SOLUTIONS

9

Demand for Goods and Supply of Labor and Capital

Solutions for *Microeconomics: An Intuitive Approach with Calculus*

Apart from end-of-chapter exercises provided in the student *Study Guide*, these solutions are provided for use by instructors. (End-of-Chapter exercises with solutions in the student *Study Guide* are so marked in the textbook.)

- Each end-of-chapter exercise begins on a new page. This is to facilitate maximum flexibility for instructors who may wish to share answers to some but not all exercises with their students.
- If you are assigning only the A-parts of exercises in *Microeconomics: An Intuitive Approach with Calculus*, you may wish to instead use the solution set created for the companion book *Microeconomics: An Intuitive Approach*.
- *Solutions to Within-Chapter Exercises are provided in the student* Study Guide.

9.1 *The following is intended to explore what kinds of income-demand relationships are logically possible.*

A: *For each of the following, indicate whether the relationship is possible or not and explain:*

(a) *A good is a necessity and has a positive income-demand relationship.*

Answer: This is possible. A good is a necessity if, as income increases by some percentage k, the percentage increase in the consumption of the good is less than k. For instance, if income increases by 10% and consumption of the good increases by 5%, the good is a necessity. Since consumption still increases with income, the income-demand curve is still positive.

(b) *A good is a necessity and has a negative income-demand relationship.*

Answer: This, too, is possible. We just defined a necessity above — as income increases by a percentage k, the percentage increase in the consumption of the good is less than k. This leaves open the possibility that the percentage "increase" in consumption is negative — i.e. that, as income increases, consumption of the good declines. This would result in a negative income-demand relationship.

(c) *A good is a luxury and has a negative income-demand relationship.*

Answer: No, this is not possible. A luxury good is a good whose consumption increases by a greater percentage than income — thus, any time income increases, its consumption also increases. Therefore, the income-demand curve must have positive slope.

(d) *A good is quasilinear and has a negative income-demand relationship.*

Answer: Quasilinear goods are goods for which there are no income effects — thus, the income-demand relationship cannot be negative.

(e) *Tastes are homothetic and one of the goods has a negative income-demand relationship.*

Answer: If tastes are homothetic, all goods are normal goods. Thus all goods must exhibit a positive income-demand relationship.

B: *Derive the income-demand relationships for each good for the following tastes:*

(a) $u(x_1, x_2, x_3) = x_1^{\alpha} x_2^{\beta} x_3^{(1-\alpha-\beta)}$ *where* α *and* β *lie between zero and 1 and sum to less than 1.*

Answer: Cobb-Douglas utility functions in which the exponents sum to 1 have the property that the demand function for each good is simply equal to that good's exponent times income divided by that good's price. (When the exponents do not sum to 1, the price in the denominator is multiplied by the sum of all the exponents.) You can verify this by solving the maximization problem

$$\max_{x_1,x_2,x_3} x_1^{\alpha} x_2^{\beta} x_3^{(1-\alpha-\beta)} \text{ subject to } p_1 x_2 + p_2 x_2 + p_3 x_3 = I \tag{9.1}$$

and you will get

$$x_1 = \frac{\alpha I}{p_1},\ x_2 = \frac{\beta I}{p_2} \text{ and } x_3 = \frac{(1-\alpha-\beta)I}{p_3}. \tag{9.2}$$

Note that this problem is somewhat easier to solve if you take the natural log of the utility function when you set up the problem; i.e. if you define the problem as

$$\max_{x_1,x_2,x_3} \alpha \ln x_1 + \beta \ln x_2 + (1-\alpha-\beta)\ln x_3 \text{ subject to } p_1 x_2 + p_2 x_2 + p_3 x_3 = I. \tag{9.3}$$

The income-demand curve we graph is then simply the inverse of these with income on the left hand side (holding prices fixed); i.e. the income demand curves are derived from

$$I = \frac{p_1 x_1}{\alpha},\ I = \frac{p_2 x_2}{\beta} \text{ and } I = \frac{p_3 x_3}{(1-\alpha-\beta)}. \tag{9.4}$$

All these have positive slopes (which is not surprising since Cobb-Douglas tastes represent normal goods.)

(b) $u(x_1, x_2) = \alpha \ln x_1 + x_2$. (Note*: To fully specify the income demand relationship in this case, you need to watch out for corner solutions.) Graph the income demand curves for* x_1 *and* x_2 *— carefully labeling slopes and intercepts.*

Answer: Solving the problem

$$\max_{x_1,x_2} \alpha \ln x_1 + x_2 \text{ subject to } p_1x_1 + p_2x_2 = I \tag{9.5}$$

in the usual way, we get the demand functions

$$x_1 = \frac{\alpha p_2}{p_1} \text{ and } x_2 = \frac{I - \alpha p_2}{p_2}. \tag{9.6}$$

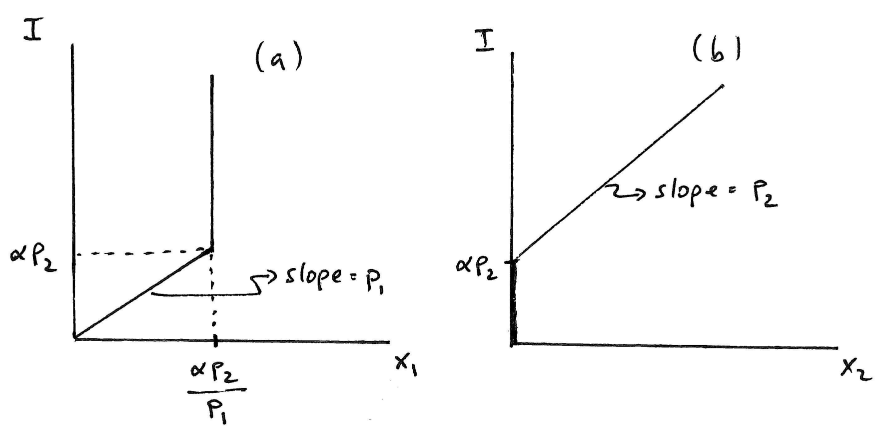
Consumption of x_1 is therefore independent of I as we expect given that x_1 is a quasilinear good. But that means that as income falls low enough, we will find ourselves at a corner solution. Put differently, the $x_1 = \alpha p_2/p_1$ quantity is not feasible for low enough incomes — in which case the consumer will simply spend all her available income on x_1. Specifically, if $I < \alpha p_2$, the consumer does not have enough income to buy $x_1 = \alpha p_2/p_1$. This implies that the real demand function for x_1 (taking into account corner solutions when I is sufficiently low) are

$$\begin{aligned} x_1 &= \frac{I}{p_1} \text{ when } I < \alpha p_2 \text{ and} \\ x_1 &= \frac{\alpha p_2}{p_1} \text{ when } I \geq \alpha p_2. \end{aligned} \tag{9.7}$$

Similarly, the real demand function for x_2 (taking into account corner solutions) is

$$\begin{aligned} x_2 &= 0 \text{ when } I < \alpha p_2 \text{ and} \\ x_2 &= \frac{I - \alpha p_2}{p_2} \text{ when } I \geq \alpha p_2. \end{aligned} \tag{9.8}$$

The income-demand curve is then derived from the inverse of these with I taken to the other side. The results are graphed in Graph 9.1.



Graph 9.1: Income Demand Curves when $u(x_1, x_2) = \alpha \ln x_1 + x_2$

9.2 *The following is intended to explore what kinds of own-price demand relationships are logically possible in a two-good model with exogenous income (unless otherwise specified).*

A: *For each of the following, indicate whether the relationship is possible or not and explain:*

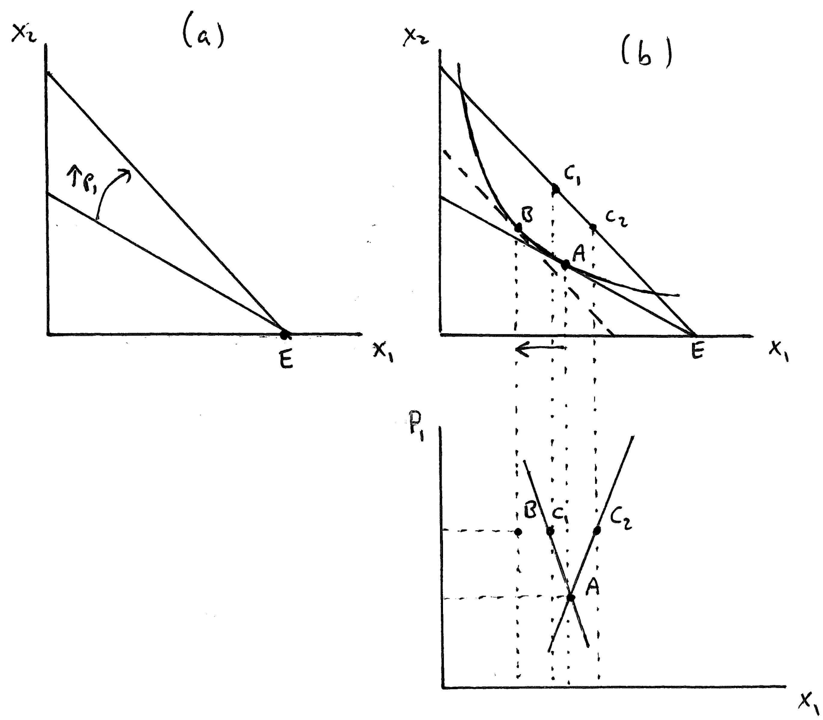
(a) *Tastes are homothetic and the own-price demand relationship is positive.*

Answer: This is not possible. When tastes are homothetic, all goods are normal goods. Price increases for normal goods result in a negative substitution effect and a negative income effect in the same direction. Thus, the own-price demand relationship must be negative.

(b) *A good is inferior and its own-price relationship is negative.*

Answer: This is possible. When a good is inferior, then an increase in the price results in a negative substitution effect and a positive income effect. If the income effect is smaller than the substitution effect, the good is "regular inferior" — and the own-price demand curve is downward sloping (i.e. the own-price demand relationship is negative). (If the income effect is larger than the substitution effect, the own-price demand curve slopes up and the good is a Giffen good.)

(c) *In a model with endogenous income, a good is normal and its own-price demand relationship is negative.*



Graph 9.2: Own-Price Demand Curves with Endogenous Incomes

Answer: Yes, this is possible. Consider the case where all income is endogenously derived from owning a quantity E of x_1. This is illustrated in panel (a) of Graph 9.2 where the shallower budget corresponds to a lower price for good x_1 and the steeper budget to a higher price for x_1. In panel (b), the substitution effect from A to B is illustrated — clearly suggesting that, as p_1 increases, consumption of x_1 declines. If x_1 is a normal good (as specified in the question), the wealth effect then points in the opposite direction. It is therefore possible that the final optimal bundle at the higher price is C_1 — which results in a negative own-price demand relationship.

(d) *In a model with endogenous income, a good is normal and its own-price demand relationship is positive.*

Answer: This is also possible — as illustrated in panel (b) of Graph 9.2. The wealth effect could be large enough to make C_2 optimal after the price increase — in which case the own-price demand curve slopes up.

B: *Suppose that tastes can be represented by the Cobb-Douglas utility function $u(x_1, x_2) = x_1^{\alpha} x_2^{(1-\alpha)}$.*

(a) *Derive the demand functions when income is exogenous and illustrate that own-price demand curves slope down.*

Answer: The demand functions are

$$x_1 = \frac{\alpha I}{p_1} \text{ and } x_2 = \frac{(1-\alpha)I}{p_2}. \tag{9.9}$$

The derivatives of these with respect to own-price are negative — thus the demand curves slope down. (Technically, we'd want to take the derivatives of the inverse demand functions with respect to the goods in order to determine the slopes of the demand curves — but the sign of slopes does not change when we invert. Thus, we can simply take the derivative of the demand functions with respect to price to determine whether curves slope up or down.)

(b) *Now suppose that all income is derived from an endowment (e_1, e_2). If $e_2 = 0$, what is the shape of the own price demand curve for x_1?*

Answer: Solving the problem

$$\max_{x_1, x_2} x_1^{\alpha} x_2^{(1-\alpha)} \text{ subject to } p_1 x_1 + p_2 x_2 = p_1 e_1 + p_2 e_2 \tag{9.10}$$

in the usual way we get

$$x_1 = \frac{\alpha(p_1 e_1 + p_2 e_2)}{p_1} = \alpha e_1 + \frac{p_2}{p_1} e_2. \tag{9.11}$$

If $e_2 = 0$, this simply reduces to $x_1 = \alpha e_1$ — i.e. demand for x_1 does not depend on price. Thus, the demand curve is perfectly vertical at quantity $x_1 = \alpha e_1$.

(c) *Continuing with part (b), what is the shape of the own price demand curve for x_1 when $e_2 > 0$?*

Answer: When $e_2 > 0$, the derivative of x_1 with respect to p_1 is

$$\frac{\partial x_1}{\partial p_1} = -\frac{p_2}{p_1^2} e_2 < 0. \tag{9.12}$$

Thus, the demand curve slopes down when $e_2 > 0$.

(d) *Suppose tastes were instead represented by the more general CES utility function. Without doing any additional math, can you guess what would have to be true about ρ in order for the own-price demand for x_1 to slope up when $e_1 > 0$ and $e_2 = 0$?*

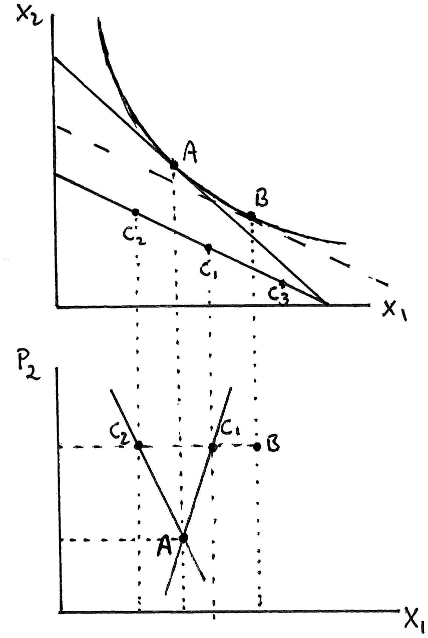
Answer: In Graph 9.2 of part A of this question, we illustrated that the own-price demand curve (for x_1) may slope up or down when all income is derived from and endowment of x_1 depending on whether the substitution effect is overcome by the positive wealth effect from a price increase. Thus, the smaller the substitution effect, the more likely it is that the own-price demand curve slopes up. We also just showed that the own-price demand curve in this case is perfectly vertical when tastes are Cobb-Douglas — which is equivalent to the CES case when $\rho = 0$. Thus, $\rho = 0$ is the borderline case where the own-price demand curve neither slopes up nor down when income is endogenously derived from an endowment of x_1. It will slope up if there is less substitutability — and down if there is more. We know that the goods become less substitutable as ρ increases in the CES utility function — thus the own-price demand curve will slope up in our scenario when $\rho > 0$.

9.3 *The following is intended to explore what kinds of cross-price demand relationships are logically possible in a two-good model with exogenous income.*

A: *For each of the following, indicate whether the relationship is possible or not and explain:*

(a) *A good is normal and its cross-price demand relationship is positive.*

Answer: Yes, this is possible. In Graph 9.3, we illustrate an increase in the price of good x_2, with A the original optimal bundle at the original lower price. The price increase results in a substitution effect to B — which suggests that consumption of x_1 should increase (since x_1 is now relatively cheaper than before). If x_1 is a normal good (as specified in the question), then the wealth effect will point in the opposite direction — as income declines from the compensated (dashed) budget to the final budget, consumption of x_1 will fall. If this negative wealth effect is smaller in absolute value than the positive substitution effect, we end up at a final bundle like C_1 — with consumption of x_1 having increased (from A) as a result of the increase in p_2. This gives a positive cross-price demand relationship in the lower panel of the graph.



Graph 9.3: Cross-price Demand

(b) *A good is normal and its cross-price relationship is negative.*

Answer: This is also possible and is also illustrated in Graph 9.3. When the wealth effect outweighs the substitution effect, the final optimal bundle might be a bundle like C_2 — which implies consumption of x_1 falls as the price of x_2 increases. This then results in the negative cross-price relationship in the lower panel of the graph.

(c) *A good is inferior and its cross-price relationship is negative.*

Answer: No, this is not possible. Consider again Graph 9.3. The substitution effect does not depend on whether the good is normal or inferior — so this effect remains exactly the same, suggesting an increase in x_1 as p_2 increases. The wealth effect, however, is now different — as income falls from the compensated (dashed) budget to the new final budget, consumption of x_1 now *increases.* Thus, we end up at a final bundle to the right of B — a point like C_3

in the graph. Since substitution and wealth effects point in the same direction, we can now say unambiguously that, when x_1 is inferior, the cross-price demand relationship must be positive.

(d) *Tastes are homothetic and one of the good's cross-price relationship is negative.*

Answer: Yes, this is possible. Homothetic tastes represent tastes over normal goods. In part (b) we already showed that the cross-price demand relationship can be negative for a normal good.

(e) *Tastes are homothetic and one of the good's cross-price relationship is positive.*

Answer: Yes, this is possible. Homothetic tastes represent tastes over normal goods. In part (a) we already showed that the cross-price demand relationship can be positive for a normal good.

B: *Now consider specific tastes represented by particular utility functions.*

(a) *Suppose tastes are represented by the function* $u(x_1,x_2) = \alpha \ln x_1 + x_2$. *What is the shape of the cross-price demand curves for* x_1 *and* x_2*?*

Answer: We already derived the demand function for this in equation (9.6) in exercise 9.1. These are

$$x_1 = \frac{\alpha p_2}{p_1} \text{ and } x_2 = \frac{I-\alpha p_2}{p_2}. \tag{9.13}$$

Thus, the demand for x_2 does not depend on p_1 — so the cross-price demand curve for x_2 is perfectly vertical at $x_2 = (I-\alpha p_2)/p_2$. The demand for x_1, on the other hand, does depend on p_2. Inverting the demand function to get p_2 on the left hand side, we get

$$p_2 = \frac{p_1}{\alpha} x_1. \tag{9.14}$$

Thus, the cross-price demand curve is upward sloping, with slope p_1/α.

(b) *Suppose instead tastes are Cobb-Douglas. What do cross-price demand curves look like?*

Answer: Cobb-Douglas tastes can be modeled by the utility function $u(x_1,x_2) = x_1^{\alpha} x_2^{(1-\alpha)}$. The demand functions that emerge from the utility maximization problem are

$$x_1 = \frac{\alpha I}{p_1} \text{ and } x_2 = \frac{(1-\alpha)I}{p_2}. \tag{9.15}$$

Since the "cross-price" does not appear in either of these functions, the demand for neither good depends on the price of the other. As a result, the cross-price demand curves are perfectly vertical.

(c) *Now suppose tastes can be represented by a CES utility function. Without doing any math, can you determine for what values of* ρ *the cross-price demand relationship is upward sloping?*

Answer: In Graph 9.3 of part A of this exercise, we showed that cross-price demand curves for normal goods can slope up or down depending on whether the substitution effect is larger or smaller than the income effect. In particular, since the substitution effect says that the consumer will buy more of x_1 when p_2 increases, the cross-price demand curve for x_1 is more likely to slope up the greater the substitution effect. In part (b) above, we showed that this curve slopes neither up or down (i.e. is perfectly vertical) when tastes are Cobb-Douglas — which is the special case of CES functions where $\rho = 0$. Within the CES family of utility functions, goods become more substitutable the smaller the value of ρ — so the substitution effect becomes relatively larger as ρ falls. Thus, if $-1 < \rho < 0$, we would expect the cross-price demand curve to slope up.

(d) *Suppose tastes can be represented by the CES function* $u(x_1,x_2) = (0.5x_1^{-\rho} + 0.5x_2^{-\rho})^{-1/\rho}$. *Verify your intuitive answer from part (c).*

Answer: The demand function for x_1 derived from utility maximization with this utility function is

$$x_1 = \frac{I}{p_1 + \left(p_1 p_2^{\rho}\right)^{1/(\rho+1)}}. \tag{9.16}$$

To determine the sign on the slope of the cross-price demand curve, we take the derivative with respect to p_2; i.e.

$$\frac{\partial x_1}{\partial p_2} = -I\left(p_1 + (p_1 p_2^{\rho})^{1/(\rho+1)}\right)^{-2}\left[\frac{\rho}{\rho+1}\right]\left(p_1 p_2^{\rho}\right)^{-\rho/(\rho+1)} p_1 p_2^{\rho-1}. \tag{9.17}$$

Note that there is a negative sign in front of I on the right hand side of this equation — and all terms in the equation are positive except for the bracketed term $[\rho/(\rho+1)]$.[1] The bracketed term is positive for $\rho > 0$ and negative for $-1 < \rho < 0$ — which implies that $\partial x_1/\partial p_2$ is positive if and only if $-1 < \rho < 0$. This is exactly what we concluded intuitively in the previous part.

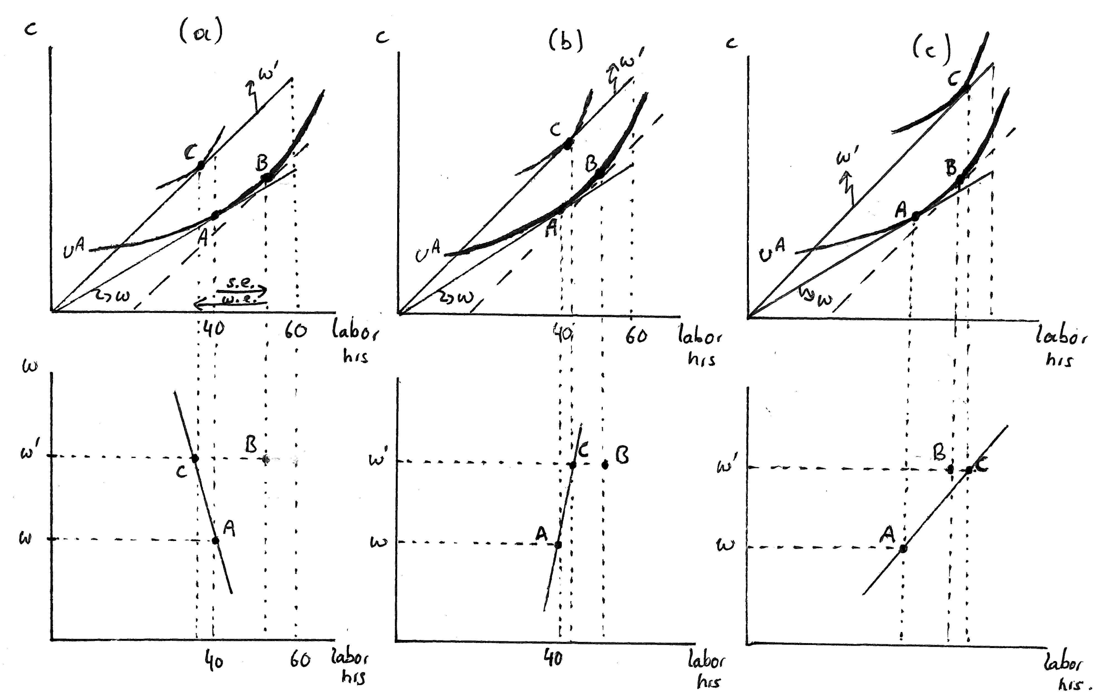
[1] Some of the exponents are negative — but that does not make the terms themselves negative.

9.4 *In Graph 9.4, we illustrated how you can derive the labor supply curve from a consumer model in which workers choose between leisure and consumption.*

A: *In end-of-chapter exercise 3.1, you were asked to illustrate a budget constraint with labor rather than leisure on the horizontal axis. Do so again, assuming that the most you can work per week is 60 hours.*

(a) *Now add to this graph an indifference curve that would make working 40 hours per week optimal.*

Answer: This is part of what is illustrated in panel (a) of Graph 9.4 where A is optimal at the original wage w.



Graph 9.4: Deriving Labor Supply Directly

(b) *Beginning with the graph you have just drawn, illustrate the same wealth and substitution effects as drawn in the top panel of Graph 9.4a for an increase in the wage.*

Answer: This is also illustrated in panel (a). The dashed budget line is the compensated budget — the budget that introduces the new wage w' but keeps the worker to the same indifference curve at the tangency at B. The move from A to B is the substitution effect. Since leisure is a normal good in this case, the wealth effect will point in the opposite direction — i.e. as the worker becomes wealthier (without a change in opportunity costs), he consumes more leisure (i.e. less labor). For the graph analogous to panel (a) in the text, this wealth effect outweighs the substitution effect — placing the new optimal bundle C to the left of A.

(c) *Then, on a second graph right below it, put weekly labor hours on the horizontal axis and wage on the vertical, and derive the labor supply curve directly from your work in the graph above. Compare the resulting graph to the lowest panel in Graph 9.4a.*

Answer: This is done in the panel directly below (a) where the bundles A, B and C are translated to a graph with labor hours on the horizontal (just as in the top panel) and wages on the vertical. The resulting downward sloping labor supply curve is identical to that derived in the text from the leisure demand curve.

(d) *Repeat this for the case where wealth and substitution effects look as they do in Graph 9.4b.*

Answer: This is done in panel (b) of Graph 9.4.

(e) *Repeat this again for the case in Graph 9.4c.*

Answer: This is done in panel (c) of Graph 9.4.

(f) True or False*: We can model the choices of workers either using our 5 standard assumptions about tastes defined over leisure and consumption, or we can model these choices using tastes defined over labor and consumption. Either way, we get the same answers so long as we let go of the monotonicity assumption in the latter type of model.*

Answer: This is true. Since labor is not a "good" that we want more of, we have tastes that do not satisfy monotonicity when we define them over consumption and labor rather than consumption and leisure. The optimization problem, however, results in exactly the same economic behavior as summarized by the labor supply curve.

B: *Now suppose that a worker's tastes over consumption and leisure can be defined by the utility function $u(c,\ell) = c^{\alpha}\ell^{(1-\alpha)}$ (and again assume that the worker has a leisure endowment of 60 hours per week).*

(a) *Derive the labor supply function by first deriving the leisure demand function.*

Answer: We first derive the leisure demand function by solving the usual optimization problem

$$\max_{c,\ell}\ c^{\alpha}\ell^{(1-\alpha)} \text{ subject to } c = w(60-\ell). \tag{9.18}$$

Solving the first order conditions of the Lagrangian for this problem then gives us the leisure demand function

$$\ell = 60(1-\alpha). \tag{9.19}$$

Subtracting the leisure demand from the leisure endowment of 60 then gives us the labor supply function

$$l(w) = 60 - 60(1-\alpha) = 60\alpha. \tag{9.20}$$

(b) *How would you define a utility function over consumption and* labor *(rather than consumption and* leisure*) such that the underlying tastes would be the same.*

Answer: Since leisure is just the number of hours not spent working, we can define it as $\ell = 60 - l$ where l is labor hours per week. Substituting this into the Cobb-Douglas utility function from before, we get $c^{\alpha}\ell^{(1-\alpha)} = c^{\alpha}(60-l)^{(1-\alpha)}$. We can therefore define the utility function

$$v(c,l) = c^{\alpha}(60-l)^{(1-\alpha)} \tag{9.21}$$

as an equivalent representation of this individual's tastes.

(c) *Which of our usual assumptions about tastes do not hold for tastes represented by the utility function you have just derived?*

Answer: The monotonicity assumption is violated — as l increases, utility falls because of the negative sign in front of l.

(d) *Using the utility function you have just given, illustrate that you can derive the same labor supply curve as before by making labor (rather than leisure) a choice variable in the optimization problem.*

Answer: The optimization problem with the new utility function and the constraint defined over labor rather than leisure is now

$$\max_{c,l}\ c^{\alpha}(60-l)^{(1-\alpha)} \ \text{ subject to } c = wl. \tag{9.22}$$

The corresponding Lagrange function is

$$\mathscr{L}(c,l,\lambda) = c^{\alpha}(60-l)^{(1-\alpha)} + \lambda(wl - c). \tag{9.23}$$

The first two first order conditions are

$$\begin{aligned} &\alpha c^{(\alpha-1)}(60-l)^{(1-\alpha)} - \lambda = 0 \\ &-(1-\alpha)c^{\alpha}(60-l)^{-\alpha} + \lambda w = 0. \end{aligned} \tag{9.24}$$

These can be solved for c to give $c = \alpha(60-l)w/(1-\alpha)$. When substituted into the budget constraint $c = lw$, we can then solve for the labor supply function $l = 60\alpha$.

9.5 Everyday Application: *Backward-Bending Labor Supply Curve*: *We have suggested in this chapter that labor economists believe that labor supply curves typically slope up when wages are low and down when wages are high. This is sometimes referred to as a* backward bending labor supply curve.

A: *Which of the following statements is inconsistent with the empirical finding of a backward bending labor supply curve?*

(a) *For the typical worker, leisure is an inferior good when wages are low and a normal good when wages are high.*

Answer: As wages increase, the substitution effect tells us that workers should work more (because taking leisure has become relatively more expensive). If leisure is an inferior good, the wealth effect also tells us that workers should work more when the wage increases. Thus, *if leisure is an inferior good*, the labor supply curve *must* slope up. If leisure is a normal good, however, the wealth effect tells us that an increase in wages should cause workers to work less. Thus, when leisure is normal, substitution and wealth effects go in the opposite direction — implying that the labor supply curve can slope up or down. Either is consistent with leisure being normal, but only an upward slope is consistent with leisure being inferior.

A backward bending labor supply curve is a labor supply curve that slopes up when wages are low and down when wages are high. If leisure is inferior when wages are low (as specified in this part of the question), this is consistent with an upward slope when wages are low. If leisure is normal when wages get high, this is consistent with either an upward or a downward slope when wages are high — and it is therefore consistent with the downward slope of the backward bending labor supply curve. Thus, the statement in this part of the question is not inconsistent with the backward bending labor supply curve.

(b) *For the typical worker, leisure is a normal good when wages are low and an inferior good when wages are high.*

Answer: (Based on the first paragraph of the answer to (a)), leisure being normal when wages are low is consistent with an upward slope of labor supply when wages are low. Leisure being inferior when wages are high, however, is inconsistent with the downward slope of the backward bending labor supply curve when wages are high. So this statement is not consistent with the backward bending labor supply behavior hypothesized by labor economists.

(c) *For the typical worker, leisure is always a normal good.*

Answer (Based on the first paragraph of the answer to (a)), leisure being a normal good is consistent with both upward and downward sloping labor supply curves. Thus, if leisure is always a normal good, it could indeed be that the labor supply curve is upward sloping for low wages and downward sloping for high wages. Thus, the statement is not inconsistent with the hypothesized backward bending labor supply curve.

(d) *For the typical worker, leisure is always an inferior good.*

Answer: The labor supply curve has to be upward sloping if leisure is inferior — but the backward bending labor supply curve hypothesizes a downward slope for high wages. Thus, leisure being always inferior is not consistent with a backward bending labor supply curve.

B: *Suppose that tastes over consumption and leisure are described by a constant elasticity of substitution utility function* $u(c,\ell) = (0.5c^{-\rho} + 0.5\ell^{-\rho})^{-1/\rho}$.

(a) *Derive the labor supply curve assuming a leisure endowment* L.

Answer: From the utility maximization problem, the *leisure demand* function is

$$\ell = \frac{L}{w^{-\rho/(\rho+1)} + 1}, \tag{9.25}$$

and the *labor supply function* $l(w)$ is then simply the leisure demand subtracted from the leisure endowment L; i.e.

$$l(w) = L - \frac{L}{w^{-\rho/(\rho+1)} + 1} = \frac{w^{-\rho/(\rho+1)}L}{w^{-\rho/(\rho+1)} + 1}. \tag{9.26}$$

(b) *Illustrate for which values of* ρ *this curve is upward sloping and for which it is downward sloping.*

Answer: It is algebraically a little easier to show how the sign of the leisure demand curve (as opposed to the labor supply curve) depends on ρ — and since the labor supply curve just has the opposite slope, we can answer the question this way. The derivative of the leisure demand curve with respect to w then is

$$\frac{\partial \ell}{\partial w} = \frac{Lw^{-(2\rho+1)/(\rho+1)}}{\left(w^{-\rho/(\rho+1)}+1\right)^2}\left[\frac{\rho}{\rho+1}\right]. \tag{9.27}$$

The non-bracketed term is unambiguously positive — which means that the equation is positive if and only if $\rho > 0$ and negative if and only if $-1 < \rho < 0$. Thus, the leisure demand curve slopes up for positive ρ and down for negative ρ. The opposite must then be true for labor supply.

You can show this also directly with the labor supply function by taking its derivative with respect to w. After a little algebraic manipulation, you can get

$$\frac{\partial l(w)}{\partial w} = -\left[\frac{\rho}{\rho+1}\right]\left(\frac{w^{-(2\rho+1)/(\rho+1)}L}{\left(w^{-\rho/(\rho+1)}+1\right)}\right)\left(1-\frac{w^{-\rho/(\rho+1)}}{\left(w^{-\rho/(\rho+1)}+1\right)^2}\right). \tag{9.28}$$

Again, all terms except for the bracketed term are positive.[2] Since there is a negative sign at the beginning of the right hand side of the equation, we can then conclude that the derivative is positive if and only if $-1 < \rho < 0$ and negative if and only if $\rho > 0$.

This should make intuitive sense: Substitution effects cause labor to increase with wages while wealth effects cause the opposite. Thus, the larger the substitution effect — i.e. the greater the substitutability between leisure and consumption — the more likely it is that the labor supply curve is upward sloping. And the elasticity of substitution between leisure and consumption increases as ρ falls. For this reason, the labor supply curve slopes up (and the leisure demand curve slopes down) if and only if ρ is below 0.

(c) *Is it possible for the backward bending labor supply curve to emerge from tastes captured by a CES utility function?*

Answer: No, it is not possible for a backward bending labor supply curve to emerge from any one CES utility function. Each such function has a fixed ρ — and, depending on what ρ is, the entire labor supply curve is either upward or downward sloping (or perfectly vertical in the case of $\rho = 0$.)

(d) *For practical purposes, we typically only have to worry about modeling tastes accurately at the margin — i.e. around the current bundles that consumers/workers are consuming. This is because low wage workers, for instance, may experience some increases in wages but not so much that they are suddenly high wage workers, and vice versa. If you were modeling worker behavior for a group of workers and you modeled each worker's tastes as CES over leisure and consumption, how would you assume ρ differs for low wage and high wage workers (assuming you are persuaded of the empirical validity of the backward bending labor supply curve)?*

Answer: We know from what we have done above that the labor supply curve is upward sloping for high elasticities of substitution (i.e. $-1 < \rho < 0$) and downward sloping for low elasticities of substitution (i.e. $\rho > 0$). If we believe in backward bending labor supply curves but we only need to worry about behavior at the margin, we could therefore model low wage workers (for whom labor supply is upward sloping on the margin) with low values of ρ and high wage workers (for whom labor supply is downward sloping at the margin) with high values of ρ.

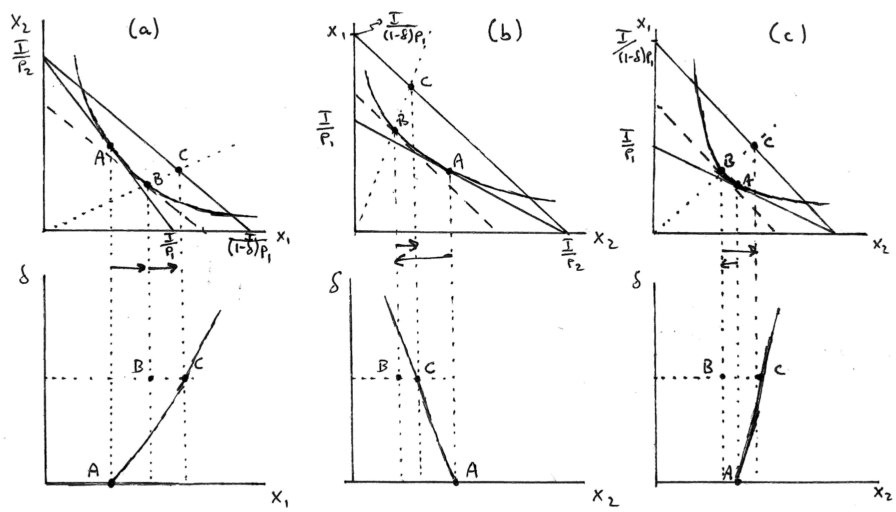
[2] The last term in parentheses is positive because the denominator in the fraction is larger than the numerator.

9.6 Business Application: *Price Discounts, Substitutes and Complements: A business might worry that pricing of one product might impact demand for another product that is also sold by the same business. Here we'll explore conditions under which such worries are more or less important before turning to some specific examples.*

A: *Suppose first that we label the two goods that a firm sells as simply x_1 and x_2. The firm considers putting a discount of δ on the price of x_1 — a discount that lowers the price from p_1 to $(1-\delta)p_1$.*

(a) *For a consumer who budgets I for consumption of x_1 and x_2, illustrate the budget before and after the discount is put in place.*

Answer: This is illustrated in panel (a) of Graph 9.5 — the discount simply lowers the price go x_1.



Graph 9.5: Price Discounts

(b) *Assuming that tastes are homothetic, derive the relationship between δ on the vertical axis and x_1 on the horizontal axis.*

Answer: This is also done in panel (a). Both income and substitution effects point in the same direction — as δ increases, consumption of x_1 increases.

(c) *Now derive the relationship between δ and x_2 — can you tell if it slopes up or down? What does your answer depend on?*

Answer: This is done in panels (b) and (c) where x_2 is put on the horizontal axis and x_1 on the vertical. In panel (b), x_1 and x_2 are relatively substitutable — which implies the substitution effect outweighs the income effect and consumption of x_2 decreases as δ increases. In panel (c), on the other hand, the two goods are relatively complementary — and so consumption of x_2 increases as δ increases. The answer therefore depends on the degree of substitutability between the two goods.

(d) *Suppose that x_1 is printers and x_2 is printer cartridges produced by the same company. Compare this to the case where x_1 is Diet Coke and x_2 is Zero Coke. In which case is there a more compelling case for discounts on x_1?*

Answer: The case is more compelling when x_1 is printers and x_2 is printer cartridges. This is for two reasons: First, the two products are strong complements; second, a printer is a one time expense that calls for repeated cartridge purchases. For this reason, printers are often sold at sharp discounts; sometimes they are given away for free. The case is less compelling when x_1 is Diet Coke and x_2 is Zero Coke. This is because the goods are closer to substitutes — and in fact it is unlikely that the consumer would continue to devote the same amount of income to the two goods.

B: *Suppose that tastes are defined by* $u(x_1, x_2) = x_1^{\alpha} x_2^{(1-\alpha)}$.

(a) *Derive the demand functions for* x_1 *and* x_2 *as a function of prices,* I *and* δ.

Answer: Solving the usual maximization problem, we get

$$x_1 = \frac{\alpha I}{(1-\delta)p_1} \text{ and } x_2 = \frac{(1-\alpha)I}{p_2}. \tag{9.29}$$

(b) *Are these upward or downward sloping in* δ*?*

Answer: The demand for x_1 is upward sloping in δ (i.e. the derivative of x_1 with respect to δ is positive), while the demand for x_2 is unaffected by δ.

(c) *Under the general specification of tastes as CES — i.e.* $u(x_1, x_2) = \left(\alpha x_1^{-\rho} + (1-\alpha)x_2^{-\rho}\right)^{-1/\rho}$, *how would your answer change as* ρ *changes?*

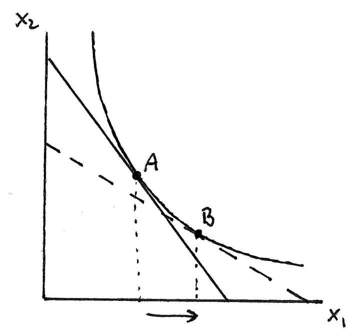
Answer: We know from what we did in part A that the relationship between δ and x_2 is more likely to be upward sloping the less substitutable x_1 and x_2 are. Substitutability decreases in the parameter ρ. When $\rho = 0$ (which is true for the Cobb-Douglas case), we have just concluded that δ has no impact on x_2. Thus, when $\rho > 0$, the relationship between ρ and δ is positive, and when $\rho < 0$ it is negative.

9.7 Business Application: *Good Apples versus Bad Apples*: *People are often amazed at the quality of produce that is available in markets far away from where that produce is grown — and that it is often the case that the average quality of produce is* higher *the farther the place where the produce originates. Here we will try to explain this as the result of producers' awareness of relative demand differences resulting from substitution effects.*

A: *Suppose you own an apple orchard that produces two types of apples: High quality apples x_1 and low quality apples x_2. The market price for a pound of high quality apples is higher than that for a pound of low quality apples — i.e. $p_1 > p_2$. You sell some of your apples locally and you ship the rest to be sold in a different market. It costs you an amount c per pound of apples to get apples to that market.*

(a) *Begin with a graph of a consumer who chooses between high and low quality apples in the local store in your town. Illustrate the consumer's budget and optimal choice.*

Answer: This is illustrated with the solid budget line and indifference curve in 9.6. The optimal bundle for the local consumer is *A*.



Graph 9.6: Good Apples and Bad Apples

(b) *The only way you are willing to ship apples to a far-away market is if you can get as much for those apples as you can get in your town — which means you will add the per-pound transportation cost c to the price you charge for your apples. How will the slope of the budget constraint for the far-away consumer differ from that for your local consumer, and what does that imply for the opportunity cost of good apples in terms of bad apples?*

Answer: The absolute value of the slope will change from p_1/p_2 to $(p_1+c)/(p_2+c)$. For instance, if the prices are $p_1 = 4$ and $p_2 = 2$, and if $c = 1$, the absolute value of the slope changes from $4/2 = 2$ to $(4+1)/(2+1) = 5/3$. Thus, the slope becomes shallower — implying the opportunity cost of good apples in terms of bad apples falls as c increases.

(c) *Apples represent a relatively small expenditure category for most consumers — which means that income effects are probably very small. In light of that, you may assume that the amount of income devoted to apple consumption is always an amount that gets the consumer to the same indifference curve in the "slice" of tastes that hold all goods other than x_1 and x_2 fixed. Can you determine where consumer demand for high quality apples is likely to be larger — in the home market or in the far-away market?*

Answer: The dashed budget line in the graph has the shallower slope that arises from the lower relative price of good apples in the far-away market. If the consumer changes the amount of income devoted to apple consumption so as to always end up on the same "slice" of the tastes holding all goods other than apples fixed, she will end up at bundle *B* — with more high quality apples and fewer low quality apples. This is a pure substitution effect — and so long as the income effect from all apples costing more is not large, the same conclusion should hold even if the consumer does not adjust her apple budget to get her all the way to the same indifference curve slice.

(d) *Explain how, in the presence of transportation costs, one would generally expect the phenomenon of finding a larger share of high quality products in markets that are far from the production source than in markets that are close.*

Answer: The example has general implications for all products that are consumed locally and also shipped to other markets. Since transportation costs are the same no matter what the quality of the product, the prices of both low and high quality products will have to increase by the same *absolute* amount in the far-away market. But since the high quality products have a higher price to start with, the same absolute increase in price implies a decrease in its *relative* price (i.e. relative to the low quality product). Thus, the existence of transportation costs implies that, although they increase the price of all goods, they increase the price of high quality goods disproportionately less. This results in a larger share of high quality goods being "exported" from local markets. As a result, grapes from Chile might be of average higher quality in US markets than in Chilean markets, just as apples in the hometown of the orchard may be of lower average quality than apples from that orchard in far-away markets.

B: *Suppose that we model our consumers' tastes as* $u(x_1, x_2) = x_1^{\alpha} x_2^{(1-\alpha)}$.

(a) *What has to be true about* α *in order for* x_1 *to be the good apples.*

Answer: It must be that the consumer gets more utility from x_1 than from the same amount of x_2 — which holds only if $\alpha > 0.5$.

(b) *Letting consumer income devoted to apple consumption be given by* I*, derive the consumer's demand for good and bad apples as a function of* p_1*,* p_2*,* I *and* c*. (Recall that* c *is the per pound transportation cost that is added to the price of apples).*

Answer: Solving the maximization problem

$$\max_{x_1, x_2} x_1^{\alpha} x_2^{(1-\alpha)} \text{ subject to } (p_1 + c)x_1 + (p_2 + c)x_2 = I, \tag{9.30}$$

we get

$$x_1 = \frac{\alpha I}{(p_1 + c)} \text{ and } x_2 = \frac{(1-\alpha)I}{(p_2 + c)}. \tag{9.31}$$

(c) *What is the ratio of demand for* x_1 *over* x_2*?*

Answer: Using the answer from above, the ratio of demand for good apples relative to bad apples is

$$\frac{x_1}{x_2} = \frac{\alpha}{(1-\alpha)} \frac{(p_2 + c)}{(p_1 + c)}. \tag{9.32}$$

(d) *Can you tell from this in which market there will be greater relative demand for good versus bad apples — the local market or the far-away market?*

Answer: Note that, when $p_2 < p_1$, an increase in c implies an increase in $(p_2 + c)/(p_1 + c)$ (or conversely, as we put it in part A, a decrease in $(p_1 + c)/(p_2 + c)$). Thus, x_1/x_2 — the amount of good apples relative to bad apples consumed by our consumer — increases as c increases. The far-away market will therefore have relatively greater demand for high quality apples.

(e) *In part A, we held the consumer's indifference curve in the graph fixed and argued that it is reasonable to approximate the consumer's behavior this way given that apple expenditures are typically a small fraction of a consumer's budget. Can you explain how what you just did in part B is different? Is it necessarily the case that consumers in far-away places will consume more high quality apples than consumers (with the same tastes) in local markets? Can we still conclude that far-away markets will have a higher* fraction *of high quality apples?*

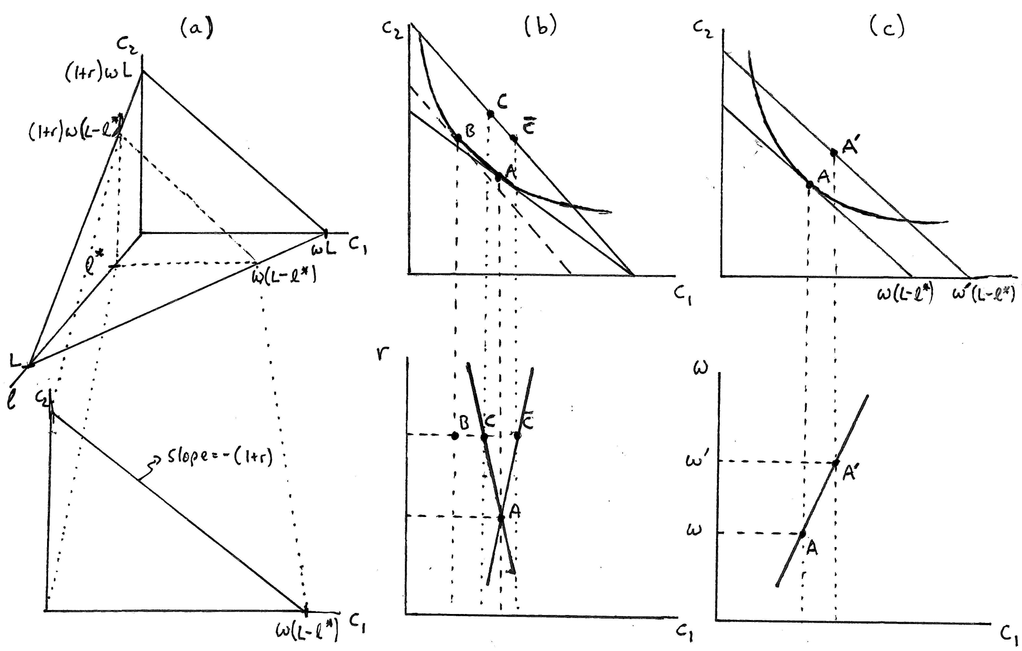
Answer: In part B, we used the demand functions for the consumer — and thus held income fixed. In other words, we did not assume that, as transportation costs push up the prices of good and bad apples, the consumer adds more money to the apple budget — rather, we assumed the apple budget stays fixed. This led us to conclude that consumers will demand relatively more high quality apples than low quality apples in the far-away market. It does not necessarily mean that consumers in far-away markets (with tastes similar to local consumers) will consume a greater *absolute* number of high quality apples — but since they demand *relatively* more high quality apples, the fraction of apples that are of high quality would be expected to be larger in the far-away market.

9.8 Policy Application: *Tax and Retirement Policy: In Chapter 3, we illustrated budgets in which a consumer faced trade-offs between working and leisuring now as well as between consuming now and consuming in the future. We can use a model of this kind to think about tax and retirement policy.*

A: *Suppose period 1 represents the period over which a worker is productive in the labor force and period 2 represents the period during which the worker expects to be retired. The worker earns a wage w and has L hours of leisure time that could be devoted to work l or leisure consumption ℓ. Earnings this period can be consumed as current consumption c_1 or saved for retirement consumption c_2 at an interest rate r. Suppose throughout that consumption in both periods is a normal good — as is leisure this period.*

(a) *Illustrate this worker's budget constraint in a 3-dimensional graph with c_1, c_2, and ℓ on the axes.*

Answer: This is illustrated in panel (a) of Graph 9.7.



Graph 9.7: Taxing Wages to Subsidize Savings

(b) *For certain types of tastes (as for those used in part B of this question), the optimal labor decision does not vary with the wage or the interest rate in this problem. Suppose this implies that taking ℓ^* in leisure is always optimal for this worker. Illustrate how this puts the worker's decision on a slice of the 3-dimensional budget you graphed in part (a).*

Answer: This slice is also illustrated in panel (a) of the graph — with the slice pulled out below the 3-dimensional graph. Since leisure is now fixed, the slice only shows the tradeoff faced between c_1 and c_2.

(c) *Assume that optimal choices always occur on the 2-dimensional slice you have identified. Illustrate how you could derive a demand curve for c_1 — i.e. a curve that shows the relationship between c_1 on the horizontal axis and the interest rate r on the vertical. Does this curve slope up or down? What does your answer depend on?*

Answer: This is done in panel (b) of Graph 9.7. The top panel begins with the original budget (slice) where A is optimal. An increase in the interest rate r causes an outward rotation of the budget — giving rise to the substitution effect from A to B and a wealth effect in the opposite direction (since consumption in all periods is assumed to be a normal good). Depending on the size of the substitution effect, the wealth effect may result in a new optimal bundle like C to the left of A or $\overline{C}$ to the right or A. When translated to the lower graph, the c_1 demand curve may therefore slope up or down depending on the size of the substitution effect.

(d) *Can you derive a similar economic relationship — except this time with w rather than r on the vertical axis? Can you be certain about whether this relationship is upward sloping (given that consumption in both periods is a normal good)?*

Answer: This is done in panel (c) of Graph 9.7. Here we again begin with the original budget (slice) where A is optimal. An increase in the wage w to w' does not change the slope of this budget — but it does change the horizontal intercept. Thus, the budget shifts out in a parallel way — giving rise to a pure wealth effect without a substitution effect. Since consumption is always assumed to be normal, this implies an unambiguous increase in c_1 — and thus an unambiguous upward slope to the economic relationship between c_1 and w in the lower panel.

(e) *Suppose that the government introduces a program that raises taxes on wages and uses the revenues to subsidize savings. Indicate first how each part of this policy — the tax on wages and the subsidy for savings (which raises the effective interest rate) — impacts current and retirement consumption.*

Answer: The increase in taxes causes a decrease in the effective wage for the worker — and we see in panel (c) of the graph we have just derived that this will cause a decrease in current consumption. It will similarly cause a decrease in retirement consumption. The subsidy of savings raises the interest rate — and we see from panel (b) of the graph we have derived that this may cause an increase or a decrease in current consumption depending on the size of the substitution effect. It also causes an unambiguous increase in retirement consumption.

(f) *Suppose the tax revenue is exactly enough to pay for the subsidy. Without drawing any further graphs, what do you think will happen to current and retirement consumption?*

Answer: The increase in the interest rate from the savings subsidy causes a substitution and wealth effect — but the tax on wages essentially removes the wealth effect. This leaves us essentially with the substitution effect — and that is unambiguously in the direction of less consumption now and more consumption in retirement.

(g) *There are two ways that programs such as this can be structured: Method 1 puts the tax revenues collected from the individual into a personal savings account that is used to finance the savings subsidy when the worker retires; Method 2 uses current tax revenues to support current retirees — and then uses tax revenues from future workers to subsidize current workers when they retire. (The latter is often referred to as "pay-as-you-go" financing.) By simply knowing what happens to current and retirement consumption of workers under such programs, can you speculate what will happen to overall savings under Method 1 and Method 2 (given that tax revenues become savings under Method 1 but not under Method 2)?*

Answer: We know from our previous answer that c_1 will fall and c_2 will rise. When tax revenues are collected and put into (forced) savings accounts for workers, this must imply that overall savings of the worker increase because less is consumed now and all is eventually consumed in the future. When tax revenues are not put into a savings account — and instead used to fund current retirees, on the other hand, personal savings will fall because the worker still treats the tax revenue collected from him as if it were savings. Since the government does not actually save the money under this method, overall savings therefore decline.

B: *Suppose the worker's tastes can be summarized by the function* $u(c_1, c_2, \ell) = \left(c_1^{\alpha}\ell^{(1-\alpha)}\right)^{\beta} c_2^{(1-\beta)}$.

(a) *Set up the budget equation that takes into account the tradeoffs this worker faces between consuming and leisuring now as well as between consuming now and consuming in the future.*

Answer: Current income is $w(L-\ell)$ and current consumption is c_1. Thus, savings will be $w(L-\ell)-c_1$ which leaves $(1+r)[w(L-\ell)-c_1]$ for consumption during retirement. The budget constraint can therefore be written as

$$c_2 = (1+r)\left[w(L-\ell)-c_1\right], \tag{9.33}$$

or equivalently,

$$c_1(1+r)+c_2 = w(L-\ell)(1+r). \tag{9.34}$$

(b) *Set up this worker's optimization problem — and solve for the optimal consumption levels in each period as well as the optimal leisure consumption this period.*

Answer: The optimization problem is then

$$\max_{c_1,c_2,\ell} \left(c_1^{\alpha}\ell^{(1-\alpha)}\right)^{\beta} c_2^{(1-\beta)} \text{ subject to } c_1(1+r)+c_2 = w(L-\ell)(1+r). \tag{9.35}$$

The algebra becomes considerably simpler (without affecting the solution to the problem) if we take the natural log of the utility function and thus write the problem as

$$\max_{c_1,c_2,\ell} \alpha\beta\ln c_1 + (1-\alpha)\beta\ln\ell + (1-\beta)\ln c_2 \text{ subject to } c_1(1+r)+c_2 = w(L-\ell)(1+r). \tag{9.36}$$

The Lagrange function then becomes

$$\mathscr{L}(c_1,c_2,\ell,\lambda) = \alpha\beta\ln c_1 + (1-\alpha)\beta\ln\ell + (1-\beta)\ln c_2 + \lambda(w(L-\ell)(1+r)-c_1(1+r)-c_2), \tag{9.37}$$

with the first three first order conditions of

$$\begin{aligned}
\frac{\partial\mathscr{L}}{\partial c_1} &= \frac{\alpha\beta}{c_1} - \lambda(1+r) = 0\\
\frac{\partial\mathscr{L}}{\partial c_2} &= \frac{(1-\beta)}{c_2} - \lambda = 0\\
\frac{\partial\mathscr{L}}{\partial \ell} &= \frac{(1-\alpha)\beta}{\ell} - \lambda w(1+r) = 0
\end{aligned} \tag{9.38}$$

The second of these implies $\lambda = (1-\beta)/c_2$. Substituting this into the first and third condition allows us to write c_1 and ℓ in terms of c_2; i.e.

$$c_1 = \frac{\alpha\beta}{(1-\beta)(1+r)}c_2 \text{ and } \ell = \frac{(1-\alpha)\beta}{(1-\beta)w(1+r)}c_2. \tag{9.39}$$

Plugging these into the budget constraint, we can then solve for c_2 — and plugging that solution back into the above expressions for c_1 and ℓ allows us to solve for optimal levels of c_1 and ℓ. We then get

$$\begin{aligned}
c_1 &= \alpha\beta wL\\
c_2 &= (1-\beta)wL(1+r)\\
\ell &= (1-\alpha)\beta L.
\end{aligned} \tag{9.40}$$

(c) *In part A we assumed that the worker would choose the same amount of work effort regardless of the wage and interest rate. Is this true for the tastes used in this part of the exercise?*

Answer: Yes — leisure is a function of neither w nor r.

(d) *How does consumption before retirement change with w and r? Can you make sense of this in light of your graphical answers in part A?*

Answer: Current consumption c_1 increases with w but does not change with r. The first of these results is consistent with our conclusion in panel (c) of Graph 9.7 where we concluded that an unambiguous wealth effect will result in an unambiguous increase in current consumption as w increases. The latter result is consistent with panel (b) of our graph where we concluded that substitution and wealth effects from an increase in r have opposite effects on current consumption. With the Cobb-Douglas tastes modeled here, the two effects exactly offset one another — causing current consumption to be unaffected by the interest rate.

(e) *In A(e) we described a policy that imposes a tax t on wages and a subsidy s on savings. Suppose that the tax lowers the wage retained by the worker to $(1-t)w$ and the subsidy raises the effective interest rate for the worker to $(r+s)$. Without necessarily redoing the optimization problem, how will the equations for the optimal levels of c_1, c_2 and ℓ change under such a policy?*

Answer: All we need to do is replace w with $(1-t)w$ and r with $(r+s)$ — which gives us

$$\begin{aligned} c_1 &= \alpha\beta(1-t)wL \\ c_2 &= (1-\beta)(1-t)wL(1+r+s) \\ \ell &= (1-\alpha)\beta L. \end{aligned} \tag{9.41}$$

(f) *Are the effects of t and s individually as you concluded in A(e)?*

Answer: Yes. In A(e) we concluded that current and retirement consumption will fall with an increase in t. Since t enters with a negative sign in our equations for c_1 and c_2, this result is verified here. We also concluded that an increase in s may cause current consumption to increase or decrease but will cause retirement consumption to unambiguously increase. Our equations tell us that, under the tastes modeled here, c_1 is unchanged with a change in s but c_2 is positively related to s. This is therefore also consistent with what we concluded in A(e).

(g) *For a given t, how much tax revenue does the government raise? For a given s, how much of a cost does the government incur? What do your answers imply about the relationship between s and t if the revenues raised now are exactly offset by the expenditures incurred next period (taking into account that the revenues can earn interest until they need to be spent)?*

Answer: Tax revenues are collected on labor income. We concluded above that leisure consumption is independent of wages and interest rates — i.e. $\ell = \beta(1-\alpha)L$. Subtracting this from the leisure endowment then gives the labor hours — i.e. $l = L - \ell = L - \beta(1-\alpha)L = (1-\beta(1-\alpha))L$. Tax revenue is therefore

$$\text{Tax Revenue } = tw\left(1-\beta(1-\alpha)\right)L. \tag{9.42}$$

The cost of the subsidy next period is s times the amount that the individual saves. Savings is just after-tax earnings this period minus consumption this period. After tax earnings are $(1-t)w\left(1-\beta(1-\alpha)\right)L$, and we previously concluded that consumption now $c_1 = \alpha\beta(1-t)wL$. Subtracting the latter from the former (and doing some algebra), we get that savings is equal to $w(1-t)(1-\beta)L$. The cost of the subsidy is then simply this times s — i.e.

$$\text{Subsidy Cost } = sw(1-t)(1-\beta)L. \tag{9.43}$$

If current tax revenue exactly pays for the retirement subsidy next period (and current tax revenue earns interest until it is needed to pay for the subsidy), then (1+r)(Tax Revenue)= Subsidy Cost; i.e.

$$(1+r)tw\left(1-\beta(1-\alpha)\right)L = sw(1-t)(1-\beta)L. \tag{9.44}$$

Solving for s, we then get that the subsidy rate that is affordable from the tax revenues that are collected is

$$s = \frac{(1+r)(1-\beta(1-\alpha))t}{(1-\beta)(1-t)}. \tag{9.45}$$

(h) *Can you now verify your conclusion from A(f)?*

Answer: From the equation $c_1 = \alpha\beta(1-t)wL$, it is immediately obvious that consumption now will decrease as a result of any program (t, s) that has $t > 0$. Retirement consumption is given by the equation $c_2 = (1-\beta)(1-t)w(1+r+s)L$ that we derived before. When we substitute for s using equation (9.45), we get (after simplifying the expression)

$$c_2 = (1+r)(1-\beta+t\alpha\beta)wL. \tag{9.46}$$

Thus, for any program (t, s) where current tax revenues exactly pay for future retirement subsidies, c_2 is increasing in t — i.e. retirement consumption goes up.

(i) *What happens to the size of personal savings that the individual worker puts away under this policy? If we consider the tax revenue the government collects on behalf of the worker (which will be returned in the form of the savings subsidy when the worker retires), what happens to the worker's overall savings — his personal savings plus the forced savings from the tax?*

Answer: We already calculated above that savings for the worker are given by

$$\text{Personal Savings } = w(1-t)(1-\beta)L. \tag{9.47}$$

This personal savings therefore decreases as t increases — i.e. the program of taxing wages and then subsidizing interest causes personal savings to fall. We also calculated the tax revenue (which we can now view as forced savings) as

$$\text{Forced Savings } = tw\left(1-\beta(1-\alpha)\right)L. \tag{9.48}$$

Adding the two forms of savings together, we get

$$\text{Total Savings } = w(1-t)(1-\beta)L + tw\left(1-\beta(1-\alpha)\right)L = w(1-\beta)L + \alpha\beta twL. \tag{9.49}$$

From equation (9.47) we can see that the first term $w(1-\beta)L$ is how much savings the worker would undertake if $t = 0$ — i.e. in the absence of the tax/subsidy program. The second term $\alpha\beta twL$ is unambiguously positive — which implies that an increase in t will result in an increase in overall savings if the tax revenues are saved on behalf of the worker.

(j) *How would your answer about the increase in actual overall savings change if the government, instead of actually saving the tax revenue on behalf of the worker, were to simply spend current tax revenues on current retirees. (This is sometimes referred to as a* pay-as-you-go *policy.)*

Answer: In that case, personal savings would be the only actual savings. Although the worker would perceive the government as forcing him to save in addition to what he saves personally, the government would not actually be saving this money but rather would be paying it out immediately to current retirees. Thus, overall savings under this system would decline rather than increase.

9.9 Policy Application: *Demand for Charities and Tax Deductibility: One of the ways in which government policy supports a variety of activities in the economy is to make contributions to those activities tax deductible. For instance, suppose you pay a marginal income tax rate t and that a fraction δ of your contributions to charity are tax deductible. Then if you give \$1 to a charity, you do not have to pay income tax on \$$\delta$ and thus you end up paying \$$\delta t$ less in taxes. Giving \$1 to charity therefore does not cost you \$1 — it only costs you \$$(1-\delta t)$.*

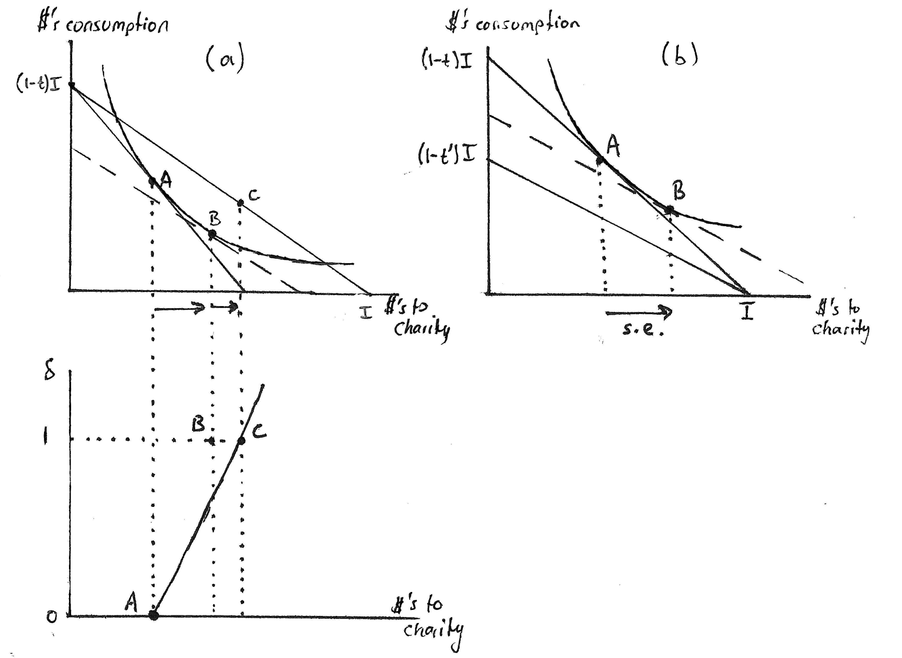
A: *In the remainder of the problem, we will refer to $\delta = 0$ as no deductibility and $\delta = 1$ as full deductibility. Assume throughout that giving to charity is a normal good.*

(a) *How much does it cost you to give \$1 to charity under no deductibility? How much does it cost under full deductibility?*

Answer: Under no deductibility, it costs you \$1 to give \$1. Under full deductibility, it costs you \$$(1-t)$ to give \$1 — because by giving \$1 to charity, you save \$t in taxes.

(b) *On a graph with "dollars given to charity" on the horizontal and "dollars spent on other consumption" on the vertical, illustrate a taxpayer's budget constraint (assuming the taxpayer pays a tax rate t on all income) under no deductibility and under full deductibility.*

Answer: This is illustrated in panel (a) of Graph 9.8 where the steeper solid line is the no-deductibility budget and the shallower solid line is the full-deductibility budget. If no money is given to charity, then the consumer will be able to spend $(1-t)I$ — her after-tax income — on consumption. If, on the other hand, she gives all her income to charity under full deductibility, she has to pay no taxes — and is thus able to contribute her before tax income I.



Graph 9.8: Tax deductibility of Charitable Contributions

(c) *On a separate graph, derive the relationship between δ (ranging from zero to 1 on the vertical) and charitable giving (on the horizontal).*

Answer: This is derived in the lower graph of panel (a) from the upper graph. Under no deductibility, the consumer optimizes at *A*. The substitution effect from the lower price for giving to charity under deductibility implies an increase in charitable giving to *B* — and the remaining income effect increases this further to *C* (given that we have assumed charitable giving is a normal good). Thus, as deductibility increases, charitably giving unambiguously increases.

(d) *Next, suppose that charitable giving is fully deductible and illustrate how the consumer's budget changes as t increases. Can you tell whether charitable giving increases or decreases as the tax rate rises?*

Answer: The change in the budget is illustrated in panel (b) of the graph. Under full deductibility, the maximum amount that a consumer can give to charity if she gives all her income remains the same as her tax rate changes — because if she gives her entire income, she owes no taxes under full deductibility. However, as t increases, she will not be able to consume as much in other consumption. The budget constraint therefore becomes shallower as t increases from t to t' — with the horizontal intercept remaining unchanged. Beginning at the lower tax rate t, the consumer optimizes at *A*. An increase in t makes giving to charity relatively cheaper — resulting in a substitution effect to *B* that implies greater charitable giving. However, there is an additional income effect — and, if charitable giving is a normal good, this effect will point in the opposite direction. Depending on which of these effects is bigger, a consumer might end up increasing or decreasing her charitable giving as her tax rate increases — the more substitutable charitable giving and personal consumption are, the more likely she is to increase her charitable giving as her tax rate increases.

(e) *Suppose that an empirical economist reports the following finding: "Increasing tax deductibility raises charitable giving, and charitable giving under full deductibility remains unchanged as the tax rate changes." Can such behavior emerge from a rationally optimizing individual?*

Answer: Yes, we have shown that it can in the answers above.

(f) *Shortly after assuming office, President Barack Obama proposed repealing the Bush tax cuts — thus raising the top income tax rate to 39.6%. At the same time, he made the controversial proposal to only allow deductions for charitable giving as if the marginal tax rate were 28%. For someone who pays the top marginal income tax under the Obama proposal, what does the proposal imply for δ? What about for someone paying a marginal tax rate of 33% or someone paying a marginal tax rate of 28%?*

Answer: The Obama proposal implies that anyone whose marginal income tax rate exceeds 28% will face a cost of 72 cents for every dollar he gives to charity; i.e. $(1-\delta t) = 0.72$. For someone who pays the top marginal tax rate, we then plug $t = 0.396$ into $(1-\delta t) = 0.72$ and solve for δ to get $\delta \approx 0.71$. When the tax rates are 33% or 28%, repeating this for $t = 0.33$ and $t = 0.28$ gives us $\delta \approx 0.85$ and $\delta = 1$. The Obama proposal therefore effectively lowers the fraction δ of charitable contributions that can be deducted by high income taxpayers.

(g) *Would you predict that the Obama proposal would reduce charitable giving?*

Answer: In part (c) we showed that as deductibility δ increases, we get unambiguously more charitable giving. By the same logic — i.e. both income and substitution effects pointing in the same direction, we conclude that charitable giving will fall as deductibility δ falls. We therefore expect the Obama proposal to result in reduced charitable giving.

(h) *Defenders of the Obama proposal point out the following: After President Ronald Reagan's 1986 Tax Reform, the top marginal income tax rate was 28% — implying that it would cost high earners 72 cents for every dollar they contribute to charity, just as it would under the Obama proposal. If that was good enough under Reagan, it should be good enough now. In what sense it the comparison right, and in what sense is it misleading?*

Answer: The first statement is absolutely correct: For high income individuals, the cost of giving $1 to charities is 72 cents under the 1986 tax reform as well as under the Obama proposal. Put differently, both proposals set the same opportunity cost for giving to charities (for high income earners) — and thus the substitution effect is the same. The difference is the income effect because the tax rates are higher under the Obama proposal than under the Reagan reform. And the income effect would predict lower charitable giving under the Obama proposal than under the terms of the 1986 tax reform.

B: *Now suppose that a taxpayer has Cobb-Douglas tastes over charitable giving (x_1) and other consumption (x_2).*

(a) *Derive the taxpayer's demand for charitable giving as a function of income I, the degree of tax deductibility δ and the tax rate t.*

Answer: Solving the problem

$$\max_{x_1,x_2} x_1^{\alpha} x_2^{(1-\alpha)} \text{ subject to } (1-\delta t)x_1 + x_2 = (1-t)I, \tag{9.50}$$

we get

$$x_1 = \frac{\alpha(1-t)I}{(1-\delta t)}. \tag{9.51}$$

(b) *Is this taxpayer's behavior consistent with the empirical finding by the economist in part A(e) of the question?*

Answer: Yes, it is. The first part of the empirical finding said that increasing tax deductibility will increase the consumer's charitable giving. The derivative of x_1 with respect to δ is indeed positive — thus, as δ increases (i.e. as deductibility increases), x_1 increases. The second part of the empirical finding is that, under full deductibility (i.e. when $\delta = 1$), a change in the tax rate has no effect on charitable giving. Setting δ equal to 1 in equation (9.51), we get

$$x_1 = \frac{\alpha(1-t)I}{(1-t)} = \alpha I. \tag{9.52}$$

Thus, under full deductibility, charitable giving is immune to the tax rate — because the income and substitution effects exactly offset each other.