CHAPTER

0

Foundational Preliminaries: Answers to Within-Chapter Exercises

0A Solutions to Within-Chapter-Exercises for Part A

Exercise 0A.1

Consider the set $[0,1)$ which includes the point 0, all the points in between 0 and 1 but not the point 1. Is this set convex? What about the union of this set with the point 1? What about the union of this set with the point 1.1?

Answer: The set $[0,1)$ is convex. So is the union of this set with 1, denoted $[0,1]$ (as illustrated in the chapter). But the union with 1.1 is not convex because any line segment connecting 1.1 with a point in $[0,1)$ is not fully contained in the union of $[0,1)$ with 1.1.

Exercise 0A.2

Is the set of all rational numbers a convex set? What about the set of all non-integers? Or the set of all irrational numbers?

Answer: The set of all rational numbers is not convex because any line segment connecting two rational numbers contain irrational numbers that are not in the set of rational numbers.

Exercise 0A.3

Describe points B, C, D and E in Graph 0.2 as a pair of real numbers.

Answer: $B=(8,2)$, $C=(6,5)$, $D=(4,5)$, and $E=(7,6)$.

Exercise 0A.4

Which of the sets in Graph 0.3 is/are not convex?

Answer: Only the set in (a) is convex.

Exercise 0A.5

Can you use points A and B to arrive at the same value for the slope? What about the points D and F or the points D and C?

Answer: From A to B, we have a positive rise of 20 and a negative run of 10 — giving us a slope of $-20/10 = -2$. From D to F, we have a positive rise of 2 and a negative run of 1 — giving us a slope of $-2/1 = -2$. From D to C, we have a positive rise of 4 and a negative run of 2 — giving us a slope of $-4/2 = -2$.

Exercise 0A.6

Is the shaded set in Graph 0.4 a convex set?

Answer: Yes.

Exercise 0A.7

Suppose the blue line in Graph 0.4 had a kink in it. This kink could point "inward" (i.e. toward to origin) or "outward" (i.e. away from the origin). For which of these would the shaded area underneath the kinked line become a non-convex set?

Answer: If the kink points "inward", we get a graph like panel (b) in Graph 2.4 of the text. Connecting B and A in that graph gives us a line segment that lies fully above the set defined by the kinked lines — implying the set below is non-convex. If the kink points "outward", we get a graph like panel (a) of Graph 2.4 in the text. The set that lies underneath is then convex.

Exercise 0A.8

Check to see that the other intercepts (at B and A) are correctly labeled based on equation (0.4).

Answer: Setting $z = 0$ and $y = 0$, the equation gives us $4x = 40$ which implies $x = 10$ — the intercept at A. Setting $x = 0$ and $z = 0$, the equation gives us $2y = 40$ which implies $y = 20$ — the intercept at B.

Exercise 0A.9

Is the plane in Graph 0.5 a convex set?

Answer: Yes.

Exercise 0A.10

Given the equation (0.4) that describes the 3-dimensional plane, what is the equation that describes the magenta line segment which intersects with the plane in panel (a)?

Answer: At that slice, $z = 10$. Setting z equal to 10 in the equation $4x+2y+z = 40$, we get $4x + 2y + 10 = 40$ or $4x + 2y = 30$. Re-writing this in terms of y, we get the equation $y = 15 - 2x$, an equation with vertical intercept of 15 and slope of -2.

Exercise 0A.11

Suppose I like eating steak and will eat more steak as my income goes up. Which way will my demand curve for steak shift as my income increases? Can you think of any goods for which my demand curve might shift in the other direction as my income increases?

Answer: As income increases, the demand curve for steak will shift to the right (or "down"). For some goods, our consumption might decrease as our income increases. For instance, perhaps we buy less pasta as we get richer — implying our demand curve for pasta shifts to the left (or "up") as income increases.

Exercise 0A.12

Coffee and sugar are *complements* for me in the sense that I use sugar in my coffee. Can you guess which way my demand curve for coffee will shift as the price of sugar increases?

Answer: When the price for sugar increases, I will buy *less* sugar as I slide up the demand curve for sugar. Since sugar and coffee are complements, I will also buy less coffee — implying that my demand curve for coffee shifts to the left (or "up").

Exercise 0A.13

Ice tea and coffee are *substitutes* for me in the sense that I like both of them but will only drink a certain total amount of liquids. Can you guess which way my demand curve for coffee will shift as the price of iced tea increases?

Answer: As the price of iced tea increases, I slide up on the demand curve for iced tea, implying I will buy less iced tea. Since iced tea and coffee are substitutes for me, I will likely buy more coffee — implying my coffee consumption goes up. So my demand curve for coffee will shift to the right (or "down").

Exercise 0A.14

How would the supply curve for a firm shift if the general wage rate in the economy increases?

Answer: The supply curve would shift to the left (or "up").

Exercise 0A.15

Would you expect the supply curve for a firm that produces x to shift when the price of some other good y (that is not used in the production of x) increases?

Answer: The increase in the price of y is unrelated to the production costs for x — and so the supply curve would not shift.

Exercise 0A.16

If the supply curve depicts the supply curve for a *market* composed of many firms, we may also see shifts in the supply curve that arise from the *entry* of new firms or the *exit* of existing firms. How would the market supply curve shift as firms enter and exit?

Answer: As firms enter, the market supply curve would shift to the right (or "down"), and as firms exit, the market supply curve would shift to the left (or "up").

Exercise 0A.17

Is consumer 2's consumption also more elastic than consumer 1's when price falls?

Answer: Yes, when price falls, the green consumer 2 will increase consumption more than the magenta consumer 1.

Exercise 0A.18

Do slopes similarly change if we measure price differently — i.e. if we measure price in euros instead of dollars?

Answer: Yes, as we change the units we use on the vertical axis, slopes will change just as they do if we change units on the horizontal axis.

Exercise 0A.19

In panel (b) of Graph 0.9, the supply curves of two producers who both produce x^* at the price p^* are illustrated. Which producer is more price elastic when price increases? What about when price decreases?

Answer: The green producer 2 is more price elastic as price increases — raising output to x'' when price increases to p' while the blue producer 1 only raises output to x'. Producer 2 is similarly more price elastic when price falls.

Exercise 0A.20

How much does the consumer spend when price is \$50? How much does she spend when price increases to \$100?

Answer: When price is \$50, the consumer spends $50(700) = \$35,000$.When price is \$100, the consumer spends $100(600) = \$60,000$.

Exercise 0A.21

What is the size of the blue shaded area in panel (a)? What about the magenta area? Is the difference between the magenta and the blue area the same as the increase in spending you calculated in exercise 0A.20?

Answer: The blue shaded area has height of 50 and length of 100 — implying a size of 5,000. The magenta area has height of 50 and length of 600 — implying a size of 50(600)=30,000. The difference between the magenta and the blue areas is therefore 25,000. In the previous exercise we concluded that the consumer spends $35,000 at the lower price and $60,000 at the higher price — a difference of $25,000.

Exercise 0A.22

A *price ceiling* is a government-enforced maximum legal price. In order for such a price ceiling to have an impact on the price at which goods are traded, would it have to be set above or below the equilibrium price p^*?

Answer: If the price ceiling is set above p^*, the equilibrium price p^* is legal — and so the equilibrium is undisturbed. But if the price ceiling is set below p^*, the equilibrium price p^* is no longer legal — implying that the price ceiling has an impact.

Exercise 0A.23

If a price ceiling changes the price at which goods are traded, would you expect a "shortage" or a "surplus" of goods to emerge? How would the magnitude of the shortage or surplus be related to the price elasticity of demand?

Answer: A price ceiling that has an impact is set below p^* — where the quantity demanded read off the magenta demand curve is greater than the quantity supplied (read off the blue supply curve). Thus, more is demanded than supplied at the price ceiling — implying a shortage.

Exercise 0A.24

A *price floor* is a government-enforced minimum legal price. Repeat the previous two questions for a price floor instead of a price ceiling.

Answer: If the price floor is set below p^*, it has no impact because p^* is still legal. But if the price floor is set above p^*, the equilibrium price p^* is no longer legal — implying the price floor has an impact on the market. At a price floor above p^*, the quantity demanded (read off the magenta demand curve) is less than the quantity supplied (read off the blue supply curve) — implying that more is supplied than demanded. Thus, we have a surplus.

Exercise 0A.25

Can you come to similar conclusions about decreases in market prices by looking at Graph 0.13?

Answer: If a price decrease is accompanied by a decline in market output (as in panel (a) of Graph 0.13), the price decrease must be driven by a decrease in demand. But if the price decrease is accompanied by an increase in market output (as in panel (b) of Graph 0.13), the price decrease is driven by an increase in supply.

Exercise 0A.26

Suppose that, instead of taxing the sale of x, the government *subsidized* consumer purchases of x. Thus, consumers will be paid an amount s for each good x they buy. Can you use Graph 0.15 to determine whether firms will benefit from such a consumer subsidy — and how the per-unit benefit for firms depends on the price elasticity of demand?

Answer: Firms will benefit because the equilibrium price will increase (as shown in Graph 0.15 where demand shifts up because of the consumer subsidy). But firms will benefit more the more price inelastic the supply curve — because the price increase will be larger the less elastic the supply curve is. Put differently, more of the consumer subsidy will be passed onto producers as the supply curve becomes more price inelastic.

Exercise 0A.27

If the goal of consumer subsidies is to raise economic output in a market, will the government be more likely to succeed in markets will high or low price elasticities of demand?

Answer: The government will be more successful if the market supply curve is more price elastic (as in panel (a) of Graph 0.15) than if the supply curve is more price inelastic (as in panel (b) of Graph 0.15).

0B Answers to Within-Chapter-Exercises for Part B

Exercise 0B.1

Consider the function $f(x,y,z) = xy + z$. How would you describe this function in terms of the notation of equation (0.5)? What value does the function assign to the points (0,1,2), (1,2,1) and (3,2,4)?

Answer: $f : \mathbb{R}^3 \to \mathbb{R}^1$.

Exercise 0B.2

How would you write the expression for the set of points that lie above the function in panel (a) of Graph 0.16? Which is different from expression (0.6): the necessary or the sufficient condition?

Answer: This would be written as

$$\left\{(x,y) \in \mathbb{R}^2_+ \mid y \geq x^2\right\}. \tag{0B.2}$$

The necessary condition (which comes before the vertical line in the expression) remains the same, but the sufficient condition has changed.

Exercise 0B.3

Is this set a convex set? What about the set described in expression (0.6) and the set defined in exercise 0B.2?

Answer: Yes, this is a convex set, as is the set described in equation (0.6) of the chapter. The set defined in the previous exercise, however, is non-convex.

Exercise 0B.4

Suppose that the quantity of the good x that is demanded is a function of not only p_x and I but also p_y, the price of some other good y. How would you express such a demand function in the notation of equation (0.5)?

Answer: $f : \mathbb{R}^3 \to \mathbb{R}^1$.

Exercise 0B.5

Following on exercise 0B.4, suppose the demand function took the form $f(p_x, p_y, I) = (I/2) + p_y - 2p_x$. How much of x will the consumer demand if $I = 100$, $p_x = 20$ and $p_y = 10$?

Answer: The consumer would demand $(100/2) + 10 - 2(20) = 50 + 10 - 40 = 20$.

Exercise 0B.6

Using the demand function from exercise 0B.5, derive the demand *curve* for when income is 100 and $p_y = 10$.

Answer: Substituting $I = 100$ and $p_y = 10$ into the function, we get $f(p_x) = 50 + 10 - 2p_x = 60 - 2p_x$. Inverting this, we solve the equation $x = 60 - 2p_x$ for p_x to get $p_x = 30 - 0.5x$. This gives us the equation for the demand curve that has p_x on the vertical axis and x on the horizontal — i.e. an equation with vertical intercept of 30 and slope of -0.5.

Exercise 0B.7

Verify the last sentence of the previous paragraph.

Answer: From the demand function $f(p_x, I) = (I/2) - 10p_x$, we get $\overline{g}(p_x) = (200/2) - 10p_x = 100 - 10p_x$ when we substitute in $I = 200$.

Exercise 0B.8

On a graph with p_x and I on the lower axes and x on the vertical axis, can you graph $f(p_x, I) = (I/2) - 10p_x$? Where in your graph are the slices that hold I fixed at 100 and 200? How do these slices relate to one another when graphed on a 2-dimensional graph with p_x on the horizontal and x on the vertical axis?

Answer: The graph is a 3-dimensional plane, with x (on the vertical axis) increasing as income increases and falling as price increases. Holding I fixed, we simply get downward sloping linear demand functions that have x (on the vertical axis) fall as the price increases. And as income increases, these 2-dimensional demand functions shift out.

Exercise 0B.9

Return to exercise 0B.5 and suppose that $I = 100$ and $p_y = 10$. What is $g(p_x)$? How does it change when p_y increases to 20? How does this translate to a shift in the demand *curve*?

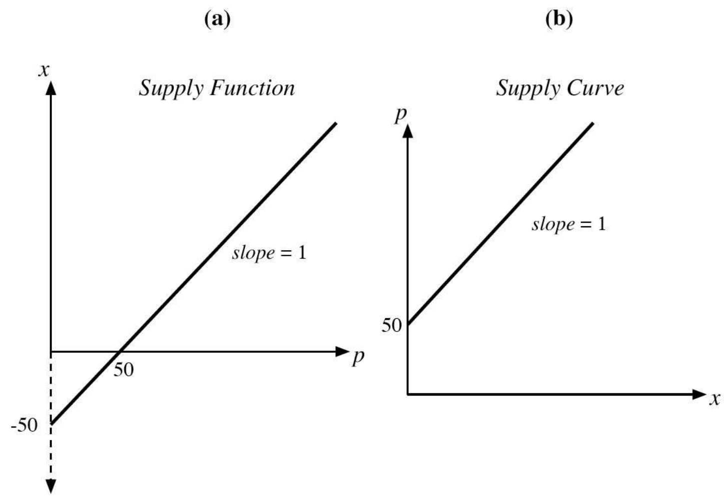
Answer: Plugging $I = 100$ and $p_y = 10$ into the function $f(p_x, p_y, I) = (I/2) + p_y - 2p_x$, we get $g(p_x) = 50 + 10 - 2p_x = 60 - 2p_x$. When p_y increases to 20, the new function becomes $g'(p_x) = 70 - 2p_x$ — which implies the slope is unchanged but the intercept shifts out. This translates to a shift of the demand curve to the right.

Exercise 0B.10

The same relationship between demand functions and demand curves exists for supply functions and supply curves. Suppose, for instance, that supply is a function of the wage rate ω and the output price p and is given by the supply function $f(\omega, p) = p - 5\omega$. Illustrate the "slice" of the supply function that holds ω fixed to

10, and then derive from it the supply curve. *(Hint: You should get two graphs analogous to Graph 0.18.)*

Answer: When ω is fixed at 10, the supply function becomes $g(p) = p - 50$ which is graphed in panel (a) of Exercise Graph 0B.10. The supply curve is the inverse — graphed in panel (b).



Exercise Graph 0B.10 : Supply Function and Supply Curve

Exercise 0B.11

What happens in your two graphs from exercise 0B.10 when ω changes to 5?

Answer: When ω is fixed at 5, the supply function becomes $g(p) = p - 25$. This shifts the supply function and curve as illustrated in Exercise Graph 0B.11.

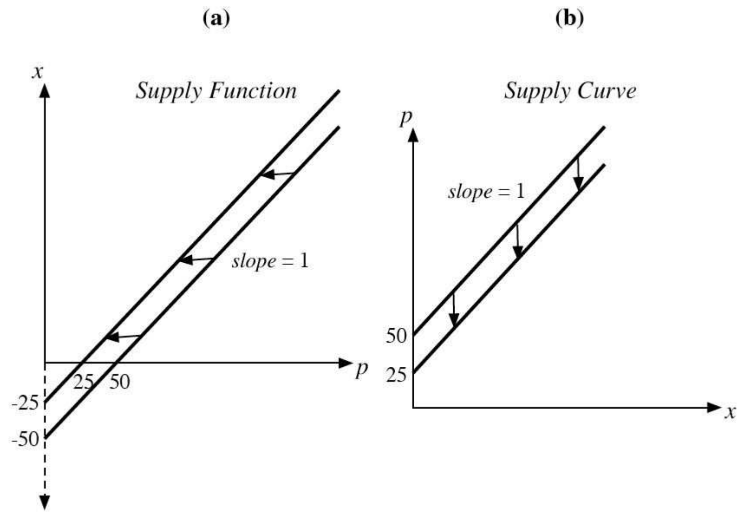
Exercise 0B.12

Can you tell how equilibrium price changes as A and B change? Can you make intuitive sense of this by linking changes in parameters to changes in Graph 0.19?

Answer: As both A and B increase, the equilibrium price rises — since both A and B appear as positive quantities in the numerator of the p^* equation. This makes intuitive sense in that an increase in A is an upward shift of the demand curve and an increase in B is an upward shift in the supply curve. Both such upward shifts will result in the intersection of D and S shifting up.

Exercise 0B.13

Can you do the same for the equilibrium quantity?



Exercise Graph 0B.11 : Shifts in Supply Function and Curve

Answer: In the equilibrium quantity equation for x^*, A enters positively but B enters negatively in the numerator. Thus, as A increases, output increases; but as B increases, output falls. This again makes intuitive sense: An increase in A is an outward shift in D — which implies a rightward shift in the intersection of S and D (and thus an increase in equilibrium output). An increase in B, on the other hand, implies an upward shift in supply which results in a leftward shift of the intersection of D and S, and thus a decrease in the equilibrium output.

Exercise 0B.14

In this Chapter 6 problem, it turns out we do not care about the value of λ. Suppose, however, we did. How would you derive it's value?

Answer: You could now plug in $x_1 = 5$ and $x_2 = 10$ into either of the first two equations and then solve for λ. In both cases, we get an approximate value of 0.03536 for λ.

Exercise 0B.15

Can you prove Rules 3 through 5 by just using the definition of exponents?

Answer: For Rule 3: x^m is the same as x multiplied m times, and x^n is the same as x multiplied n times. Thus, $x^m x^n$ must be the same as x multiplied $(m+n)$ times. For Rule 4: We are simply dividing x multiplied m times by x multiplied n times. If $m > n$, that means n of the m multiplied x's in the numerator cancel, leaving us with $(m-n)$ multiplied x's in the numerator. If $m < n$, all the x's in the numerator cancel, leaving us with $(n-m)$ multiplied x's in the denominator — i.e. $1/(x^{(n-m)})$ which can simply be written as $x^{-(n-m)}$ or $x^{(m-n)}$. For Rule 5: When we write $(x^m)^n$, we are saying that we will multiply x^m by x^m a total of n times. So we are left with x^{mn}.

Exercise 0B.16

Simplify the following: $\frac{x^{3/2}y^{1/2}}{x^{1/2}y^{-1/2}}$.

Answer: This expression reduces to xy as we subtract the exponents in the denominator from those in the numerator.

Exercise 0B.17

Simplify the following: $\left(\frac{x^{4/5}y^2}{x^{-1/2}y^4}\right)^{-2}$.

Answer: Subtracting exponents in the denominator from those in the numerator inside the parentheses, we get $\left(x^{13/10}y^{-2}\right)^{-2}$, and multiplying the exponents inside the parentheses by -2, we end up with $x^{-13/5}y^4$.

Exercise 0B.18

Solve the following quadratic equation by factoring: $2x^2+2x-12=0$.

Answer: First, we can divide both sides of the equation by 2 to get $x^2+x-6=0$. We can then factor this to get $(x+3)(x-2)=0$ — implying the solutions $x=-3$ and $x=2$.

Exercise 0B.19

An equation is still quadratic if $b=0$. Solve the following quadratic equation by factoring: $x^2-4=0$.

Answer: This is factored as $(x-2)(x+2)=0$ — giving us the solutions $x=2$ and $x=-2$.

Exercise 0B.20

Does the quadratic equation $x^2+4x+4=0$ have two solutions?

Answer: This equation factors to $(x+2)(x+2)=0$, which gives us $x=-2$ — a single solution.

Exercise 0B.21

Use the quadratic formula to verify your answer to exercises 0B.18, 0B.19 and 0B.20.

Answer: In exercise 0B.18, $a=2$, $b=2$ and $c=-12$. Thus, the quadratic formula gives the solutions

$$\frac{-2+\sqrt{2^2-4(2)(-12)}}{2(2)}=\frac{-2+\sqrt{100}}{4}=2, \qquad (0B.21.i)$$

and

$$\frac{-2-\sqrt{2^2-4(2)(-12)}}{2(2)} = \frac{-2-\sqrt{100}}{4} = -3. \qquad (0B.21.ii)$$

In exercise 0B.19, $a = 1$, $b = 0$ and $c = -4$. The quadratic formula then gives solutions

$$\frac{\sqrt{-4(1)(-4)}}{2(1)} = \frac{4}{2} = 2, \qquad (0B.21.iii)$$

and

$$\frac{-\sqrt{-4(1)(-4)}}{2(1)} = \frac{4}{2} = -2. \qquad (0B.21.iv)$$

And in exercise 0B.20, $a = 1$, $b = 4$ and $c = 4$. The quadratic formula then implies solutions

$$\frac{-4+\sqrt{4^2-4(1)(4)}}{2(1)} = \frac{-4}{2} = -2 \qquad (0B.21.v)$$

and

$$\frac{-4-\sqrt{4^2-4(1)(4)}}{2(1)} = \frac{-4}{2} = -2. \qquad (0B.21.vi)$$

Exercise 0B.22

Solve the quadratic equation $3x^2 + 8x + 4 = 0$.
Answer: With $a = 3$, $b = 8$ and $c = 4$, we get

$$\frac{-8+\sqrt{8^2-4(3)(4)}}{2(3)} = \frac{-8+\sqrt{16}}{6} = -\frac{4}{6} = -\frac{2}{3} \qquad (0B.22.i)$$

and

$$\frac{-8-\sqrt{8^2-4(3)(4)}}{2(3)} = \frac{-8-\sqrt{16}}{6} = -\frac{12}{6} = -2. \qquad (0B.22.ii)$$

Exercise 0B.23

Re-write the following as an expression with a single ln term: $\ln(2x) + \ln(y) - \ln(x)$.
Answer: $\ln(2x^2 y)$.

Exercise 0B.24

Simplify the following: $\ln e^{(x^2+xy)}$.
Answer: $x^2 + xy$.

Exercise 0B.25

Solve the following for x: $\ln e^{(x^2+4)} = \ln e^{4x}$.
Answer: The equation can be simplified to read $x^2+4 = 4x$ and can be written in the form $x^2-4x+4 = 0$. This in turn can be factored and written as $(x-2)(x-2) = 0$ which implies $x = 2$.

Exercise 0B.26

Show that Rule 2 follows from Rule 3.
Answer: Employing Rule 3 with $n = 1$, we get $\frac{d(\gamma x)}{dx} = (1)\gamma x^{(1-1)} = \gamma$.

Exercise 0B.27

Differentiate the following with respect to x: $f(x) = 3x^3 + 2x^2 + x + 4$.
Answer: We get

$$\frac{df(x)}{dx} = 9x^2 + 4x + 1. \qquad (0B.27)$$

Exercise 0B.28

Use the product rule to differentiate the function $(x+3)(2x-2)$ with respect to x.
Answer: We get

$$(1)(2x-2) + (x+3)(2) = 2x-2+2x+6 = 4x+4. \qquad (0B.28)$$

Exercise 0B.29

Multiply the function in exercise 0B.28 out and solve for its derivative with respect to x without using the product rule. Do you get the same answer?
Answer: Multiplying through we get $2x^2 + 4x - 6$. Taking the derivative, we get $4x + 4$ as in the previous exercise.

Exercise 0B.30

Use the quotient rule to solve for the first derivative of $\frac{(x+3)}{2x-2}$.
Answer: The quotient rule gets us

$$\frac{(1)(2x-2)-(x+3)(2)}{(2x-2)^2} = \frac{2x-2-2x-6}{(2x-2)^2} = -\frac{8}{(2x-2)^2} \qquad (0B.30)$$

Exercise 0B.31

Multiply out $(x^3-2)^2$ and take the derivative without using the chain rule. Do you get the same result?

Answer: Multiplying out $(x^3-2)^2$, we get $(x^3-2)(x^3-2) = x^6-4x^3+4$. Taking the derivative, we get $6x^5-12x^2$ as in the chapter when we used the chain rule.

Exercise 0B.32

In exercise 0B.30, you used the quotient rule to evaluate the derivative of $\frac{(x+3)}{2x-2}$. This function could alternatively be written as $(x+3)(2x-2)^{-1}$. Can you combine the product rule and the chain rule to solve again for the derivative with respect to x? Is your answer the same as in exercise 0B.30?

Answer: Combining the product and chain rule, we get

$$(1)(2x-2)^{-1}+(x+3)(-1)(2x-2)^{-2}(2) = \frac{1}{(2x-2)} - \frac{2(x+3)}{(2x-2)^2} \qquad (0B.32.i)$$

which can then be simplified as

$$\frac{2x-2}{(2x-2)^2} - \frac{2x+6}{(2x-2)^2} = -\frac{8}{(2x-2)^2}, \qquad (0B.32.ii)$$

the same answer as we previously derived using just the quotient rule.

Exercise 0B.33

Derive the derivative of $f(x) = \ln(x^2)$ using the chain rule. Then, use Rule 1 from our logarithm section to re-write $f(x)$ in such a way that you don't have to use the chain rule. Check whether you get the same answer.

Answer: Using the chain rule, we get

$$\frac{1}{x^2}(2x) = \frac{2}{x}. \qquad (0B.33.i)$$

We can also write $\ln(x^2)$ as $\ln x + \ln x$, and the derivative of this (without using the chain rule) is

$$\frac{1}{x} + \frac{1}{x} = \frac{2}{x}. \qquad (0B.33.ii)$$

Exercise 0B.34

We have seen that price elasticities vary along linear demand curve. Consider now the demand curve given by the equation $x(p) = \alpha/p$. Is the same true for this demand function?

Answer: The price elasticity now is

$$\varepsilon_d = \frac{dx(p)}{dp}\frac{p}{x(p)} = \frac{-\alpha}{p^2}\left(\frac{p}{\alpha/p}\right) = -\frac{\alpha}{p^2}\left(\frac{p^2}{\alpha}\right) = -1. \qquad (0B.34)$$

Thus, for the demand function $x(p) = \alpha/p$, the price elasticity of demand is constant at -1 throughout.

Exercise 0B.35

Verify that this is correct.

Answer: Applying the product and chain rules, we get

$$\frac{df(x)}{dx} = \frac{1}{2}x^{-1/2}(20-2x)^{1/2} - x^{1/2}\left(-\frac{1}{2}\right)(20-2x)^{-1/2}(-2) \qquad (0B.35)$$

which simplifies to $\frac{1}{2}x^{-1/2}(20-2x)^{1/2} - x^{1/2}(20-2x)^{-1/2}$.

Exercise 0B.36

Evaluate the following statement: A derivative of zero is a necessary but not a sufficient condition for us to identify a maximum of a function.

Answer: The derivative of the function will be zero at every maximum — so the condition is a necessary condition for a maximum. But not all points at which the derivative is zero represent a maximum (because the derivative is also zero at a minimum). It is therefore not a sufficient condition for a maximum.