2} \right)^2 \quad \text{and} \quad x_2(I) = \frac{-1 + (1 + 4I)^{1/2}}{2} \quad (7.4)$$

As income increases, consumption of both goods therefore increases (since  $I$  enters positively into both equations). However, it does not increase at a constant rate. Taking the derivative of  $x_2(I)$  with respect to  $I$ , we get

$$\frac{dx_2(I)}{dI} = \frac{1}{(1 + 4I)^{1/2}}, \quad (7.5)$$

which is a decreasing function of  $I$ . Thus, as income increases, the fraction devoted to consumption of  $x_2$  decreases — making  $x_2$  a necessity (and thus  $x_1$  a luxury good).

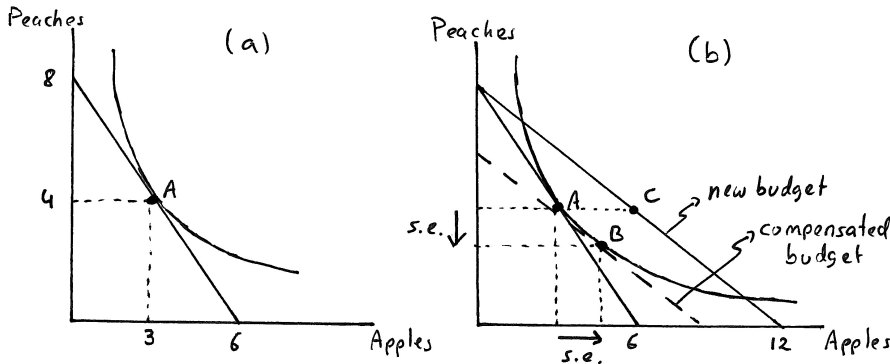
<sup>1</sup>Combining the first 2 first order conditions, we get  $x_1 = x_2^2$ , and substituting this into the budget constraint, we get  $x_2^2 + x_2 - I = 0$ . To solve this, we apply the quadratic formula which gives two answers for  $x_2$ . However, one of these is clearly negative.

**7.2** Suppose you have an income of \$24 and the only two goods you consume are apples ( $x_1$ ) and peaches ( $x_2$ ). The price of apples is \$4 and the price of peaches is \$3.

**A:** Suppose that your optimal consumption is 4 peaches and 3 apples.

(a) Illustrate this in a graph using indifference curves and budget lines.

Answer: This is illustrated in panel (a) of Graph 7.1, with the optimal bundle denoted A.



Graph 7.1: Apples and Peaches

(b) Now suppose that the price of apples falls to \$2 and I take enough money away from you to make you as happy as you were originally. Will you buy more or fewer peaches?

Answer: Panel (b) of the graph illustrates this substitution effect (s.e.) with the compensated budget that is tangent at B. As with all substitution effects, the consumer consumes more of what has become less expensive (apples) and less of what has become relatively more expensive (peaches).

(c) In reality, I do not actually take income away from you as described in (b) but your income stays at \$24 after the price of apples falls. I observe that, after the price of apples fell, you did not change your consumption of peaches. Can you conclude whether peaches are an inferior or normal good for you?

Answer: The compensated and new budgets are parallel to one another — with the new budget simply containing more income than the compensated budget. Since B has fewer peaches than C, we know that peach consumption increases with an increase in income. Therefore we can conclude that peaches are normal goods for you.

**B:** Suppose that your tastes can be characterized by the function  $u(x_1, x_2) = x_1^\alpha x_2^{(1-\alpha)}$ .

(a) What value must  $\alpha$  take in order for you to choose 3 apples and 4 peaches at the original prices?

Answer: Solving the optimization problem

$$\max_{x_1, x_2} u(x_1, x_2) = x_1^\alpha x_2^{(1-\alpha)} \quad \text{subject to } 4x_1 + 3x_2 = 24, \quad (7.6)$$

we get

$$x_1 = 6\alpha \quad \text{and} \quad x_2 = 8(1 - \alpha). \quad (7.7)$$

In order for  $x_1$  (apples) to be 3 and  $x_2$  (peaches) to be 4, this implies  $\alpha = 0.5$ .

(b) What bundle would you consume under the scenario described in A(b)?

Answer: At the original bundle (3, 4), your utility was  $u(3, 4) = 3^{0.5} 4^{0.5} \approx 3.464$ . To determine what bundle you would consume if enough income were taken away to make you just as happy after the price of apples falls to \$2, you would solve the problem

$$\min_{x_1, x_2} 2x_1 + 3x_2 \quad \text{subject to} \quad x_1^{0.5} x_2^{0.5} = 3.464. \quad (7.8)$$

The Lagrange function for this problem is

$$\mathcal{L}(x_1, x_2, \lambda) = 2x_1 + 3x_2 + \lambda(3.464 - x_1^{0.5} x_2^{0.5}), \quad (7.9)$$

and the first order conditions are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= 2 - 0.5\lambda x_1^{-0.5} x_2^{0.5} = 0, \\ \frac{\partial \mathcal{L}}{\partial x_2} &= 3 - 0.5\lambda x_1^{0.5} x_2^{-0.5} = 0, \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= 3.464 - x_1^{0.5} x_2^{0.5} = 0. \end{aligned} \quad (7.10)$$

Solving the first two equations for  $x_1$  gives  $x_1 = 3x_2/2$ , and plugging this into the third, we get  $x_2 \approx 2.828$ . Plugging this back into  $x_1 = 3x_2/2$  then gives us  $x_1 \approx 4.243$ . Point  $B$  in Graph 7.1 is therefore  $(4.243, 2.828)$ .

- (c) *How much income can I take away from you and still keep you as happy as you were before the price change?*

Answer: The most I can take from you is an amount that will allow you to purchase bundle  $B$  at the new prices. You will need  $2(4.243) + 3(2.828) \approx 16.97$ . Since you started with an income of \$24, this implies I can take \$7.03 from you.

- (d) *What will you actually consume after the price increase?*

Answer: Solving the optimization problem

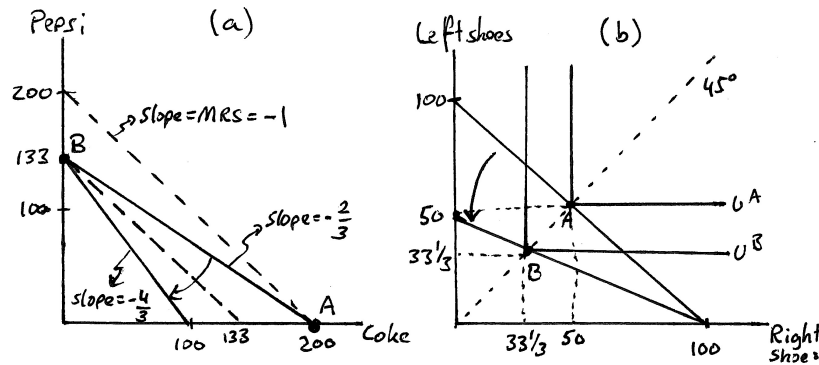
$$\max_{x_1, x_2} u(x_1, x_2) = x_1^{0.5} x_2^{0.5} \quad \text{subject to} \quad 2x_1 + 3x_2 = 24, \quad (7.11)$$

we get  $x_1 = 6$  and  $x_2 = 4$ . This is bundle  $C$  in Graph 7.1.

**7.3** Consider once again my tastes for Coke and Pepsi and my tastes for right and left shoes (as described in end-of-chapter exercise 6.2).

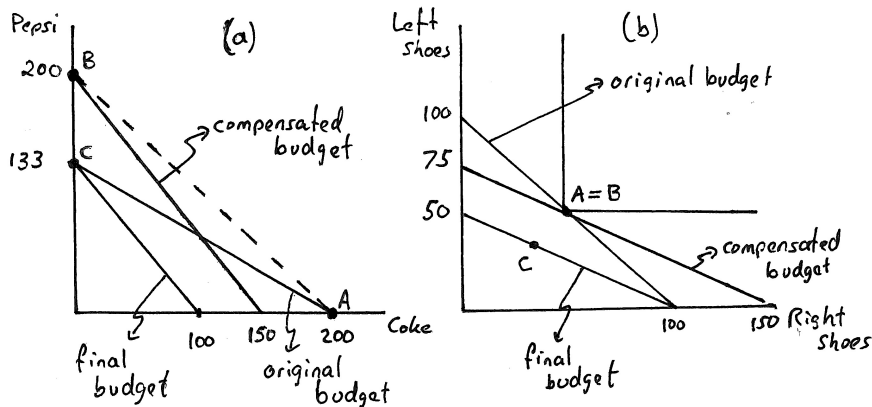
**A:** On two separate graphs — one with Coke and Pepsi on the axes, the other with right shoes and left shoes — replicate your answers to end-of-chapter exercise 6.2A(a) and (b). Label the original optimal bundles A and the new optimal bundles C.

**Answer:** The graphs from end-of-chapter exercise 6.2A(a) and (b) are replicated in Graph 7.2. Note that indifference curves in panel (a) are dashed while budget lines are solid. Also, note that in this replicated graph, B is the final optimum and should now be labeled C.



Graph 7.2: Replicated from End-of-Chapter exercise 6.2

(a) In your Coke/Pepsi graph, decompose the change in behavior into income and substitution effects by drawing the compensated budget and indicating the optimal bundle B on that budget.



Graph 7.3: Inc. and Subst. Effects for Perfect Substitutes and Complements

Answer: In panel (a) of Graph 7.3 (previous page), the original optimum occurs on the dashed indifference curve at bundle  $A$  while the final optimum occurs on the final budget at  $C$ . (To keep the picture uncluttered, the final indifference curve is left out.) The compensated budget has the same slope as the final budget but sufficient income to reach the original dashed indifference curve — which occurs at  $B$ . Thus, the substitution effect takes us from  $A$  to  $B$ , and the income effect to  $C$ . This should make sense: For the good whose price has changed (coke), the entire change is due to the substitution effect because the goods are perfect substitutes.

(b) Repeat (a) for your right shoes/left shoes graph.

Answer: Panel (b) of the graph shows the analogous for perfect complements. The compensated budget has the same slope as the final budget but must be “tangent” to the original indifference curve. This happens at  $A$  — which means the usual  $B$  that includes the substitution effect lies right on top of  $A$ . Thus, there is no substitution effect — which again should make sense since there is no substitutability between the two goods.

**B:** Now consider the following utility functions:  $u(x_1, x_2) = \min\{x_1, x_2\}$  and  $u(x_1, x_2) = x_1 + x_2$ .

(a) Which of these could plausibly represent my tastes for Coke and Pepsi, and which could represent my tastes for right and left shoes?

Answer: The first could represent tastes for right and left shoes while the second could represent tastes for Coke and Pepsi.

(b) Use the appropriate function from above to assign utility levels to bundles  $A$ ,  $B$  and  $C$  in your graph from 7.3A(a).

Answer: The appropriate function in this case is  $u(x_1, x_2) = x_1 + x_2$ . The three bundles are  $A=(200,0)$ ,  $B=(0,200)$  and  $C=(0,133)$ . Thus, the utility levels assigned to each of these bundles is  $u(A) = 200 = u(B)$  and  $u(C) = 133$ .

(c) Repeat this for bundles  $A$ ,  $B$  and  $C$  for your graph in 7.3A(b).

Answer: The appropriate function now is  $u(x_1, x_2) = \min\{x_1, x_2\}$  and the three bundles are  $A=B=(50,50)$  and  $C=(33.33,33.33)$ . The utility values associated with these bundles are  $u(A) = u(B) = 50$  and  $u(C) = 33.33$ .

**7.4** Return to the case of our beer and pizza consumption from end-of-chapter exercise 6.3.

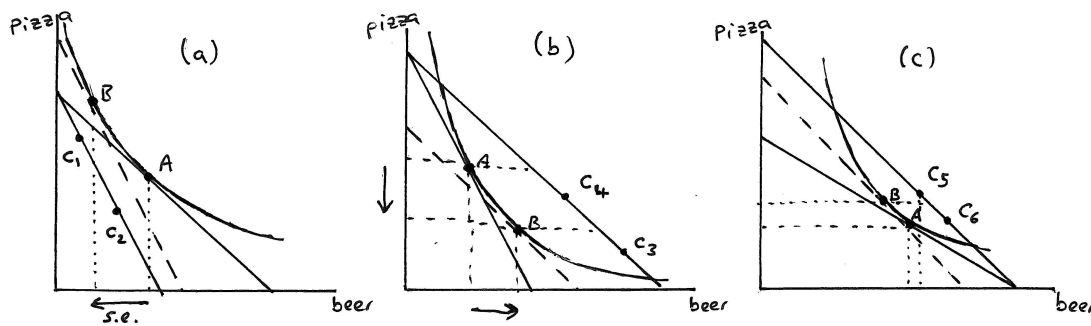
**A:** Again suppose you consume only beer and pizza (sold at prices  $p_1$  and  $p_2$  respectively) with an exogenously set income  $I$ . Assume again some initial optimal (interior) bundle  $A$ .

(a) In 6.3A(b), can you tell whether beer is normal or inferior? What about pizza?

Answer: Beer is normal since its consumption goes up with income. Pizza is borderline normal/inferior (or quasilinear) since its consumption is unchanged with income.

(b) When the price of beer goes up, I notice that you consume less beer. Can you tell whether beer is a normal or an inferior good?

Answer: No, you cannot tell. Panel (a) of Graph 7.4 illustrates the original consumption bundle  $A$  and the substitution effect that takes us to  $B$ . As always, the substitution effect tells us to consume less of what has become relatively more expensive (beer). The income effect could now go either way — push us toward even less beer consumption if beer is normal or toward more (than at  $B$ ) if beer is inferior. As long as beer is not a Giffen good, we will end up to the left of  $A$  — i.e. with less beer consumption — whether beer is normal or inferior. Two possibilities are illustrated — if we end up at  $C_1$ , beer is normal, and if we end up at  $C_2$ , it is inferior.



Graph 7.4: Beer and Pizza

(c) When the price of beer goes down, I notice you buy less pizza. Can you tell whether pizza is a normal good?

Answer: No, you cannot. Panel (b) of Graph 7.4 illustrates the substitution effect as the move from  $A$  to  $B$  — which involves a shift away from pizza and toward beer consumption (since beer has become relatively cheaper). The compensated budget (tangent at  $B$ ) and the final budget are then parallel and just involve an increase in income. We could end up at a bundle like  $C_3$  — with even less pizza consumption than at  $B$ . That would imply pizza consumption decreases with income, making pizza an inferior good. Or we could end up at a bundle like  $C_4$  — still below  $A$  but above  $B$ . In that case, we end up consuming less pizza as a result of the price change (i.e. less than at  $A$ ) but pizza is still a normal good (since pizza consumption increases (from  $B$ ) with an increase in income.)

(d) When the price of pizza goes down, I notice you buy more beer. Is beer an inferior good for you? Is pizza?

Answer: Beer is definitely a normal good but you can't tell for sure whether pizza is normal or inferior. Panel (c) of the graph illustrates the substitution effect again as the movement from  $A$  to  $B$  — toward more pizza (which has become cheaper) and less beer (which is now relatively more expensive). If beer consumption goes up as a result of the price change, this means that  $C$  is to the right of  $A$  — which means that beer consumption increases from the compensated to the final budget when income increases. Thus beer is normal. But it could be that the final optimum lies at a bundle like  $C_5$  or at a bundle like  $C_6$ . At  $C_5$ , pizza



consumption increases with income (from the compensated budget) — thus making pizza a normal good, but at  $C_6$ , pizza consumption decreases with income — thus making pizza inferior. Both are consistent with beer consumption increasing when the price of pizza goes down.

(e) Which of your conclusions in part (d) would change if you knew pizza and beer are very substitutable?

Answer: Suppose the price of pizza goes down and beer consumption goes up as in (d). For pizza to be a normal good, we have to end up at a bundle  $C$  that contains more pizza than  $B$  if pizza is to be a normal good. But if the substitution effect is large,  $B$  will contain sufficiently much pizza such that there will be no bundles on the new budget constraint that have more pizza than bundle  $B$  and more beer than bundle  $A$ . Thus, if we know the substitution effect is sufficiently big, we will know that pizza must be an inferior good.

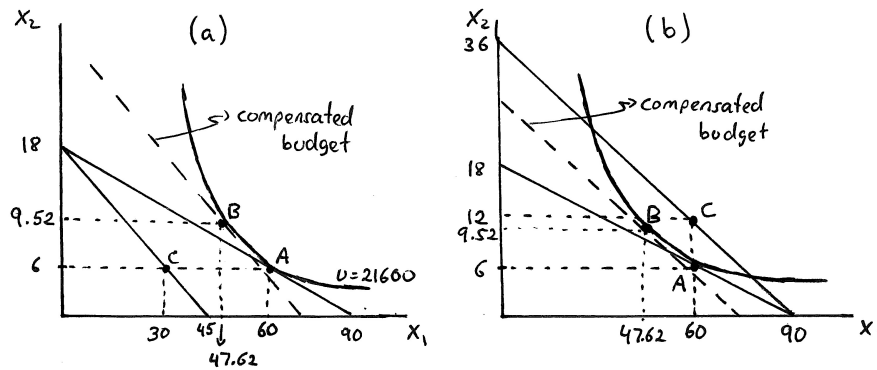
**B:** Suppose, as you did in end-of-chapter exercise 6.3B that your tastes over beer ( $x_1$ ) and pizza ( $x_2$ ) can be summarized by the utility function  $u(x_1, x_2) = x_1^2 x_2$ . If you have not already done so, calculate the optimal quantity of beer and pizza consumption as a function of  $p_1$ ,  $p_2$  and  $I$ .

(a) Illustrate the optimal bundle  $A$  when  $p_1=2$ ,  $p_2=10$  and weekly income  $I=180$ . What numerical label does this utility function assign to the indifference curve that contains bundle  $A$ ?

Answer: The answer we calculated in end-of-chapter exercise 6.3B is

$$x_1 = \frac{2I}{3p_1} \text{ and } x_2 = \frac{I}{3p_2}. \quad (7.12)$$

When  $p_1=2$ ,  $p_2=10$  and weekly income  $I=180$ , this implies  $x_1 = 60$  and  $x_2 = 6$ . The utility label attached to the indifference curve that contains this bundle is  $u(60, 6) = 60^2(6) = 21600$ . The solution is illustrated in panel (a) of Graph 7.5 as  $A$ .



Graph 7.5: Beer and Pizza: Part II

(b) Using your answer above, show that both beer and pizza are normal goods when your tastes can be summarized by this utility function.

Answer: We simply have to check how demand for the two goods changes as income changes, which we can do by simply taking the derivative of our expressions for  $x_1$  and  $x_2$ . Thus,

$$\frac{\partial x_1}{\partial I} = \frac{2}{3p_1} \text{ and } \frac{\partial x_2}{\partial I} = \frac{1}{3p_2}, \quad (7.13)$$

with both of these greater than zero. Thus, as income increases, consumption of both goods increases — which implies the goods are both normal.

- (c) Suppose the price of beer goes up to \$4. Illustrate your new optimal bundle and label it C.

Answer: Plugging the new price for  $p_1$  into  $x_1 = 2I/3p_1$  gives us  $x_1 = 2(180)/(3(4)) = 30$ . Since the expression for  $x_2$  does not contain  $p_1$ , consumption of  $x_2$  remains unchanged at 6. Point C = (30, 6) is also illustrated in panel (a) of the graph.

- (d) How much beer and pizza would you buy if you had received just enough of a raise to keep you just as happy after the increase in the price of beer as you were before (at your original income of \$180)? Illustrate this as bundle B.

Answer: We know that the utility label on the indifference curve containing the original bundle is 21,600. To determine the bundle B that lies on the same indifference curve but at the new prices, we solve the problem

$$\min_{x_1, x_2} 4x_1 + 10x_2 \quad \text{subject to} \quad x_1^2 x_2 = 21600. \quad (7.14)$$

Solving the first two first order conditions for  $x_1$ , we get  $x_1 = 5x_2$ . Plugging this into  $x_1^2 x_2 = 21600$ , we get  $(5x_2)^2 x_2 = 21600$  which solves to  $x_2 = 9.52$ . Plugging this back into  $x_1 = 5x_2$  then gives us  $x_1 = 47.62$ . Thus,  $B = (47.62, 9.52)$  which is also illustrated in panel (a) of the graph.

- (e) How large was your salary increase in (d)?

Answer: At the new prices, bundle B costs  $4(47.62) + 10(9.52) \approx 285.73$ . Given you started with an income of \$180, this implies your salary increase was about \$105.73.

- (f) Now suppose the price of pizza ( $p_2$ ) falls to \$5 (and suppose the price of beer and your income are \$2 and \$180 as they were originally at bundle A.) Illustrate your original budget, your new budget, the original optimum A and the new optimum C in a graph.

Answer: Plugging the new values into the equations  $x_1 = 2I/3p_1$  and  $x_2 = I/3p_2$ , we get  $x_1 = 60$  and  $x_2 = 12$ . Thus, consumption of  $x_1$  is unchanged but consumption of  $x_2$  doubles as illustrated in panel (b) of Graph 7.5.

- (g) Calculate the income effect and the substitution effect for both pizza and beer consumption from this change in the price of pizza. Illustrate this in your graph.

Answer: The original utility level at A is 21,600. To calculate the substitution effect, we need to solve

$$\min_{x_1, x_2} 2x_1 + 5x_2 \quad \text{subject to} \quad x_1^2 x_2 = 21600. \quad (7.15)$$

The first two first order conditions can be solved to yield  $x_1 = 5x_2$ . Plugging this back into  $x_1^2 x_2 = 21600$ , we get  $(5x_2)^2 x_2 = 21600$  which solves to  $x_2 = 9.52$ . Plugging this back into  $x_1 = 5x_2$  gives us  $x_1 = 47.62$ . The substitution effect is therefore the move from  $A = (60, 6)$  to  $B = (47.62, 9.52)$ , and the income effect is the move from  $B$  to  $C = (60, 12)$ .

Note that B is the same in panel (b) of the graph as it is in panel (a). This is because in both cases we have the same original indifference curve through the original bundle A, and in both cases the compensated budget has the slope  $-2/5$ . In one case, the slope arises from an increase in  $p_1$ , in the other case from a drop in  $p_2$  — but in both cases we are fitting the same slope to the same indifference curve.

- (h) True or False: Since income and substitution effects point in opposite directions for beer, beer must be an inferior good.

Answer: False. When income increases from the compensated budget to the final budget, consumption of beer increases. Thus, beer is a normal good. While it is true that income and substitution effects point in the same direction for a normal good when *that* good's price changes, in this case it is the *other* good's price that changes. In that case, income and substitution effects point in opposite directions when the good is normal.

**7.5** Return to the analysis of my undying love for my wife expressed through weekly purchases of roses (as introduced in end-of-chapter exercise 6.4).

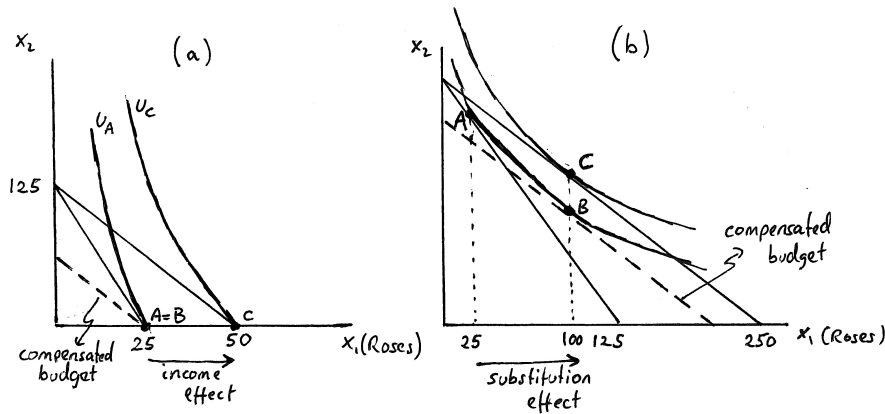
**A:** Recall that initially roses cost \$5 each and, with an income of \$125 per week, I bought 25 roses each week. Then, when my income increased to \$500 per week, I continued to buy 25 roses per week (at the same price).

- (a) From what you observed thus far, are roses a normal or an inferior good for me? Are they a luxury or a necessity?

**Answer:** As income went up, my consumption remained unchanged. This would typically indicate that the good in question is borderline normal/inferior — or quasilinear. Since the consumption at the lower income is at a corner solution, however, we cannot be certain that the good is not inferior, with the  $MRS$  at the original optimum larger in absolute value than the  $MRS$  at the new (higher income) optimum. Regardless, roses must be a necessity — whether they are borderline inferior/normal or inferior, the percentage of income spent on roses declines as income increases.

- (b) On a graph with weekly roses consumption on the horizontal and “other goods” on the vertical, illustrate my budget constraint when my weekly income is \$125. Then illustrate the change in the budget constraint when income remains \$125 per week and the price of roses falls to \$2.50. Suppose that my optimal consumption of roses after this price change rises to 50 roses per week and illustrate this as bundle C.

**Answer:** This is illustrated in panel (a) of Graph 7.6 (on the next page) where A is the original corner solution, C is the new corner solution and the dashed line is the compensated budget.



Graph 7.6: Love and Roses

- (c) Illustrate the compensated budget line and use it to illustrate the income and substitution effects.

**Answer:** This is also illustrated in panel (a) of the graph. In this case, there is no substitution effect (in terms of roses) and only an income effect.

- (d) Now consider the case where my income is \$500 and, when the price changes from \$5 to \$2.50, I end up consuming 100 roses per week (rather than 25). Assuming quasilinearity in roses, illustrate income and substitution effects.

**Answer:** This is illustrated in panel (b) of Graph 7.6 where the dashed line is again the compensated budget line. Unlike in panel (a), the entire change in roses consumption is now due to a substitution effect rather than an income effect.

- (e) True or False: Price changes of goods that are quasilinear give rise to no income effects for the quasilinear good unless corner solutions are involved.

Answer: This is true. We will often make the statement that income effects disappear if we assume quasilinearity of a good — because then a good is borderline normal/inferior, which implies consumption remains unchanged as income changes. This is true so long as the consumer is at an interior solution. If quasilinear tastes lead to corner solutions, then this may give rise to income effects as we see in panel (a) of the graph.

**B:** Suppose again, as in 6.4B, that my tastes for roses ( $x_1$ ) and other goods ( $x_2$ ) can be represented by the utility function  $u(x_1, x_2) = \beta x_1^\alpha + x_2$ .

- (a) If you have not already done so, assume that  $p_2$  is by definition equal to 1, let  $\alpha = 0.5$  and  $\beta = 50$ , and calculate my optimal consumption of roses and other goods as a function of  $p_1$  and  $I$ .

Answer: Solving the optimization problem

$$\max_{x_1, x_2} 50x_1^{0.5} + x_2 \quad \text{subject to} \quad I = p_1 x_1 + x_2, \quad (7.16)$$

we get

$$x_1 = \frac{625}{p_1^2} \quad \text{and} \quad x_2 = I - \frac{625}{p_1}. \quad (7.17)$$

- (b) The original scenario you graphed in 7.5A(b) contains corner solutions when my income is \$125 and the price is initially \$5 and then \$2.50. Does your answer above allow for this?

Answer: Substituting  $I = 125$  and  $p_1 = 5$  into our equations (7.17) for  $x_1$  and  $x_2$  from above, we get  $x_1 = 625/(5^2) = 25$  and  $x_2 = 125 - (625/5) = 0$ . This is exactly the original corner solution in the scenario in part A.

Changing the price to  $p_1 = 2.5$ , we get  $x_1 = 625/(2.5^2) = 100$  and  $x_2 = 125 - (625/2.5) = -125$ . Given that the solution from our Lagrange method now gives us a negative consumption level for  $x_2$ , we know that the true optimum is the corner solution where all income is spent on  $x_1$  — i.e. the bundle (50,0) just as described in the scenario in A.

At the original price, it turns out that the  $MRS$  at the corner solution is exactly equal to the slope of the budget line. At the lower price, the  $MRS$  is large in absolute value than the budget line — which means the indifference curve cuts the budget line at the corner from above. The tangency of an indifference curve with this budget line therefore does not happen until  $x_2$  is negative — which the Lagrange method finds but which is not economically meaningful.

- (c) Verify that the scenario in your answer to 7.5A(d) is also consistent with tastes described by this utility function — i.e. verify that A, B and C are as you described in your answer.

Answer: Using equations (7.17), we get  $x_1 = 625/(5^2) = 25$  and  $x_2 = 500 - (625/5) = 375$  when  $p_1 = 5$  (and  $I = 500$ ), and we get  $x_1 = 625/(2.5^2) = 100$  and  $x_2 = 500 - (625/2.5) = 250$  when  $p_1 = 2.5$ . These correspond to A and C in panel (b) of Graph 7.6.

To calculate B in the graph, we need to first find the utility level associated with the original bundle A — i.e.  $u(25, 375) = 50(25^{0.5}) + 375 = 625$ . We then need to find what bundle the consumer would buy if she was given enough money to reach that same indifference curve at the new price; i.e. we need to solve the problem

$$\min_{x_1, x_2} 2.5x_1 + x_2 \quad \text{subject to} \quad 625 = 50x_1^{0.5} + x_2. \quad (7.18)$$

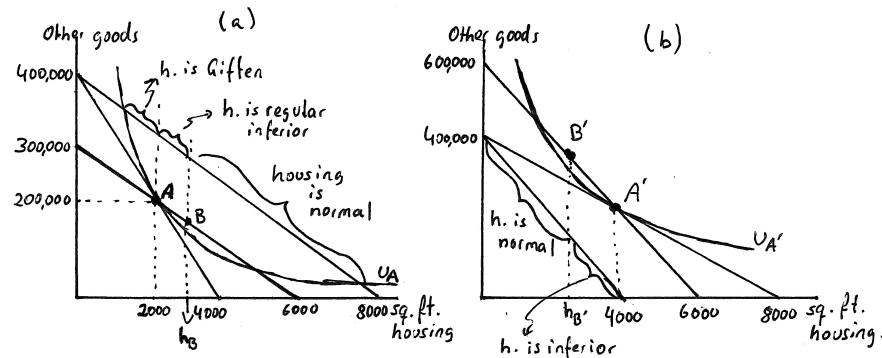
Solving the first order conditions, we then get  $x_1 = 100$  and  $x_2 = 125$  — consistent with panel (b) of the graph.

**7.6 Everyday Application: Housing Price Fluctuations: Part 2:** Suppose, as in end-of-chapter exercise 6.9, you have \$400,000 to spend on “square feet of housing” and “all other goods”. Assume the same is true for me.

**A:** Suppose again that you initially face a \$100 per square foot price for housing, and you choose to buy a 2000 square foot house.

- (a) Illustrate this on a graph with square footage of housing on the horizontal axis and other consumption on the vertical. Then suppose, as you did in exercise 6.9, that the price of housing falls to \$50 per square foot after you bought your 2000 square foot house. Label the square footage of the house you would switch to  $h_B$ .

**Answer:** In panel (a) of Graph 7.7, bundle  $A$  lies on the original budget constraint that extends from \$400,000 on the vertical axis to 4000 square feet on the horizontal. Since this is the optimal bundle for that budget, the indifference curve  $u_A$  is tangent at that point. This implies that the new budget — which extends from \$300,000 to 6000 square feet — will cut the indifference curve  $u^A$  from below at  $A$ . Thus, a number of new bundles that lie above the indifference curve  $u^A$  become available — all of which contain more housing.  $B$  is one possible bundle that could be a new optimum.



Graph 7.7: Housing Prices

- (b) Is  $h_B$  smaller or larger than 2000 square feet? Does your answer depend on whether housing is normal, regular inferior or Giffen?

**Answer:** As already explained above, the square footage of the new house is larger than 2000 square feet. This does not depend on whether housing is normal, regular inferior or Giffen — it is the result of an almost pure substitution effect. (It is “almost” pure because a pure substitution effect as we have defined it would involve no change in utility. In fact, as we will explore in exercise 7.11, this is what is known as a “Slutsky substitution effect” which differs from the “Hicksian” substitution we typically assume in that it compensates individuals to be able to buy the original bundle rather than to attain their original level of happiness.)

- (c) Now suppose that the price of housing had fallen to \$50 per square foot before you bought your initial 2000 square foot house. Denote the size of house you would have bought  $h_C$ .

**Answer:** The budget constraint you would have faced is also graphed in panel (a) of Graph 7.7 as the line from \$400,000 to 8000 square feet. Since this has the same relative price as the budget line from \$300,000 to 6000 square feet, the only difference between these two constraints is that one has more income. If  $B$  is the optimal bundle at the lower income, then the new optimal bundle will lie to the right of  $B$  if housing is a normal good and to the left of  $B$  if it is inferior. Furthermore, if the new optimal bundle were to lie to the left of  $A$ , then housing consumption would be decreasing with a decrease in price — which would make housing a Giffen good.

- (d) Is  $h_C$  larger than  $h_B$ ? Is it larger than 2000 square feet? Does your answer depend on whether housing is a normal, regular inferior or Giffen good?

Answer: As indicated in the graph, the answer depends on what kind of good housing is. If housing is a normal good (as it almost certainly is for most people), then  $h_C > h_B > h_A$ . If it were a regular inferior good, then  $h_B > h_C > h_A$ , and, in the extremely unlikely event that it were a Giffen good,  $h_B > h_A > h_C$ .

- (e) Now consider me. I did not buy a house until the price of housing was \$50 per square foot — at which time I bought a 4000 square foot house. Then the price of housing rises to \$100 per square foot. Would I sell my house and buy a new one? If so, is the new house size  $h_{B'}$  larger or smaller than 4000 square feet? Does your answer depend on whether housing is normal, regular inferior or Giffen for me?

Answer: My original budget constraint is graphed in panel (b) of Graph 7.7 as the line from \$400,000 to 8000 square feet. Since bundle  $A'$  is optimal on that budget, the indifference curve  $u_{A'}$  is tangent at  $A'$ . When the price increases to \$100 per square foot, my new budget extends from \$600,000 to 6000 square feet and runs through  $A'$ . Since it is steeper than the original budget, this implies it cuts the indifference curve  $u_{A'}$  from above at  $A'$ . This results in a set of bundles to the left of  $A'$  that now lie above the original indifference curve  $u_{A'}$ . The new optimal bundle  $B'$  then lies somewhere in this set — which implies  $h_{B'}$  is less than 4000 square feet. This is again an almost pure substitution effect — and does not depend on whether housing is normal, regular inferior or Giffen.

- (f) Am I better or worse off?

Answer: Since  $B'$  lies above  $u_{A'}$ , I have moved to a higher indifference curve and am therefore better off.

- (g) Suppose I had not purchased at the low price but rather purchased a house of size  $h_{C'}$  after the price had risen to \$100 per square foot. Is  $h_{C'}$  larger or smaller than  $h_{B'}$ ? Is it larger or smaller than 4000 square feet? Does your answer depend on whether housing is normal, regular inferior or Giffen for me?

Answer: In this case, my budget constraint would have run from \$400,000 to 4000 square feet (as depicted in panel (b) of the graph). This budget reflects the same relative prices as the budget which runs from \$600,000 to 6000 square feet — which means that those two budgets just differ in the amount of income available. Since  $B'$  is optimal at the higher income, housing is a normal good if the optimal bundle at the lower income lies to the left of  $B'$  and inferior if it lies to the right. (If it were to lie to the right of  $A'$ , housing would be a Giffen good — but in this case there are no bundles available to the right of  $A'$  under the new budget. We would therefore not be able to identify housing as a Giffen good.)

**B:** Suppose both you and I have tastes that can be represented by the utility function  $u(x_1, x_2) = x_1^{0.5} x_2^{0.5}$ , where  $x_1$  is square feet of housing and  $x_2$  is “dollars of other goods”.

- (a) Calculate the optimal level of housing consumption  $x_1$  as a function of per square foot housing prices  $p_1$  and income  $I$ .

Answer: We would solve the problem

$$\max_{x_1, x_2} x_1^{0.5} x_2^{0.5} \text{ subject to } p_1 x_1 + x_2 = I \quad (7.19)$$

to get

$$x_1 = \frac{I}{2p_1} \text{ and } x_2 = \frac{I}{2}. \quad (7.20)$$

- (b) Verify that your initial choice of a 2000 square foot house and my initial choice of a 4000 square foot house was optimal under the circumstances we faced (assuming we both started with \$400,000.)

Answer: At  $p_1 = 100$  and  $I = 400,000$ , the above equations give us  $x_1 = 400000/(2(100)) = 2000$  and  $x_2 = 400000/2 = 200,000$ . At  $p_1 = 50$  and  $I = 400,000$ , the above equations give us  $x_1 = 400000/(2(50)) = 4000$  and  $x_2 = 400000/2 = 200,000$ .

(c) Calculate the values of  $h_B$  and  $h_C$  as they are described in A(a) and (c).

Answer: To calculate  $h_B$ , we can use the fact (which can be seen from panel (a) of Graph 7.7) that your available income (if you were to sell your original house at the new price) is now \$300,000 when the price falls to \$50 per square foot. Plugging these values into our equation for  $x_1$ , we get  $x_1 = 300000 / (2(50)) = 3000$ . Thus, you would sell your 2000 square foot house and buy a 3000 square foot house.

To calculate  $h_C$ , we can simply plug in  $I = 400,000$  and  $p_1 = 50$  to get  $x_1 = 400000 / (2(50)) = 4000$  — i.e. you would have purchased a 4000 square foot house at \$50 per square foot had you waited for the price to fall.

(d) Calculate  $h_{B'}$  and  $h_{C'}$  as those are described in A(d) and (f).

Answer: To calculate  $h_{B'}$ , we use  $I = 600000$  and  $p_1 = 100$  to get  $x_1 = 600000 / (2(100)) = 3000$ . To calculate  $h_{C'}$ , we use  $I = 400000$  and  $p_1 = 100$  to get  $x_1 = 400000 / (2(100)) = 2000$ .

(e) Verify your answer to A(e).

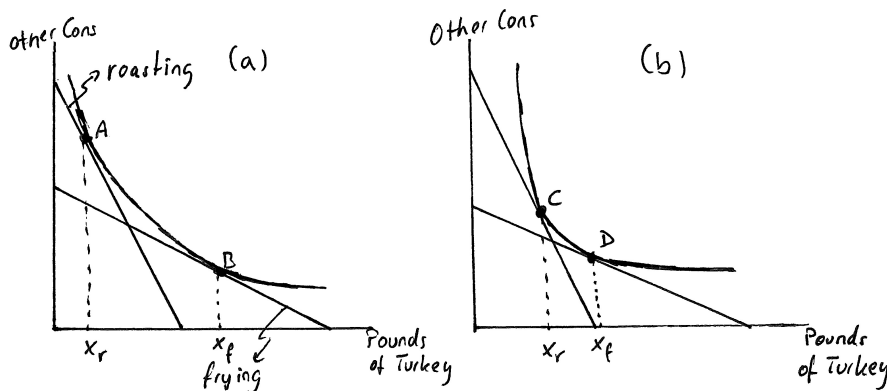
Answer: Utility at  $A'$  is given by  $u_{A'} = u(4000, 200000) = 28,284$ . Utility at  $B'$  is  $u_{B'} = u(3000, 300000) = 30,000$ . Thus, utility is higher at  $B'$  than at  $A'$ .

**7.7 Everyday Application: Turkey and Thanksgiving.** Every Thanksgiving, my wife and I debate about how we should prepare the turkey we will serve (and will then have left over). My wife likes preparing turkeys the conventional way — roasted in the oven where it has to cook at 350 degrees for 4 hours or so. I, on the other hand, like to fry turkeys in a big pot of peanut oil heated over a powerful flame outdoors. The two methods have different costs and benefits. The conventional way of cooking turkeys has very little set-up cost (since the oven is already there and just has to be turned on) but a relatively large time cost from then on. (It takes hours to cook.) The frying method, on the other hand, takes some set-up (dragging out the turkey fryer, pouring gallons of peanut oil, etc. — and then later the cleanup associated with it), but turkeys cook predictably quickly in just 3.5 minutes per pound.

**A:** As a household, we seem to be indifferent between doing it one way or another — sometimes we use the oven, sometimes we use the fryer. But we have noticed that we cook much more turkey — several turkeys, as a matter of fact, when we use the fryer than when we use the oven.

- (a) Construct a graph with “pounds of cooked turkeys” on the horizontal and “other consumption” on the vertical. (“Other consumption” here is not denominated in dollars as normally but rather in some consumption index that takes into account the time it takes to engage in such consumption.) Think of the set-up cost for frying turkeys and the waiting cost for cooking them as the main costs that are relevant. Can you illustrate our family’s choice of whether to fry or roast turkeys at Thanksgiving as a choice between two “budget lines”?

**Answer:** This is illustrated in panel (a) of Graph 7.8. The set-up cost of the turkey fryer results in a lower intercept for the frying budget on the vertical axis — but the lower cost of cooking turkey results in a shallower slope.



Graph 7.8: Frying versus Roasting Turkey

- (b) Can you explain the fact that we seem to eat more turkey around Thanksgiving whenever we pull out the turkey fryer as opposed to roasting the turkey in the oven?

**Answer:** Since we are indifferent between frying and roasting, our optimal bundle on the two budget lines must lie on the same indifference curve. This is also illustrated in panel (a) of the graph — where it is immediately apparent that we will cook more turkey when frying than when roasting because of the lower opportunity cost.

- (c) We have some friends who also struggle each Thanksgiving with the decision of whether to fry or roast — and they, too, seem to be indifferent between the two options. But we have noticed that they only cook a little more turkey when they fry than when they roast. What is different about them?

**Answer:** A possible picture for my friend’s family is illustrated in panel (b) of the graph — where the indifference curve is not as flat — making the two goods less substitutable. Since the effect we are demonstrating is a pure substitution effect, it makes sense that with less



substitutability between the goods, the difference in behavior is smaller for the two turkey cooking options.

**B:** Suppose that, if we did not cook turkeys, we could consume 100 units of “other consumption” — but the time it takes to cook turkeys takes away from that consumption. Setting up the turkey fryer costs  $c$  units of consumption and waiting 3.5 minutes (which is how long it takes to cook 1 pound of turkey) costs 1 unit of consumption. Roasting a turkey involves no set-up cost, but it takes 5 times as long to cook per pound. Suppose that tastes can be characterized by the CES utility function  $u(x_1, x_2) = (0.5x_1^{-\rho} + 0.5x_2^{-\rho})^{-1/\rho}$  where  $x_1$  is pounds of turkey and  $x_2$  is “other consumption”.

(a) What are the two budget constraints I am facing?

Answer: Costs are denominated in “units of consumption” — which implies that  $p_2$ , the price of consuming “other goods”, is by definition 1. The price of cooking 1 pound of turkey ( $p_1$ ) is then either 1 if we fry or 5 if we roast. This gives us the budget constraints

$$5x_1 + x_2 = 100 \text{ when roasting, and } x_1 + x_2 = 100 - c \text{ when frying.} \quad (7.21)$$

(b) Can you calculate how much turkey someone with these tastes will roast (as a function of  $\rho$ )? How much will the same person fry? (Hint: Rather than solving this using the Lagrange method, use the fact that you know the MRS is equal to the slope of the budget line — and recall from chapter 5 that, for a CES utility function of this kind,  $MRS = -(x_2/x_1)^{\rho+1}$ .)

Answer: At the optimum, we set the MRS equal to the ratio  $-p_1/p_2$ . Setting MRS equal to the ratio of prices then implies

$$\left(\frac{x_2}{x_1}\right)^{\rho+1} = 5 \text{ when roasting, and } \left(\frac{x_2}{x_1}\right)^{\rho+1} = 1 \text{ when frying.} \quad (7.22)$$

Solving for  $x_2$ , we get  $x_2 = 5^{1/(\rho+1)}x_1$  when roasting and  $x_2 = x_1$  when frying. Substituting these into the appropriate budget constraints from equation (7.21) and solving for  $x_1$ , we get

$$x_1 = \frac{100}{5 + 5^{1/(\rho+1)}} \text{ when roasting, and } x_1 = \frac{100 - c}{2} \text{ when frying.} \quad (7.23)$$

(c) Suppose my family has tastes with  $\rho = 0$  and my friend's with  $\rho = 1$ . If each of us individually roasts turkeys this Thanksgiving, how much will we each roast?

Answer: My family will roast

$$x_1 = \frac{100}{5 + 5^1} = 10, \quad (7.24)$$

and my friend's family will roast

$$x_1 = \frac{100}{5 + 5^{1/2}} = 13.82. \quad (7.25)$$

(d) How much utility will each of us get (as measured by the relevant utility function)? (Hint: In the case where  $\rho = 0$ , the exponent  $1/\rho$  is undefined. Use the fact that you know that when  $\rho = 0$  the CES utility function is Cobb-Douglas.)

Answer: To calculate utilities, we first have to calculate how much of  $x_2$  each of us consumes. Just plugging our answers above into the first budget constraint in equation (7.21), we get  $x_2 = 50$  for my family and  $x_2 = 30.9$  for my friends. For my family,  $\rho = 0$  — which means we can use the Cobb-Douglas utility function  $x_1^{0.5}x_2^{0.5}$  instead of the CES functional form. Plugging  $(x_1, x_2) = (10, 50)$  into  $x_1^{0.5}x_2^{0.5}$  gives us utility of 22.36. For my friend's family, plugging  $(x_1, x_2) = (13.82, 30.90)$  into his utility function (with  $\rho = 1$ ), we get utility of 19.1.

(e) Which family is happier?

Answer: We can't know since we generally do not believe that we are measuring utility in units that can be compared across people.

- (f) If we are really indifferent between roasting and frying, what must  $c$  be for my family? What must it be for my friend's family? (Hint: Rather than setting up the usual minimization problem, use your answer to (b) determine  $c$  by setting utility equal to what it was for roasting).

Answer: We know from our answer in (b) that, when frying,  $x_1 = (100 - c)/2$  regardless of  $\rho$ . Plugging this into our frying budget constraint  $x_1 + x_2 = 100 - c$ , this implies that  $x_2 = (100 - c)/2$  regardless of  $\rho$ . When  $\rho = 0$ , we can then plug these into the Cobb-Douglas version of the utility function and set it equal to the utility of 22.36 that we determined above my family gets when roasting turkeys; i.e.

$$\left(\frac{100 - c}{2}\right)^{0.5} \left(\frac{100 - c}{2}\right)^{0.5} = \left(\frac{100 - c}{2}\right) = 22.36. \quad (7.26)$$

Solving for  $c$ , we get  $c = 55.28$ . For my friend's family, we can similarly substitute  $x_1 = (100 - c)/2$  and  $x_2 = (100 - c)/2$  into his CES utility function (with  $\rho = 1$ ) and set it equal to the utility he gets from roasting — which we calculated above to be 19.1. Thus,

$$\left[0.5\left(\frac{100 - c}{2}\right)^{-1} + 0.5\left(\frac{100 - c}{2}\right)^{-1}\right]^{-1} = \frac{100 - c}{2} = 19.1. \quad (7.27)$$

Solving for  $c$ , we get  $c = 61.8$ .

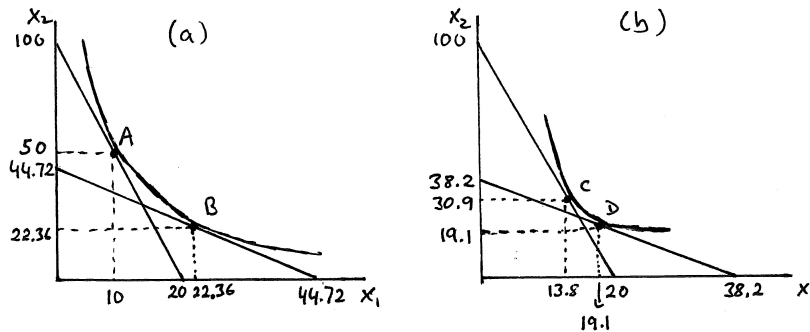
- (g) Given your answers so far, how much would we each have fried had we chosen to fry instead of roast (and we were truly indifferent between the two because of the different values of  $c$  we face)?

Answer: Given that we calculated  $c = 55.28$  for my family and  $c = 61.8$  for my friend's, we get that  $x_1 = (100 - 55.28)/2 = 22.36$  pounds for my family and  $x_1 = (100 - 61.8)/2 = 19.1$  pounds for my friend's family.

- (h) Compare the size of the substitution effect you have calculated for my family and that you calculated for my friend's family and illustrate your answer in a graph with pounds of turkey on the horizontal and other consumption on the vertical. Relate the difference in the size of the substitution effect to the elasticity of substitution.

Answer: My family goes from roasting 10 pounds of turkey to frying 22.23 pounds — a substitution effect of 12.36 pounds. My friend's family goes from roasting 13.82 pounds to frying 19.1 pounds — a substitution effect of 5.28 pounds. The difference, of course, is the greater substitutability that is built into my utility function with  $\rho = 0$  as opposed to my friend's with  $\rho = 1$ . To be precise, my elasticity of substitution is 1 whereas my friend's is 0.5.

The results are graphed in Graph 7.9, with panel (a) representing my family and panel (b) representing my friend's.



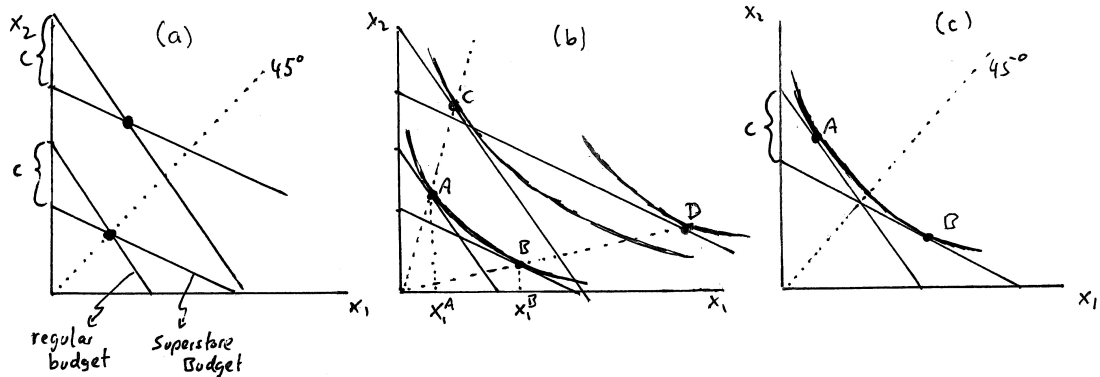
Graph 7.9: Frying versus Roasting Turkey: Part II

**7.8 Business Application: Sam's Club and the Marginal Consumer.** Superstores like Costco and Sam's Club serve as wholesalers to businesses but also target consumers who are willing to pay a fixed fee in order to get access to the lower wholesale prices offered in these stores. For purposes of this exercise, suppose that you can denote goods sold at Superstores as  $x_1$  and "dollars of other consumption" as  $x_2$ .

**A:** Suppose all consumers have the same homothetic tastes over  $x_1$  and  $x_2$  but they differ in their income. Every consumer is offered the same option of either shopping at stores with somewhat higher prices for  $x_1$  or paying the fixed fee  $c$  to shop at a Superstore at somewhat lower prices for  $x_1$ .

- (a) On a graph with  $x_1$  on the horizontal axis and  $x_2$  on the vertical, illustrate the regular budget (without a Superstore membership) and the Superstore budget for a consumer whose income is such that these two budgets cross on the 45 degree line. Indicate on your graph a vertical distance that is equal to the Superstore membership fee  $c$ .

**Answer:** In panel (a) of Graph 7.10 (on the next page), the Superstore budget has shallower slope (because of the lower price of  $x_1$ ) but a lower vertical intercept (because of the fixed membership fee). The lower two budgets in the graph are such that they intersect on the 45 degree line.



Graph 7.10: Sam's Club

- (b) Now consider a consumer with twice that much income. Where will this consumer's two budgets intersect relative to the 45 degree line?

**Answer:** This is also illustrated in panel (a). When income is doubled, the vertical intercept of the regular budget doubles — but the vertical intercept of the Superstore budget more than doubles because the fixed fee remains the same. (If the initial income is  $I$ , the initial intercept of the Superstore budget is  $(I - c)$ . When income doubles, the new intercept is  $(2I - c)$  — which is greater than  $2(I - c)$ .) For this reason, the two budget lines will cross above the 45 degree line when income doubles.

- (c) Suppose consumer 1 (from part (a)) is just indifferent between buying and not buying the Superstore membership. How will her behavior differ depending on whether or not she buys the membership.

**Answer:** In panel (b) of the graph, Consumer 1 will then consume at bundle A if she does not buy the membership and at bundle B if she does. This is a pure substitution effect — with greater consumption when price is lower.

- (d) If consumer 1 was indifferent between buying and not buying the Superstore membership, can you tell whether consumer 2 (from part (b)) is also indifferent? (Hint: Given that tastes are homothetic and identical across consumers, what would have to be true about the intersection of the two budgets for the higher income consumer in order for the consumer to also be indifferent between them?)

Answer: Consumer 2 will then definitely buy the membership. This is also illustrated in panel (b) of Graph 7.10 where  $C$  is the optimal bundle on the regular budget and  $D$  is the optimal bundle on the Superstore budget for the higher income consumer. (These optimal bundles lie along rays from the origin going through  $A$  and  $B$  because we are assuming that tastes are homothetic). Because of the different relationship between the two budgets for the lower and higher income consumers (as identified in panel (a)),  $D$  lies on a higher indifference curve than  $C$  — implying that consumer 2 will buy the membership.

- (e) True or False: *Assuming consumers have the same homothetic tastes, there exists a “marginal” consumer with income  $\bar{I}$  such that all consumers with income greater than  $\bar{I}$  will buy the Superstore membership and no consumer with income below  $\bar{I}$  will buy that membership.*

Answer: This is true. Higher income consumers whose two budgets will intersect above the 45 degree line will be better off on the Superstore budget (as illustrated in panel (b)). For analogous reasons, lower income consumers will face that intersection point below the 45 degree line — causing the regular budget to yield an optimum with greater utility than the Superstore budget.

- (f) True or False: *By raising  $c$  and/or  $p_1$ , the Superstore will lose relatively lower income customers and keep high income customers.*

Answer: True. Suppose we begin again with Consumer 1 who is indifferent and whose budget lines are illustrated again in panel (c) of Graph 7.10. An increase in  $c$  will cause the shallower Superstore budget to shift in parallel — causing the two budgets to intersect below the 45 degree line and leaving Consumer 1 better off on the regular budget (where she can still consume at  $A$ ). If  $p_1$  increases in the Superstore, the slope of the Superstore budget becomes steeper — again causing the intersection point to fall below the 45 degree line and leaving Consumer 1 better off at  $A$  under the regular budget. Thus, the marginal consumer will cease shopping at the Superstore if  $c$  or  $p_1$  are increased. Because of the homotheticity assumption, we also know that the new marginal consumer will again have her budgets intersect on the 45 degree line — and we have seen in panel (a) that this intersection point moves up on the regular budget as income increases. If an increase in  $c$  or  $p_1$  have caused the intersection point to slide below the 45 degree line for the original marginal consumer, then an increase in income will cause it to slide back up. Thus, there exists some higher income level at which we will find our new marginal consumer.

- (g) *Suppose you are a Superstore manager and you think your store is overcrowded. You’d like to reduce the number of customers while at the same time increasing the amount each customer purchases. How would you do this?*

Answer: You would want to increase  $c$  — which will raise the income of your marginal consumer and reduce the overall number of consumers with memberships. Then, in order to get your members to shop more, you would lower  $p_1$  — but not so much that membership again goes up by too much. You can see that this is possible by again looking at panel (c) of the graph. By increasing  $c$ , you insure that this marginal consumer will no longer be a member. You can then lower price (which will make the new budget shallower) and keep the marginal consumer from coming back to your store so long as you don’t lower the prices too much.

**B:** *Suppose you manage a Superstore and you are currently charging an annual membership fee of \$50. Since  $x_2$  is denominated in dollar units,  $p_2 = 1$ . Suppose that  $p_1 = 1$  for those shopping outside the Superstore but your store sells  $x_1$  at 0.95. Your statisticians have estimated that your consumers have tastes that can be summarized by the utility function  $u(x_1, x_2) = x_1^{0.15} x_2^{0.85}$ .*

- (a) *What is the annual discretionary income (that could be allocated to purchasing  $x_1$  and  $x_2$ ) of your “marginal” consumer?*

Answer: The marginal consumer is indifferent between buying and not buying the membership. If she does not buy the membership, her budget is  $x_1 + x_2 = I$  — and she would optimize by solving

$$\max_{x_1, x_2} x_1^{0.15} x_2^{0.85} \text{ subject to } x_1 + x_2 = I. \quad (7.28)$$

This gives us  $x_1 = 0.15I$  and  $x_2 = 0.85I$ . Thus, without membership, our consumer gets utility

$$u(0.15I, 0.85I) = (0.15I)^{0.15} (0.85I)^{0.85} \approx 0.655I. \quad (7.29)$$

If she becomes a member at the Superstore, her budget will be  $0.95x_1 + x_2 = I - 50$ . She will then solve the maximization problem

$$\max_{x_1, x_2} x_1^{0.15} x_2^{0.85} \text{ subject to } 0.95x_1 + x_2 = I - 50. \quad (7.30)$$

This gives us  $x_1 = 0.15(I - 50)/0.95 \approx 0.158(I - 50)$  and  $x_2 = 0.85(I - 50)$ . Plugging these into the utility function gives

$$u(0.158(I - 50), 0.85(I - 50)) = (0.158(I - 50))^{0.15} (0.85(I - 50))^{0.85} \approx 0.660(I - 50). \quad (7.31)$$

In order for the consumer to be indifferent, it must then be that the utility under the regular budget equals the utility under the Superstore budget — i.e.  $0.655I = 0.660(I - 50)$ . Solving for  $I$ , we get that the income of the marginal consumer is  $I \approx \$6,600$ . (Without rounding along the way, the figure is \$6,524.)

- (b) *Can you show that consumers with more income than the marginal consumer will definitely purchase the membership while consumers with less income will not? (Hint: Calculate the income of the marginal consumer as a function of  $c$  and show what happens to income that makes a consumer marginal as  $c$  changes.)*

Answer: We can solve the same problems as in B(a) above but now let the membership fee be equal to  $c$  rather than \$50. The optimal bundle under the regular budget does not change; the optimal bundle under the Superstore budget is now  $(0.158(I - c), 0.85(I - c))$ , giving us utility under the Superstore constraint of approximately  $0.66(I - c)$ . Setting that equal to the utility under the regular budget ( $0.655I$ ) and solving for  $I$  we get

$$I \approx 132c. \quad (7.32)$$

Thus, as  $c$  increases, the income of the marginal consumer increases, and as  $c$  decreases, the income of the marginal consumer decreases.

- (c) *If the membership fee is increased from \$50 to \$100, how much could the Superstore lower  $p_1$  without increasing membership beyond what it was when the fee was \$50 and  $p_1$  was 0.95?*

Answer: In order to keep membership constant, the marginal consumer who buys the membership before and after the changes must be the same. The optimal bundle under the regular budget would again be unchanged — i.e.  $x_1 = 0.15I$  and  $x_2 = 0.85I$ , giving utility of  $u \approx 0.655I$ . Letting the new Superstore price be  $p_1$ , a consumer with a membership would solve

$$\max_{x_1, x_2} x_1^{0.15} x_2^{0.85} \text{ subject to } p_1 x_1 + x_2 = I - 100. \quad (7.33)$$

This gives us  $x_1 = 0.15(I - 100)/p_1$  and  $x_2 = 0.85(I - 100)$ . Plugging this into the utility function, we get that

$$u\left(\frac{0.15(I - 100)}{p_1}, 0.85(I - 100)\right) \approx \frac{0.655}{p_1^{0.15}} (I - 100). \quad (7.34)$$

We calculated in B(a) that the marginal consumer when  $p_1 = 0.95$  and  $c = 50$  had income of approximately \$6,600. In order for this consumer to be indifferent once again after the membership fee goes up to \$100, it has to be that her utility without the membership is again equal to her utility with it; i.e.

$$0.655(6600) = \frac{0.655}{p_1^{0.15}} (6600 - 100). \quad (7.35)$$

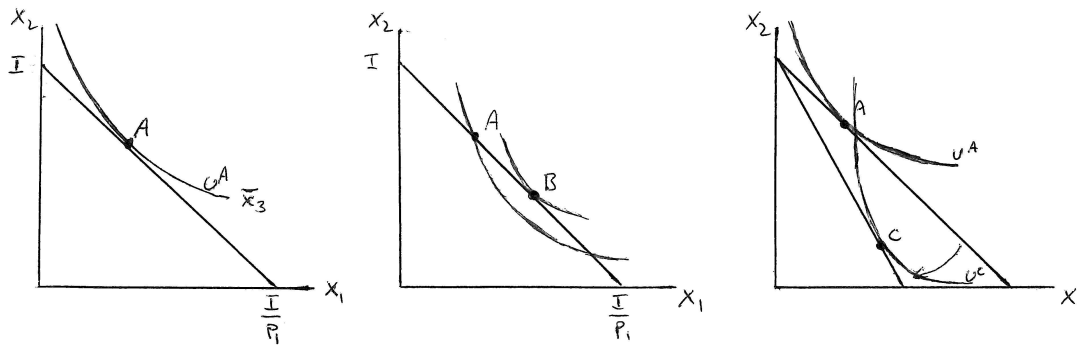
Solving this for  $p_1$ , we get  $p_1 \approx 0.903$ .

**7.9 Business Application: Are Gucci products Giffen Goods?** We defined a Giffen good as a good that consumers (with exogenous incomes) buy more of when the price increases. When students first hear about such goods, they often think of luxury goods such as expensive Gucci purses and accessories. If the marketing departments for firms like Gucci are very successful, they may find a way of associating price with “prestige” in the minds of consumers — and this may allow them to raise the price and sell more products. But would that make Gucci products Giffen goods? The answer, as you will see in this exercise, is no.

**A:** Suppose we model a consumer who cares about the “practical value and style of Gucci products”, dollars of other consumption and the “prestige value” of being seen with Gucci products. Denote these as  $x_1$ ,  $x_2$  and  $x_3$  respectively.

- (a) The consumer only has to buy  $x_1$  and  $x_2$  — the prestige value  $x_3$  comes with the Gucci products. Let  $p_1$  denote the price of Gucci products and  $p_2 = 1$  be the price of dollars of other consumption. Illustrate the consumer's budget constraint (assuming an exogenous income  $I$ ).

Answer: This is just like any typical budget constraint and is illustrated as part of panel (a) of Graph 7.11.



Graph 7.11: Gucci Products and Prestige

- (b) The prestige value of Gucci purchases —  $x_3$  — is something an individual consumer has no control over. If  $x_3$  is fixed at a particular level  $\bar{x}_3$ , the consumer therefore operates on a 2-dimensional slice of her 3-dimensional indifference map over  $x_1$ ,  $x_2$  and  $x_3$ . Draw such a slice for the indifference curve that contains the consumer's optimal bundle A on the budget from part (a).

Answer: The 2-dimensional slice of the indifference map will look exactly like our typical indifference maps over 2 goods. The optimal bundle A is illustrated as the bundle at the tangency of an indifference curve from this slice with the budget constraint from part (a).

- (c) Now suppose that Gucci manages to raise the prestige value of its products — and thus  $x_3$  that comes with the purchase of Gucci products. For now, suppose they do this without changing  $p_1$ . This implies you will shift to a different 2-dimensional slice of your 3-dimensional indifference map. Illustrate the new 2-dimensional indifference curve that contains A. Is the new MRS at A greater or smaller in absolute value than it was before?

Answer: This is illustrated in panel (b) of the graph. The increase in prestige implies the consumer is willing to pay more for any additional Gucci products — thus the MRS increases in absolute value.

- (d) Would the consumer consume more or fewer Gucci products after the increase in prestige value?

Answer: All the bundles that lie above the indifference curve through  $A$  in panel (b) of the graph contain more Gucci products. The consumer will now optimize at some new bundle such as  $B$ .

- (e) Now suppose that Gucci manages to convince consumers that Gucci products become more desirable the more expensive they are. Put differently, the prestige value  $x_3$  is linked to  $p_1$ , the price of the Gucci products. On a new graph, illustrate the change in the consumer's budget as a result of an increase in  $p_1$ .

Answer: This change in the budget is no different than it would usually be — and is illustrated as part of panel (c) of Graph 7.11.

- (f) Suppose that our consumer increases her purchases of Gucci products as a result of the increase in the price  $p_1$ . Illustrate two indifference curves — one that gives rise to the original optimum  $A$  and another that gives rise to the new optimum  $C$ . Can these indifference curves cross?

Answer: This is illustrated in panel (c) of the Graph. Since the indifference curve  $u^C$  is drawn from a different 2-dimensional slice of the 3-dimensional indifference curve over  $x_1, x_2$  and  $x_3$  than the indifference curve  $u^A$ , the two indifference curves can indeed cross.

- (g) Explain why, even though the behavior is consistent with what we would expect if Gucci products were a Giffen good, Gucci products are not a Giffen good in this case.

Answer: Gucci products in this example are really bundles of 2 products — the physical product itself, and the prestige value that comes with the product. When price increases, the prestige value increases — which means we are no longer dealing with the same product as before (even though the physical characteristics of the product remain the same). Thus, while the consumer is indeed buying more Gucci products after the price increase, she is also buying more prestige that is bundled with the physical product. In terms of our 3-dimensional indifference curves, she is shifting to a different  $x_3$  level because  $p_1$  is higher. Holding all else fixed, she would not buy more Gucci products as price increases — it is only because she is buying more prestige at the higher price that it looks like she is buying more as price increases.

- (h) In a footnote in the chapter we defined the following: A good is a Veblen good if preferences for the good change as price increases — with this change in preferences possibly leading to an increase in consumption as price increases. Are Gucci products a Veblen good in this exercise?

Answer: Yes — as price increases, tastes (i.e. the indifference map in 2 dimensions) change in the sense that we are shifting to a different slice of the true 3-D indifference surfaces. The resulting increased consumption of Gucci products as price increases is due to this “change in tastes” — or, to put it more accurately, to the change in the product that looks like a change in tastes when we graph our 2-dimensional indifference curves. This is different from Giffen behavior where the indifference map does not change with an increase in price — but consumption does.

**B:** Consider the same definition of  $x_1$ ,  $x_2$  and  $x_3$  as in part A. Suppose that the tastes for our consumer can be captured by the utility function  $u(x_1, x_2, x_3) = \alpha x_3^2 \ln x_1 + x_2$ .

- (a) Set up the consumer's utility maximization problem — keeping in mind that  $x_3$  is not a choice variable.

Answer: The maximization problem is

$$\max_{x_1, x_2} \alpha x_3^2 \ln x_1 + x_2 \text{ subject to } p_1 x_1 + x_2 = I. \quad (7.36)$$

- (b) Solve for the optimal consumption of  $x_1$  (which will be a function of the prestige value  $x_3$ ).

Answer: The Lagrange function for this problem is

$$\mathcal{L}(x_1, x_2, \lambda) = \alpha x_3^2 \ln x_1 + x_2 + \lambda(I - p_1 x_1 - x_2). \quad (7.37)$$

Solving this the usual way, we get

$$x_1 = \frac{\alpha x_3^2}{p_1} \text{ and } x_2 = I - \alpha x_3^2. \quad (7.38)$$

- (c) Is  $x_1$  normal or inferior? Is it Giffen?

Answer:  $x_1$  does not vary with income — thus making it quasilinear. Put differently,  $x_1$  is borderline between normal and inferior. At the same time,  $x_1$  falls with  $p_1$  — implying that consumers will buy less  $x_1$  as  $p_1$  increases all else being equal. Thus,  $x_1$  is not a Giffen good.

- (d) Now suppose that prestige value is a function of  $p_1$ . In particular, suppose that  $x_3 = p_1$ . Substitute this into your solution for  $x_1$ . Will consumption increase or decrease as  $p_1$  increases?

Answer: This implies that

$$x_1 = \frac{\alpha p_1^2}{p_1} = \alpha p_1. \quad (7.39)$$

Thus, consumption of  $x_1$  increases as  $p_1$  increases.

- (e) How would you explain that  $x_1$  is not a Giffen good despite the fact that its consumption increases as  $p_1$  goes up?

Answer: In order for  $x_1$  to be a Giffen good, consumption of  $x_1$  would have to increase with an increase in  $p_1$  all else remaining equal. We showed in (b) that this is not the case — all else (including prestige) remaining constant, an increase in  $p_1$  leads to a decrease in  $x_1$ . The only reason that  $x_1$  increases as  $p_1$  increases is that we allow  $p_1$  to change the prestige value of Gucci products — and thus the very nature of those products.

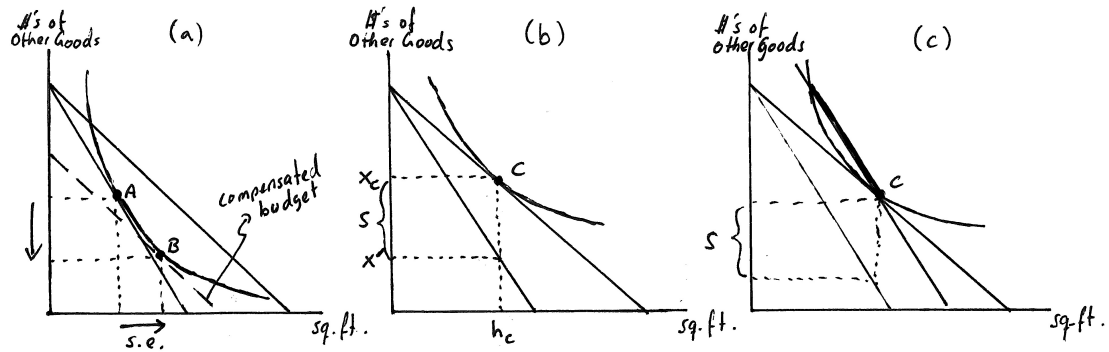


**7.10 Policy Application: Tax Deductibility and Tax Credits:** In end-of-chapter exercise 2.17, you were asked to think about the impact of tax deductibility on a household's budget constraint.

**A:** Suppose we begin in a system in which mortgage interest is not deductible and then tax deductibility of mortgage interest is introduced.

- (a) Using a graph (as you did in exercise 2.17) with "square feet of housing" on the horizontal axis and "dollars of other consumption" on the vertical, illustrate the direction of the substitution effect.

**Answer:** This is illustrated in panel (a) of Graph 7.12. Bundle  $A$  is the original optimal bundle before the introduction of deductibility of mortgage interest. The price of owner occupied housing falls with the implicit subsidy introduced by making mortgage interest deductible, leading to a shallower slope of the budget. Bundle  $B$  is the bundle the consumer would consume if she faced the lower price of housing but had enough income taken away to make her just as happy as she was before. The move from  $A$  to  $B$  is the substitution effect — which (as always) tells us the consumer will consume more of what has become cheaper, less of what has become relatively more expensive.



Graph 7.12: Tax Deductibility of Mortgage Interest

- (b) What kind of good would housing have to be in order for the household to consume less housing as a result of the introduction of the tax deductibility program?

**Answer:** In order to consume less housing, the new optimal bundle would have to be to the left of  $A$  — which would imply that a drop in the price of housing leads to less consumption. That is the definition of a Giffen good.

- (c) On a graph that contains both the before and after deductibility budget constraints, how would you illustrate the amount of subsidy the government provides to this household?

**Answer:** This is illustrated in panel (b) of the graph. In order to know how much of an implicit subsidy the government is providing, we need to know how much housing is consumed *after* the subsidy. (Knowing how much is consumed *before* the subsidy is not helpful since the subsidy is based on the level of consumption when the subsidy is in place.) Thus, we start at bundle  $C$  in panel (b) of the graph. We can then see that the consumer is able to consume  $x_C$  dollars of other goods while consuming  $h_C$  in housing. If the government were not subsidizing housing consumption through the tax code, the consumer would only be able to consume  $x'$  dollars of other goods while consuming  $h_C$  in housing. The difference — labeled  $S$  — is the amount the government is paying by introducing tax deductibility. (Note that it is not necessary to assume that  $h_C$  is how much housing the consumer would have consumed in the absence of tax deductibility in order for this logic to work — i.e. bundle  $A$  can lie anywhere on the lower budget constraint.)

- (d) Suppose the government provided the same amount of money to this household but did so instead by simply giving it to the household as cash back on its taxes (without linking it to housing consumption). Will the household buy more or less housing?

Answer: If the government gave the amount  $S$  as simply cash, the slope of the budget would go back to what it was before tax deductibility — but it would shift up by  $S$ . This means it will intersect  $C$  and therefore will cut the indifference curve that is tangent at  $C$  from above. The darkened portion of the budget line in panel (c) of Graph 7.12 then represents all the bundles that are better than  $C$  — with all these bundles containing less housing. Thus, the household will buy less housing (than under deductibility) if the government gives  $S$  as cash.

- (e) Will the household be better or worse off?

Answer: The household will be better off since it will optimize at a new bundle that lies on a higher indifference curve than  $C$ . This should make sense — if you get money that you can spend in any way you choose, you'll probably be better off than if you get the same amount of money that is tied to particular behavior.

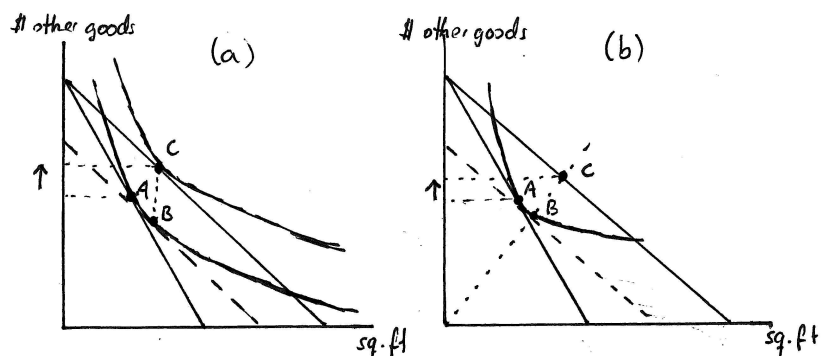
- (f) Do your answer to (d) and (e) depend on whether housing is normal, reg inferior or Giffen?

Answer: No — nothing we said in the answer above depends on what kind of good housing is. This is because this is a case where the change in behavior is “almost” entirely a substitution effect. (In fact it is a pure “Slutsky substitution effect” which we differentiate from the usual (“Hicksian”) substitution effect in exercise 7.11.)

- (g) Under tax deductibility, will the household spend more on other consumption before or after tax deductibility is introduced? Discuss your answer in terms of income and substitution effects and assume that “other goods” is a normal good.

Answer: Going back to panel (a) of Graph 7.12, the fact that “other goods” is normal means that consumption of other goods will go up with income. Income increases from the compensated budget to the final budget (i.e. between the two budgets whose slopes are the same). Since  $B$  is optimal on the compensated budget,  $C$  must happen on the final budget with higher “other good” consumption than at  $B$ . But that leaves open the possibility that  $C$  contains less “other goods” consumption than  $A$ . Thus, knowing that “other goods consumption” is normal does not tell us enough to determine whether “other consumption” will increase or decrease with tax deductibility of mortgage interest. If the substitution effect is small (and  $A$  lies close to  $B$ ), then it is likely that other good consumption will increase. But if the substitution effect is large (and  $A$  lies far from  $B$ ), then the reverse is likely.

- (h) If you observed that a household consumes more in “other goods” after the introduction of tax deductibility, could that household's tastes be quasilinear in housing? Homothetic?



Graph 7.13: Tax Deductibility of Mortgage Interest: Part II

Answer: Yes to both. This is illustrated in panels (a) and (b) of Graph 7.13 (previous page). The key for the result to hold is that the substitution effect is small — i.e. that the indifference curves make the two goods relatively complementary, at least at A.

**B:** Households typically spend about a quarter of their after-tax income  $I$  on housing. Let  $x_1$  denote square feet of housing and let  $x_2$  denote other consumption.

- (a) If we represent a household's tastes with the Cobb-Douglas function  $u(x_1, x_2) = x_1^\alpha x_2^{(1-\alpha)}$ , what should  $\alpha$  be?

Answer: Solving the optimization problem

$$\max_{x_1, x_2} x_1^\alpha x_2^{(1-\alpha)} \text{ subject to } p_1 x_1 + x_2 = I, \quad (7.40)$$

we get

$$x_1 = \frac{\alpha I}{p_1} \text{ and } x_2 = (1 - \alpha)I. \quad (7.41)$$

Thus, in order for a household with Cobb-Douglas tastes to spend a quarter of its income on housing, it must be that  $\alpha$ , the exponent on housing, is 0.25. (This is generally true for Cobb-Douglas tastes: When the exponents in the Cobb-Douglas utility function sum to 1, they represent the share of the budget that a consumer will optimally allocate to the different goods.)

- (b) Using your answer about the value of  $\alpha$ , and letting the price per square foot of housing be denoted as  $p_1$ , derive the optimal level of housing consumption (in terms of  $I$ ,  $p_1$ , and  $t$ ) under a tax deductibility program that implicitly subsidizes a fraction  $t$  of a household's housing purchase.

Answer: We would solve

$$\max_{x_1, x_2} x_1^{0.25} x_2^{0.75} \text{ subject to } (1 - t)p_1 x_1 + x_2 = I, \quad (7.42)$$

to get

$$x_1 = \frac{0.25I}{(1 - t)p_1} \text{ and } x_2 = 0.75I. \quad (7.43)$$

- (c) What happens to housing consumption and other good consumption under tax deductibility as a household's tax bracket (i.e. their tax rate  $t$ ) increases?

Answer: We can see that housing consumption increases as  $t$  increases by simply taking the derivative of  $x_1$  from equation (7.43); i.e.

$$\frac{\partial x_1}{\partial t} = \frac{0.25I}{(1 - t)^2 p_1} > 0. \quad (7.44)$$

Consumption of other goods is unaffected by  $t$  — and thus would remain unchanged as  $t$  increases.

- (d) Determine the portion of changed housing consumption that is due to the income effect and the portion that is due to the substitution effect.

Answer: The utility before tax deductibility is

$$u\left(\frac{0.25I}{p_1}, 0.75I\right) = \left(\frac{0.25I}{p_1}\right)^{0.25} (0.75I)^{0.75} \approx \frac{0.57I}{p_1^{0.25}}. \quad (7.45)$$

To determine how much in housing would have been consumed at the new price  $(1 - t)p_1$  if enough income was taken away to keep utility the same, we solve

$$\min_{x_1, x_2} (1 - t)p_1 x_1 + x_2 \text{ subject to } x_1^{0.25} x_2^{0.75} = \frac{0.57I}{p_1^{0.25}}. \quad (7.46)$$

The first two first order conditions can be used to solve for  $x_2 = 3(1 - t)p_1 x_1$ . Substituting this into the constraint, we can then (with a bit of algebra) solve to get

$$x_1 = \frac{0.25I}{(1-t)^{0.75}p_1}. \quad (7.47)$$

Thus, we begin at  $x_1 = 0.25I/p_1$  before tax deductibility is introduced. The substitution effect of tax deductibility then increases housing consumption to  $0.25I/[(1-t)^{0.75}p_1]$ , and the income effect increases it further to  $0.25I/[(1-t)p_1]$ .

- (e) Calculate the amount of money the government is spending on subsidizing this household's mortgage interest.

Answer: We have calculated that, under tax deductibility, the household will consume  $x_1 = 0.25I/[(1-t)p_1]$  and  $x_2 = 0.75I$ . This is equivalent to our bundle  $C$  in the graphs. Had this household tried to consume  $x_1 = 0.25I/[(1-t)p_1]$  in the absence of tax deductibility, it would have cost  $p_1x_1 = 0.25I/(1-t)$  and would therefore have left only

$$x_2 = I - \frac{0.25I}{(1-t)} = \frac{I-tI}{(1-t)} - \frac{0.25I}{(1-t)} = \frac{0.75I-tI}{(1-t)} \quad (7.48)$$

for other consumption. Since other consumption is equal to  $0.75I$  under deductibility, the government's implicit subsidy is

$$0.75I - \frac{0.75I-tI}{(1-t)} = \frac{0.75I-0.75tI}{(1-t)} - \frac{0.75I-tI}{(1-t)} = \frac{0.25tI}{(1-t)}. \quad (7.49)$$

- (f) Now suppose that, instead of a deductibility program, the government simply gives the amount you calculated in (e) to the household as cash. Calculate the amount of housing now consumed and compare it to your answer under tax deductibility.

Answer: If the government gives  $0.25tI/(1-t)$  in cash, then income increases from  $I$  to

$$I + \frac{0.25tI}{(1-t)} = \frac{I-tI}{(1-t)} + \frac{0.25tI}{(1-t)} = \frac{(1-0.75t)I}{(1-t)}. \quad (7.50)$$

We know that when income is  $I$ , housing consumption is  $0.25I/p_1$  in the absence of deductibility. Plugging the new income level in for  $I$ , we then get that housing consumption under the cash subsidy is

$$x_1 = \frac{0.25(1-0.75t)I}{(1-t)p_1}. \quad (7.51)$$

This can also be written as

$$x_1 = \frac{0.25I}{(1-t)p_1} - \frac{0.25(0.75)tI}{(1-t)p_1} = \frac{0.25I}{(1-t)p_1} - \frac{3tI}{16(1-t)p_1}. \quad (7.52)$$

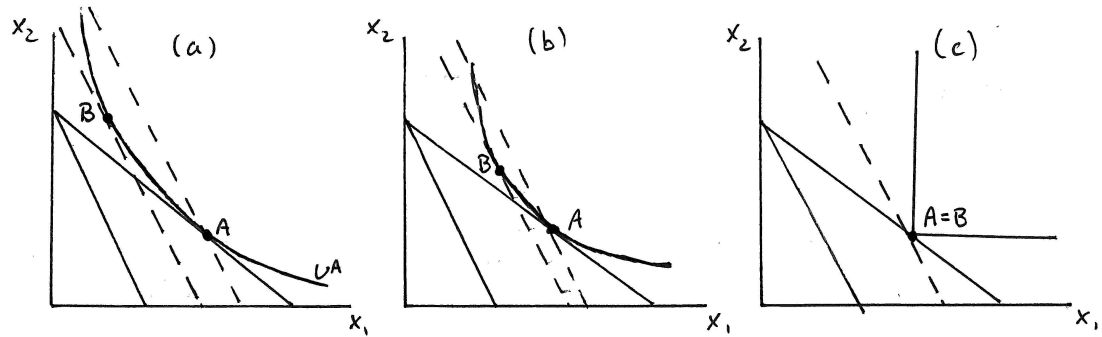
The first term  $0.25I/(1-t)p_1$  is what we derived as the housing consumption under tax deductibility. Thus, when the government gives the same amount of subsidy as cash, housing consumption increases less by  $3tI/[16(1-t)p_1]$ .

**7.11 Policy Application: Substitution Effects and Social Security Cost of Living Adjustments:** In end-of-chapter exercise 6.16, you investigated the government's practice for adjusting social security income for seniors by insuring that the average senior can always afford to buy some average bundle of goods that remains fixed. To simplify the analysis, let us again assume that the average senior consumes only two different goods.

**A:** Suppose that last year our average senior optimized at the average bundle  $A$  identified by the government, and begin by assuming that we denominate the units of  $x_1$  and  $x_2$  such that last year  $p_1 = p_2 = 1$ .

(a) Suppose that  $p_1$  increases. On a graph with  $x_1$  on the horizontal and  $x_2$  on the vertical axis, illustrate the compensated budget and the bundle  $B$  that, given your senior's tastes, would keep the senior just as well off at the new price.

**Answer:** In panel (a) of Graph 7.14, bundle  $A$  lies on the original (solid line) budget. The price increase causes an inward rotation of that budget in the absence of compensation. To compensate the person so that he will be as happy as before, we have to raise income to the lower dashed line in the graph — the line that is tangent to  $B$  that lies on the indifference curve  $u^A$ .



Graph 7.14: Hicks and Slutsky Social Security Compensation

(b) In your graph, compare the level of income the senior requires to get to bundle  $B$  to the income required to get him back to bundle  $A$ .

**Answer:** The income required (at the new prices) to get to  $A$  is represented by the second dashed line in panel (a) of the graph.

(c) What determines the size of the difference in the income necessary to keep the senior just as well off when the price of good 1 increases as opposed to the income necessary for the senior to still be able to afford bundle  $A$ ?

**Answer:** The greater the substitutability of the two goods, the greater will be the difference between the two ways of compensating the person. This is illustrated across the three panels in Graph 7.14 where the degree of substitutability falls from left to right.

(d) Under what condition will the two forms of compensation be identical to one another?

**Answer:** The difference between the two compensation schemes disappears entirely in panel (c) of the graph when there is no substitutability between the goods (i.e. when they are perfect complements).

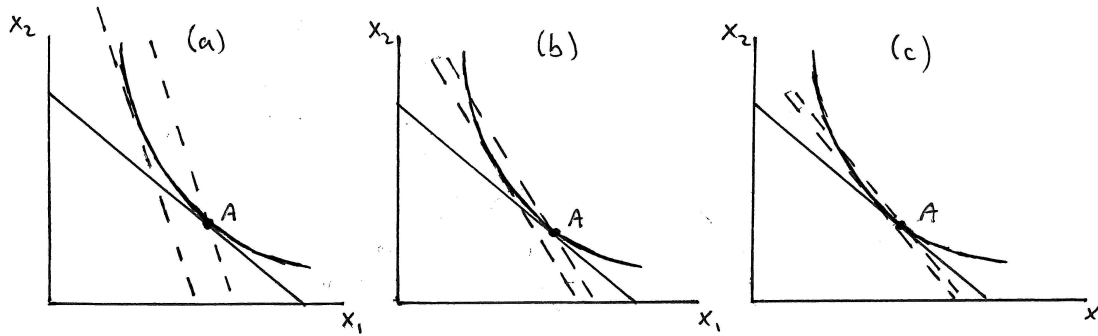
(e) You should recognize the move from  $A$  to  $B$  as a pure substitution effect as we have defined it in this chapter. Often this substitution effect is referred to as the Hicksian substitution effect — defined as the change in behavior when opportunity costs change but the consumer receives sufficient compensation to remain just as happy. Let  $B'$  be the consumption bundle the average senior would choose when compensated so as to be able to afford the original bundle

A. The movement from A to B' is often called the Slutsky substitution effect — defined as the change in behavior when opportunity costs change but the consumer receives sufficient compensation to be able to afford to stay at the original consumption bundle. True or False: The government could save money by using Hicksian rather than Slutsky substitution principles to determine appropriate cost of living adjustments for social security recipients.

Answer: The answer is true. The government in fact uses Slutsky compensation as it calculates cost of living adjustments — because it fixes a particular consumption bundle and then adjusts social security checks to make sure that seniors can still afford that bundle. For this reason, you will frequently hear proposals to adjust the way in which cost of living adjustments are calculated — with these proposals attempting to get closer to Hicksian compensation.

(f) True or False: Hicksian and Slutsky compensation get closer to one another the smaller the price changes.

Answer: This is true. Larger price changes result in larger substitution effects — and the difference between Hicksian and Slutsky substitution is entirely due to the substitution effect. This is illustrated in the three panels of Graph 7.15 where, going from left to right, the size of the price change (as evidenced in the steepness of the slope of the compensated budget) decreases.



Graph 7.15: Hicks and Slutsky Social Security Compensation: Part II

**B:** Now suppose that the tastes of the average senior can be captured by the Cobb-Douglas utility function  $u(x_1, x_2) = x_1 x_2$ , where  $x_2$  is a composite good (with price by definition equal to  $p_2 = 1$ ). Suppose the average senior currently receives social security income  $I$  (and no other income) and with it purchases bundle  $(x_1^A, x_2^A)$ .

(a) Determine  $(x_1^A, x_2^A)$  in terms of  $I$  and  $p_1$ .

Answer: Solving the usual maximization problem with budget constraint  $p_1 x_1 + x_2 = I$ , we get

$$x_1^A = \frac{I}{2p_1} \text{ and } x_2^A = \frac{I}{2}. \quad (7.53)$$

(b) Suppose that  $p_1$  is currently \$1 and  $I$  is currently \$2000. Then  $p_1$  increases to \$2. How much will the government increase the social security check given how it is actually calculating cost of living adjustments? How will this change the senior's behavior?

Answer: The government compensates so as to make it possible for the senior to keep affording the same bundle as before. With the values  $p_1 = 1$  and  $I = 2000$ ,  $x_1^A = x_2^A = 1000$ . When the price of  $x_1$  goes to \$2, this same bundle costs  $2(1000) + 1000 = \$3,000$ . Thus, the government is compensating the senior by increasing the social security check by \$1,000.

With an income of \$3,000, equations (7.53) then tell us that the senior will consume  $x_1 = 3000/(2(2)) = 750$  and  $x_2 = 3000/2 = 1,500$ . Thus, even though the government makes it possible for the senior to consume bundle  $A$  again after the price change, the senior will substitute away from  $x_1$  because its opportunity cost is now higher.

- (c) *How much would the government increase the social security check if it used Hicksian rather than Slutsky compensation? How would the senior's behavior change?*

Answer: If the government used Hicksian compensation, it would first need to calculate the bundle  $B$  on the original indifference curve that would make the senior just as well off at the higher price as he was at  $A$ . At  $A$ , the senior gets utility  $u^A = x_1^A x_2^A = 1000(1000) = 1,000,000$ . The government would then have to solve the problem

$$\min_{x_1, x_2} 2x_1 + x_2 \text{ subject to } x_1 x_2 = 1,000,000. \quad (7.54)$$

Solving the first two first order conditions, we get  $x_2 = 2x_1$ . Substituting this into the constraint and solving for  $x_1$ , we get  $x_1 \approx 707.1$ , and plugging this back into  $x_2 = 2x_1$ , we get  $x_2 = 1414.2$ . This bundle  $B = (707.1, 1414.2)$  costs  $2(707.1) + 1414.2 = 2828.4$ . Thus, under Hicksian compensation, the government would increase the senior's social security check by \$828.40 rather than \$1,000.

- (d) *Can you demonstrate mathematically that Hicksian and Slutsky compensation converge to one another as the price change gets small — and diverge from each other as the price change gets large?*

Answer: We start with  $p_1 = 1$  (and continue to assume  $p_2 = 1$ ).<sup>2</sup> Then suppose  $p_1$  increases to  $p_1 > 1$  (or falls to  $p_1 < 1$ ). Slutsky compensation requires that we continue to be able to purchase  $A = (1000, 1000)$  — so we have to make sure the senior has income of  $1000p_1 + 1000$ . Since the senior starts with an income of \$2,000, this implies that Slutsky compensation is  $1000p_1 + 1000 - 2000 = 1000p_1 - 1000 = 1000(p_1 - 1)$ .

Hicksian compensation, on the other hand, requires we calculate the substitution effect to  $B$  as we did in the previous part for  $p_1 = 2$ . Setting up the same problem but letting the new price of good 1 be denoted  $p_1$  rather than 2, we can calculate  $B = (x_1^B, x_2^B) = (1000/p_1^{0.5}, 1000p_1^{0.5})$ . This bundle costs

$$p_1 \frac{1000}{p_1^{0.5}} + 1000p_1^{0.5} = 2000p_1^{0.5}. \quad (7.55)$$

Given that the senior starts with \$2000, this means that Hicksian compensation must be equal to  $2000p_1^{0.5} - 2000 = 2000(p_1^{0.5} - 1)$ .

The difference between Slutsky compensation and Hicksian compensation, which we will call  $D(p_1)$  is then

$$\begin{aligned} D(p_1) &= 1000(p_1 - 1) - 2000(p_1^{0.5} - 1) = 1000p_1 - 1000 - 2000p_1^{0.5} + 2000 \\ &= 1000 + 1000p_1(1 - 2p_1^{-0.5}). \end{aligned} \quad (7.56)$$

As  $p_1$  approaches 1, the second term in the equation goes to  $-1000$  — making the expression go to zero; i.e. the difference between the two types of compensation goes to zero as the price increase (or decrease) gets small. In fact, it is easy to see that this difference reaches its lowest point when  $p_1 = 1$  and increases when  $p_1$  rises above 1 as well as when  $p_1$  falls below 1: Simply take the derivative of  $D(p_1)$  which is

$$\frac{dD(p_1)}{dp_1} = 1000(1 - 2p_1^{-0.5}) + 1000p_1(p_1^{-1.5}) = 1000(1 - p_1^{-0.5}). \quad (7.57)$$

Then note that  $dD/dp_1 < 0$  when  $0 < p_1 < 1$ ,  $dD/dp_1 = 0$  when  $p_1 = 1$  and  $dD/dp_1 > 0$  when  $p_1 > 1$ . This implies a U-shape for  $D(p_1)$ , with the U reaching its bottom at  $p_1 = 1$  when  $D(p_1) = 0$ . Put into words, the difference between Slutsky and Hicks compensation is positive for any price not equal to the original price, with the difference increasing the greater the deviation in price from the original price.

<sup>2</sup>We could start with any other price and change either  $p_1$  or  $p_2$  and the same logic will hold.

- (e) *We know that Cobb-Douglas utility functions are part of the CES family of utility functions — with the elasticity of substitution equal to 1. Without doing any math, can you estimate, for an increase in  $p_1$  above 1, the range of how much Slutsky compensation can exceed Hicksian compensation with tastes that lie within the CES family? (Hint: Consider the extreme cases of elasticities of substitution.)*

Answer: We know that if the two goods are perfect complements (with elasticity of substitution equal to 0), then there is no difference between the two compensation mechanisms (because, as we demonstrated in part A of the question, the difference is due entirely to the substitution effect). Thus, one end of the range of how much Slutsky compensation can exceed Hicksian compensation is zero.

The other extreme is the case of perfect substitutes. In that case, it is rational for the consumer to choose bundle  $A$  initially since the prices are identical and the indifference curve therefore lies on top of the budget line (making all bundles on the budget line optimal). But any deviation in price will result in a corner solution. Thus, if  $p_1$  increases, the consumer can remain just as well off as she was originally by simply not consuming  $x_2$ . Thus, Hicksian compensation is zero while Slutsky compensation still aims to make bundle  $A$  affordable — i.e. Slutsky compensation is still  $1000(p_1 - 1)$  as we calculated in part (d). So in this extreme case, Slutsky compensation exceeds Hicksian compensation by  $1000(p_1 - 1)$ .

Depending on the elasticity of substitution, Slutsky compensation may therefore exceed Hicksian compensation by as little as 0 (when the elasticity is 0) to as much as  $1000(p_1 - 1)$  (when the elasticity is infinite).

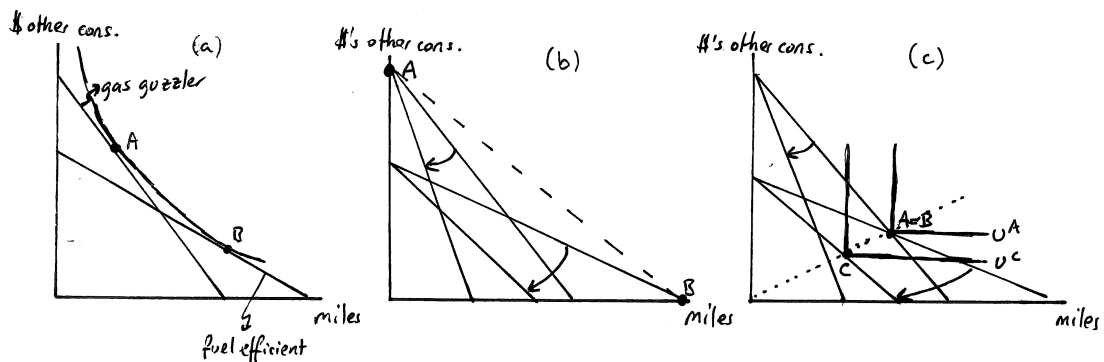


**7.12 Policy Application: Fuel Efficiency, Gasoline Consumption and Gas Prices:** Policy makers frequently search for ways to reduce consumption of gasoline. One straightforward option is to tax gasoline — thereby encouraging consumers to drive less in their cars and switch to more fuel efficient cars.

**A:** Suppose that you have tastes for driving and for other consumption, and assume throughout that your tastes are homothetic.

- (a) On a graph with monthly miles driven on the horizontal and “monthly other consumption” on the vertical axis, illustrate two budget lines: One in which you own a gas-guzzling car — which has a low monthly payment (that has to be made regardless of how much the car is driven) but high gasoline use per mile; the other in which you own a fuel efficient car — which has a high monthly payment that has to be made regardless of how much the car is driven but uses less gasoline per mile. Draw this in such a way that it is possible for you to be indifferent between owning the gas-guzzling and the fuel-efficient car.

**Answer:** These are illustrated in panel (a) of Graph 7.16 (on the next page) where the gas guzzler budget has a steeper slope (because of the higher opportunity cost of driving from the greater gasoline use) and higher intercept (because of the lower monthly payments for the car.)



Graph 7.16: Cars, Gasoline Taxes and Fuel Efficiency

- (b) Suppose you are indeed indifferent. With which car will you drive more?

**Answer:** Your optimal consumption bundle is A under the gas guzzler budget and B under the fuel efficient budget — and because of the implicit substitution effect, you will drive more in the fuel efficient car.

- (c) Can you tell with which car you will use more gasoline? What does your answer depend on?

**Answer:** No, you cannot tell. In the fuel efficient car, you will use less gasoline per mile but you will also drive more miles. Whether or not you use more gasoline in the fuel efficient car or the gas guzzler depends on which effect dominates. This depends on how far apart A and B are — i.e. on how large the substitution effect is. If driving and other consumption are sufficiently substitutable, then you will use more gasoline when you drive the fuel efficient car; if, on the other hand, driving and other goods are sufficiently complementary, you will use less gasoline in the fuel efficient car.

- (d) Now suppose that the government imposes a tax on gasoline — and this doubles the opportunity cost of driving both types of cars. If you were indifferent before the tax was imposed, can you now say whether you will definitively buy one car or the other (assuming you waited to buy a car until after the tax is imposed)? What does your answer depend on? (Hint: It may be helpful to consider the extreme cases of perfect substitutes and perfect complements before deriving your general conclusion to this question.)

Answer: As suggested in the hint, the answer is easiest to see if you start by looking at extremes. In panel (b) of the graph, we assume that driving and other goods are perfect substitutes. In order for someone to be indifferent between the two budgets, the optimal bundles must lie on opposite corners of the two budgets. (The indifference curve is illustrated as the dashed line connecting  $A$  and  $B$ ). If the price of gasoline now increases due to the tax,  $A$  remains possible but  $B$  does not — which means that now the person would choose  $A$ . In other words, conditional on buying one of the two types of cars, the person would choose the gas guzzler. The example is a bit artificial in the sense that someone who will not end up driving at all would presumably not buy a car at all — but you can see how the logic also holds for tastes that are close to perfect substitutes where the consumer would choose interior solutions.

In panel (c) of the graph, we go to the other extreme — with tastes over miles and other consumption modeled as perfect complements. In that case,  $A=B$  to begin with — and when gasoline prices go up, the fuel efficient car becomes unambiguously better (with the optimum at  $C$  and all bundles on the after-tax gas guzzling budget falling below  $u^C$ .)

Upon reflection, this should make intuitive sense. If miles and other consumption are relatively complementary, then it makes sense to switch to a more fuel efficient car because we want to keep driving quite a bit even if the price of gasoline increases. If, on the other hand, miles and other consumption are relatively substitutable, then one way to respond to a price increase is to substitute away from gasoline altogether and just drive very little. With only a little driving each month, it's better to pay the lower fixed cost of the gas guzzler even if each mile costs more.

- (e) *The empirical evidence suggests that consumers shift toward more fuel efficient cars when the price of gasoline increases. True or False: This would tend to suggest that driving and other good consumption are relatively complementary.*

Answer: True — based on the explanation to part (d) above.

- (f) *Suppose an increase in gasoline taxes raises the opportunity cost of driving a mile with a fuel efficient car to the opportunity cost of driving a gas guzzler before the tax increase. Would someone with homothetic tastes drive more or less in the fuel efficient car after the tax increase than she would in a gas guzzler prior to the tax increase?*

Answer: The person would drive less in the fuel efficient car after the tax increase than in the gas guzzler before the tax increase. You can illustrate this simply by drawing the gas guzzler budget before the tax and the fuel efficient budget after the tax. You should get both budgets to have the same slope (because of the same opportunity cost of driving) — but the fuel efficient car has lower intercept because of the higher monthly payments. This is then a pure income effect — with the new optimal bundle on the after-tax fuel efficient budget lying on the same ray from the origin as the original optimal bundle before-tax gas guzzler budget. The new bundle then necessarily lies to the left of the original.

**B:** *Suppose your tastes were captured by the utility function  $u(x_1, x_2) = x_1^{0.5} x_2^{0.5}$ , where  $x_1$  stands for miles driven and  $x_2$  stands for other consumption. Suppose you have \$600 per month of discretionary income to devote to your transportation and other consumption needs and that the monthly payment on a gas-guzzler is \$200. Furthermore, suppose the initial price of gasoline is \$0.10 per mile in the fuel efficient car and \$0.20 per mile in the gas-guzzler.*

- (a) *Calculate the number of monthly miles driven if you own a gas-guzzler.*

Answer: With a \$200 monthly payment, the remaining discretionary income is \$400 — leaving us with a budget constraint of  $0.2x_1 + x_2 = 400$ . Solving the optimization problem

$$\max_{x_1, x_2} x_1^{0.5} x_2^{0.5} \text{ subject to } 400 = 0.2x_1 + x_2, \quad (7.58)$$

we get  $x_1 = 1000$  and  $x_2 = 200$ . Thus, you would drive 1000 miles per month.

- (b) *Suppose you are indifferent between the gas-guzzler and the fuel efficient car. How much must the monthly payment for the fuel efficient car be?*

Answer: The utility you receive with the gas guzzler is  $1000^{0.5} 200^{0.5} \approx 447.2$ . To be indifferent between the gas guzzler and the fuel efficient car, we first calculate the bundle that you

must be consuming if all your income after paying the higher monthly payment is spent; i.e. we solve

$$\min_{x_1, x_2} 0.1x_1 + x_2 \text{ subject to } x_1^{0.5} x_2^{0.5} = 447.2. \quad (7.59)$$

Solving the first two first order conditions, we get  $x_2 = 0.1x_1$ . Plugging this into the constraint and solving for  $x_1$ , we get  $x_1 \approx 1414.21$ , and plugging this back into  $x_2 = 0.1x_1$  we get  $x_2 \approx 141.42$ . Thus, what we usually call bundle  $B$  is  $(1414.21, 141.42)$ . This bundle costs  $0.1(1414.21) + 141.42 = 282.84$ . Since we start with an income of \$600, this implies that the monthly payment for the fuel efficient car is  $\$600 - \$282.84 = \$317.16$ .

- (c) *Now suppose that the government imposes a tax on gasoline that doubles the price per mile driven of each of the two cars. Calculate the optimal consumption bundle under each of the new budget constraints.*

Answer: These will be  $(x_1, x_2) = (500, 200)$  with the gas guzzler and  $(x_1, x_2) = (707.11, 141.42)$  with the fuel efficient car. (You can solve for this by setting up the optimization problem again. Alternatively, you can recognize that, with Cobb-Douglas tastes, consumption of each good is independent of the price of the other. Thus, consumption of  $x_2$  remains as before since the price of  $x_2$  has not changed — which means the remainder is spent on  $x_1$ . Thus, miles driven falls by half for each type of car.)

- (d) *Do you now switch to the fuel efficient car?*

Answer: Given that we just calculated the optimal bundle for each car type, we can calculate the utility you will get under each budget. Plugging  $(500, 200)$  in the utility function, we get  $u = 316.23$  under the gas guzzling budget; and plugging in  $(707.11, 141.42)$ , we also get  $u = 316.23$ . Thus, you will still be indifferent between the two car types.

- (e) *Consider the utility function you have worked with so far as a special case of the CES family  $u(x_1, x_2) = (0.5x_1^{-\rho} + 0.5x_2^{-\rho})^{-1/\rho}$ . Given what you concluded in A(d) of this question, how would your answer to B(d) change as  $\rho$  changes?*

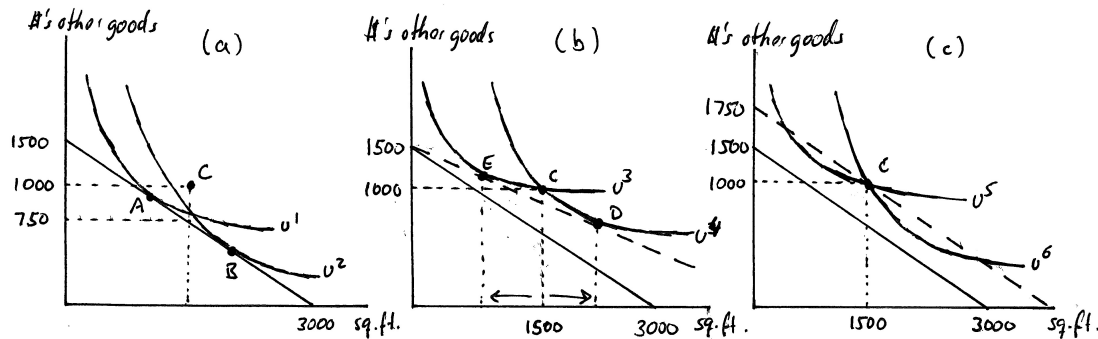
Answer: We concluded in part A that we are more likely to switch to the fuel efficient car the more complementary “miles driven” ( $x_1$ ) is to “other consumption” ( $x_2$ ) — and more likely to switch to the gas guzzler the more substitutable these goods are. As it turns out, our calculations here have demonstrated that Cobb-Douglas tastes (with  $\rho = 0$  and elasticity of substitution equal to 1) encompass exactly that amount of substitutability that will keep us indifferent between the gas guzzler and the fuel efficient car as the price of gasoline increases. For elasticities of substitution greater than 1 — i.e. for  $-1 \leq \rho < 0$  — we would switch to the gas guzzler; for elasticities less than 1 — i.e. for  $\rho > 0$ , we would switch to the fuel efficient car.

**7.13 Policy Application: Public Housing and Housing Subsidies:** In exercise 2.14, you considered two different public housing programs in parts A(a) and (b) — one where a family is simply offered a particular apartment for a below-market rent and another where the government provides a housing price subsidy that the family can use anywhere in the private rental market.

**A:** Suppose we consider a family that earns \$1500 per month and either pays 50 cents per square foot in monthly rent for an apartment in the private market or accepts a 1500 square foot government public housing unit at the government's price of \$500 per month.

- (a) On a graph with square feet of housing and "dollars of other consumption", illustrate two cases where the family accepts the public housing unit — one where this leads them to consume less housing than they otherwise would, another where it leads them to consume more housing than they otherwise would.

**Answer:** The budget constraint in the absence of public housing is drawn in panel (a) of Graph 7.17. Bundle A is optimal under tastes with indifference curve  $u^1$  while bundle B is optimal under tastes with indifference curve  $u^2$ . (Since these indifference curves cross, they of course cannot come from the same indifference map — and thus come from different indifference maps representing different tastes.) The public housing unit permits the household to consume C — the 1500 square foot public housing unit costing \$500 (and thus leaving the household with \$1000 of other consumption). Both the household that optimizes at A and the one that optimizes at B in the absence of the public housing option will choose C if it becomes available. For household 1 this implies that public housing increases its housing consumption, but for household 2 it implies that public housing decreases housing consumption.



Graph 7.17: Public Housing and Rental Subsidies

- (b) If we use the household's own judgment about its well-being, is it always the case that the option of public housing makes the households who choose to participate better off?

**Answer:** Yes — the household would not choose the option unless it thought it is better off. In panel (a) of the graph, both households end up on higher indifference curves when choosing C.

- (c) If the policy goal behind public housing is to increase the housing consumption of the poor, is it more or less likely to succeed the less substitutable housing and other goods are?

**Answer:** The less substitutable housing and other goods are, the sharper the tangency at the optimum on the original budget line. And the sharper the tangency, the less likely it is that a household can consume more than 1500 square feet of housing in the absence of public housing and still become better off at C in our graph. For instance, in panel (a) of the graph it is possible for A to be optimal and C to be better even if housing and other goods are perfect complements — but this is not true for B.

- (d) *What is the government's opportunity cost of owning a public housing unit of 1500 square feet? How much does it therefore cost the government to provide the public housing unit to this family?*

Answer: The government could charge the market price of \$0.50 per square foot for the 1500 square foot public housing unit. It is therefore giving up \$750 in rent by not renting it on the open market — and it is collecting only \$500 from the public housing participant. Thus, the cost the government incurs is \$250 per month. You can also see this in panel (a) of our graph — as the vertical difference between  $C$  and the budget line.

- (e) *Now consider instead a housing price subsidy under which the government tells qualified families that it will pay some fraction of their rental bills in the private housing market. If this rental subsidy is set so as to make the household just as well off as it was under public housing, will it lead to more or less consumption of housing than if the household chooses public housing?*

Answer: Panel (b) of Graph 7.17 illustrates that such a subsidy could lead to more or less consumption of housing.

- (f) *Will giving such a rental subsidy cost more or less than providing the public housing unit? What does your answer depend on?*

Answer: It may cost more or less. If the household consumes less housing under the rental subsidy (as with indifference curve  $u^3$ ), it will definitely cost less. (In the graph, the cost is the vertical difference between  $E$  and the original budget constraint — which must be smaller than the \$250 difference between  $C$  and the original constraint.) But if the rental subsidy results in more housing consumption than public housing (as with indifference curve  $u^D$ ), it may cost the government more or less depending on just how much more housing is consumed.

- (g) *Suppose instead that the government simply gave cash to the household. If it gave sufficient cash to make the household as well off as it is under the public housing program, would it cost the government more or less than \$250? Can you tell whether under such a cash subsidy the household consumes more or less housing than under public housing?*

Answer: It will definitely cost the government less (or at least no more) but we can't tell whether it will result in greater or lesser housing consumption. This is illustrated in panel (c) of Graph 7.17 where the dashed budget line results from the government giving \$250 in cash — the same as it spends under the public housing program. Unless the slope of the indifference curve at  $C$  just happens to be the same as the slope of the budget line, the new budget line will cut the indifference curve that contains  $C$  either from above (as in  $u^5$ ) or from below (as in  $u^6$ ). Either way, the household would be able to make itself better off by reaching a higher indifference curve. Thus, except for the special case where the budget line has the same slope as the indifference curve at  $C$ , it will cost the government less than \$250 to make the household as well off as it is under public housing. Put differently, there are smaller budgets with the same slope that are tangent to  $u^5$  and  $u^6$ . But at those tangencies, housing consumption will fall below 1500 square feet in the case of  $u^5$  and rise above 1500 in the case of  $u^6$ .

**B:** Suppose that household tastes over square feet of housing ( $x_1$ ) and dollars of other consumption ( $x_2$ ) can be represented by  $u(x_1, x_2) = \alpha \ln x_1 + (1 - \alpha) \ln x_2$ .

- (a) *Suppose that empirical studies show that we spend about a quarter of our income on housing. What does that imply about  $\alpha$ ?*

Answer: These are Cobb-Douglas tastes (equivalent to  $u(x_1, x_2) = x_1^\alpha x_2^{(1-\alpha)}$  which, when transformed by the natural log, turns into the one given in the problem). When the exponents of a Cobb-Douglas utility function sum to 1, the exponents denote the fraction of income spent on each good. Thus, if households spend a quarter of their income on housing, then  $\alpha = 0.25$ .

- (b) *Consider a family with income of \$1,500 per month facing a per square foot price of  $p_1 = 0.50$ . For what value of  $\alpha$  would the family not change its housing consumption when offered the 1500 square foot public housing apartment for \$500?*

Answer: 1500 square feet cost \$750 — which is half of the household's income of \$1,500. Given what we said about exponents in Cobb-Douglas utility functions representing budget shares,  $\alpha$  would have to be 0.5 in order for the household to spend half its income on housing.

- (c) Suppose that this family has  $\alpha$  as derived in B(a). How much of a rental price subsidy would the government have to give to this family in order to make it as well off as the family is with the public housing unit?

Answer: With the public housing unit, the family consumes the bundle (1500, 1000) — which gives utility

$$u(1500, 1000) = 0.25 \ln 1500 + 0.75 \ln 1000 \approx 7.009. \quad (7.60)$$

If you solve the maximization problem

$$\max_{x_1, x_2} 0.25 \ln x_1 + 0.75 \ln x_2 \quad \text{subject to} \quad p_1 x_1 + x_2 = 1500, \quad (7.61)$$

you get

$$x_1 = \frac{0.25(1500)}{p_1} = \frac{375}{p_1} \quad \text{and} \quad x_2 = 0.75(1500) = 1125. \quad (7.62)$$

Plugging these back into the utility function, we get

$$\begin{aligned} u\left(\frac{375}{p_1}, 1125\right) &= 0.25 \ln \frac{375}{p_1} + 0.75 \ln 1125 \\ &= 0.25 \ln 375 - 0.25 \ln p_1 + 0.75 \ln 1125 \approx 6.751 - 0.25 \ln p_1. \end{aligned} \quad (7.63)$$

In order for  $p_1$  to be subsidized to a point where it makes the household indifferent between getting the subsidy and participating in the public housing program, this utility has to be equal to the utility of public housing (which is 7.009); i.e.

$$6.751 - 0.25 \ln p_1 = 7.009. \quad (7.64)$$

Solving this for  $p_1$ , we get  $p_1 \approx 0.356$ . Thus, the subsidy that would make the household indifferent requires that the government pay a fraction of about 0.288 of rental housing (which reduces the price from 0.5 to 0.356).

- (d) How much housing will the family rent under this subsidy? How much will it cost the government to provide this subsidy?

Answer: The household would rent

$$x_1 = \frac{0.25(1500)}{0.356} \approx 1053, \quad (7.65)$$

which is less than it consumes under public housing. A house with 1053 square feet costs  $1053(0.5) = 526.50$  to rent — and the government under this subsidy pays 28.8% of this cost — i.e. the program costs  $0.288(526.5) \approx 151.63$ .

- (e) Suppose the government instead gave the family cash (without changing the price of housing). How much cash would it have to give the family in order to make it as happy?

Answer: We already determined that the utility of participating in the public housing program is 7.009. You can find the amount of income necessary to get to that utility level in different ways. One way is to solve the minimization problem

$$\min_{x_1, x_2} 0.5x_1 + x_2 \quad \text{subject to} \quad 0.25 \ln x_1 + 0.75 \ln x_2 = 7.009. \quad (7.66)$$

The first two first order conditions give us  $x_2 = 1.5x_1$ . Substituting into the constraint, we get

$$7.009 = 0.25 \ln x_1 + 0.75 \ln(1.5x_1) = \ln x_1 + 0.75 \ln(1.5) \approx \ln x_1 + 0.304. \quad (7.67)$$

Solving for  $x_1$ , we get  $x_1 \approx 816.5$ , and substituting back into  $x_2 = 1.5x_1$ ,  $x_2 \approx 1224.75$ . This bundle costs  $0.5(816.5) + 1224.75 = 1633$ . Since the household starts with \$1,500, this implies that a monthly cash payment of \$133 would make the household as well off as the public housing program (that costs \$250 per month).

- (f) *If you are a policy maker whose aim is to make this household happier at the least cost to the taxpayer, how would you rank the three policies? What if your goal was to increase housing consumption by the household?*

Answer: We have calculated that the public housing policy costs \$250 per month, the rent subsidy costs approximately \$156 per month and the cash subsidy costs \$133 per month. All three policies result in the same level of household utility. So if increasing happiness at the least cost is the goal, the cash subsidy would be best, followed by the rental subsidy and then the public housing program.

We also calculated that housing consumption will be 1500 square feet under public housing, 1053 square feet under the rental subsidy and 816.5 square feet under the cash subsidy. If the goal is to increase housing consumption, the public housing program dominates the rental subsidy which dominates the cash subsidy.