CHAPTER

10

Consumer Surplus and Dead Weight Loss

This chapter brings together all the concepts from consumer theory — and in the process illustrates the difference between (uncompensated) demand and marginal willingness to pay (or compensated demand). We already developed (uncompensated) demand in Chapter 9 where we brought our A, B and C points from the consumer diagram into a new graph that had quantity on the horizontal and price on the vertical axis. But we only used the A and C points — with B playing no real role other than allowing us to see how big the substitution effect versus the income effect was. *Compensated* demand curves arise from *compensated* budgets, and thus we now turn our attention to point B. In particular, we see that the compensated demand curve connects A and B (rather than A and C) and that this curve can also be viewed as our *marginal willingness to pay* curve. (The same distinction between compensated and uncompensated curves can be made for the supply curves that emerge from the worker and saver diagrams — but we leave that until later (Chapter 19) in the text.) We then find that it is this marginal willingness to pay curve that we can use to measure consumer surplus and changes in consumer welfare, not the uncompensated demand curve from Chapter 9. The two curves are the same only in one very special case.

Chapter Highlights

The main points of the chapter are:

1. We can quantify in dollar terms the value people place on participating in a market — and we define this as the **consumer surplus**. We can similarly quantify the value people place on either getting a lower price or having to accept a higher price.

2. To do this, we need to know the **marginal willingness to pay** for each of the goods a consumer consumes — and this is closely related to the changing MRS along the indifference curve on which the consumer operates.

3. The **marginal willingness to pay** is derived from a single indifference curve — and thus **incorporates only substitution effects**. It is the **same as the uncompensated demand curve only if there are no income effects** — only if tastes are quasilinear in the good we are modeling.

4. **Price-distorting taxes (and subsidies) are inefficient** in the consumer model **to the extent to which they give rise to substitution effects**. Therefore, the inefficiency goes away if the degree of substitutability between goods is zero.

5. The **deadweight loss** from taxes (or subsidies) can be measured as a **distance in the consumer diagram** or as **an area along the marginal willingness to pay curve**.

6. To say that a policy is **inefficient** is the same as to say that there exists in principle a way to compensate those who lose from the policy with the winnings from those who gain. That is not the same as saying that such a policy exists in practice.

7. If you are reading the B-part of the chapter: the **compensated (Hicksian) demand function** is derived from the expenditure minimization problem while the uncompensated budget is derived from the utility maximization problem. The **Slutzky equation** then illustrates the relationship of the slope of the uncompensated demand curve to the slope of the compensated demand (or marginal willingness to pay) curve. This is one of several links — summarized in our **duality** picture at the end of the chapter — between concepts emerging from the utility maximization and the expenditure minimization problems.

10A Solutions to Within-Chapter Exercises for Part A

Exercise 10A.1

As a way to review material from previous chapters, can you identify assumptions on tastes that are *sufficient* for me to know for sure that my indifference curve will be tangent to the budget line at the optimum?

Answer: This will be true so long as, in addition to the usual assumptions about tastes, we assume that both goods are “essential” as defined in Chapter 5. This implies that indifference curves never cross the axes — and thus corner solutions are not possible.

Exercise 10A.2

Demonstrate that own-price demand curves are the same as marginal willingness to pay curves for goods that can be represented by quasilinear tastes.

Answer: This is easy to see once you realize that points B and C in the lower panel of Graph 10.2 in the text will lie on top of one another when there are no income effects — i.e. when tastes are quasilinear in gasoline. C lies to the left of B when gasoline is normal, to the right when gasoline is inferior — so the only time they lie exactly on top of one another is if gasoline is borderline normal/inferior.

Exercise 10A.3

Using the graphs in Graph 9.2 of the previous chapter, determine under what condition own price demand curves are steeper and under what conditions they are shallower than marginal willingness to pay curves.

Answer: Once we realize that $MWTP$ curves connect A and B in the lower panels of the Graph 9.2 in the chapter, it is easy to see the relationship by simply connecting these and forming the $MWTP$ curves. In panel (a) where x is a normal good, $MWTP$ is steeper than own-price demand; in panel (b) where x is (regular) inferior, $MWTP$ is shallower than own-price demand.

Exercise 10A.4

What does the $MWTP$ or compensated demand curve look like if the two goods are perfect complements?

Answer: When the two goods are perfect complements, the entire change in behavior from a price change is an income effect — with no substitution effect. Since $MWTP$ curves only incorporate substitution effects, this implies that the $MWPT$ curve has to be perfectly vertical. This is illustrated in Exercise Graph 10A.4.

Exercise 10A.5

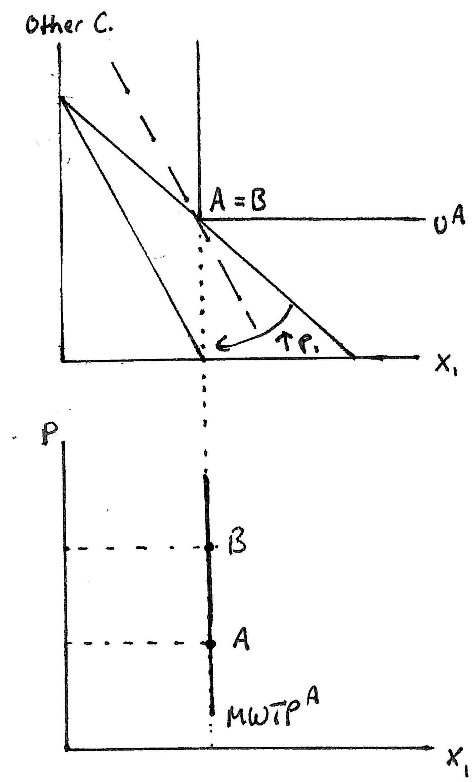
How would Graph 10.4a change if x_1 were an inferior rather than a normal good?

Answer: Bundle B would then lie to the right rather than the left of A — causing the own-price demand curve corresponding to the lower income (I^B) to lie to the *right* of the demand curve corresponding to the higher income (I^A). When a good is inferior, the demand curve therefore shifts out when income falls and in when income increases, the reverse of what is true when the good is normal.

Exercise 10A.6

How would Graph 10.4b change if x_1 were inferior rather than normal?

Answer: If x_1 is inferior, then B must again lie to the *right* of A (just as in the previous exercise). Thus, the marginal willingness to pay curve corresponding to



Exercise Graph 10A.4 : *MWTP* for Perfect Complements

higher utility (i.e. u^A) must lie to the left of the *MWTP* curve corresponding to lower utility (i.e. u^B).

Exercise 10A.7

On the lower panel of Graph 10.5b, where does the *MWTP* curve corresponding to the indifference curve that contains bundle *C* lie?

Answer: It lies exactly on top of the $MWTP(u^A)$ curve that is depicted in the graph. Since there are no income effects, every *MWTP* curve must lie on the uncompensated demand curve that now incorporates only substitution effects (because x_1 is quasilinear).

Exercise 10A.8

How do the upper and lower panels of Graph 10.5a change when gasoline is an inferior good?

Answer: If gasoline is an inferior good, then C lies to the right (instead of the left) of B in the top graph — which causes it to lie to the right (instead of left) of B in the lower panel. This implies that the uncompensated demand curve that connects A and C is now steeper (rather than shallower) than the compensated demand (or $MWTP$) curve that connects A and B.

Exercise 10A.9

Can you think of a scenario under which a consumer does not change her consumption of a good when it is taxed but there still exists an inefficiency from taxation?

Answer: This would require the existence of a substitution effect (which causes the inefficiency) that is exactly offset by an income effect. This is illustrated in Exercise Graph 10A.9 and occurs if a good was borderline regular inferior/Giffen. In the top panel, A represents the bundle consumed under the higher (tax-inclusive) price while bundle C represents the bundle consumed at the lower (before-tax) price. The two bundles contain the same quantity of x_1 — so behavior relative to consumption of the taxed good does not change. However, there is clearly an underlying substitution effect that is masked by an offsetting income effect. This substitution effect causes the tax revenue (labeled TR) to be less than what could have been raised by a lump sum tax that makes the consumer no worse off (labeled L). The difference between the two is the deadweight loss.

The lower panel of the graph illustrates the relevant compensated demand curve along which this deadweight loss can be measured (as described in the next section.) Note that the uncompensated demand curve would connect A and C — and would be perfectly vertical with no change in behavior.

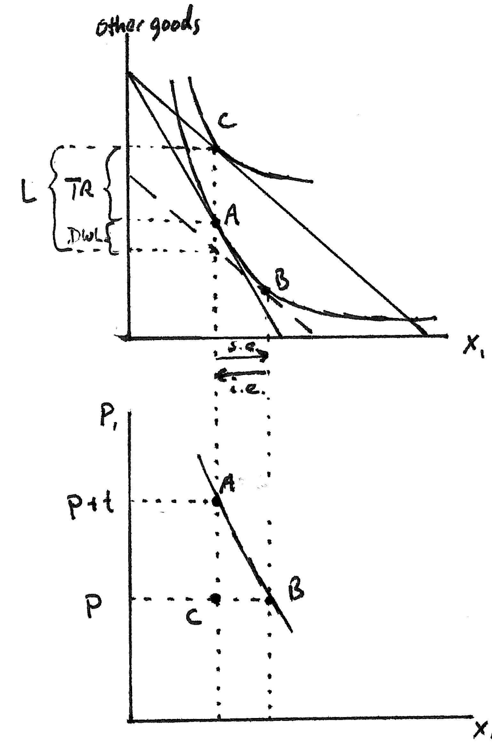
Exercise 10A.10

On a graph with consumption on the vertical axis and leisure on the horizontal, illustrate the deadweight loss of a tax on all consumption (other than the consumption of leisure).

Answer: This is illustrated in Exercise Graph 10A.10 where A is the consumption bundle after the tax is imposed and TR is the tax revenue collected from the tax. Were we to employ a non-distortionary tax instead, we could shift the before-tax budget in parallel all the way to the tangency at B and make the consumer no worse off. Through such a lump-sum tax, we could raise L in revenue — more than we raise under the distortionary tax. The difference between L and TR is the deadweight loss DWL.

Exercise 10A.11

Using Graph 10.8a, verify that the relationship between own price demand and marginal willingness to pay is as depicted in panels (a) through (c) of Graph 10.9.

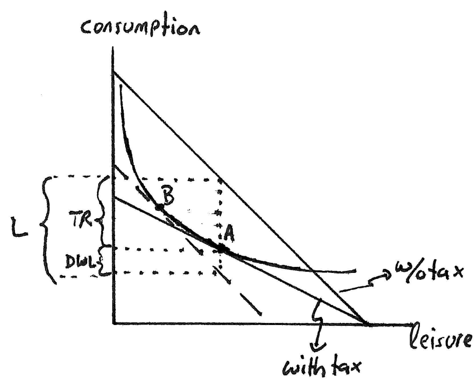


Exercise Graph 10A.9 : DWL without a Change in Behavior

Answer: In order to plot the uncompensated demand curve into the graph that has the compensated demand (or $MWTP$) curve, all we have to do is determine where C lies — i.e. where would the consumer consume in the absence of the distortionary and the lump sum tax. Starting at B, this simply means asking where the consumer would consume if she had more income (because the budget that is tangent at B is parallel to the no-tax budget). If h is a normal good, then such an increase in income implies more consumption of h — which would put C to the right of B. If h is quasilinear, then additional income does not affect consumption of h and would thus put C at the same level of h as B. Finally, if housing is inferior, an increase in income causes less consumption of h — thus causing C to lie to the left of B. This verifies the placement of C relative to B in the three panels of the graph in the text.

Exercise 10A.12

The two proposals also result in different levels of tax revenue. Which proposal actually results in higher revenue for the government? Does this strengthen



Exercise Graph 10A.10 : DWL from a tax on all Consumption

or weaken the policy proposal "to broaden the base and lower the rates"?

Answer: The tax proposal that imposes $2t$ on the single family housing market results in tax revenues equal to $2t$ times q_{p+2t}. The alternative proposal raises t times q_{p+t} in each market, or $2t$ times q_{p+t}. Thus, tax revenue under the latter proposal is a bit larger, t times $(q_{p+t} - q_{p+2t})$ to be exact. The proposal with less deadweight loss therefore also produces more revenue — which strengthens the advice to broaden the base and lower the rates.

10B Solutions to Within-Chapter Exercises for Part B

Exercise 10B.1

In Graph 10.5b we illustrated that $MWTP$ curves and own price demand curves are the same when tastes are quasilinear. Suppose tastes can be modeled with the quasilinear utility function $u(x_1,x_2) = \alpha \ln x_1 + x_2$. Verify a generalization of the intuition from Graph 10.5b — that demand functions and compensated demand functions are identical for x_1 in this case.

Answer: Solving the utility maximization problem

$$\max_{x_1,x_2} \alpha \ln x_1 + x_2 \text{ subject to } p_1x_1 + p_2x_2 = I, \tag{10B.1.i}$$

we get the (uncompensated) demand function $x_1 = \alpha p_2/p_1$. We get exactly the same function when we solve the expenditure minimization problem

$$\min_{x_1,x_2} p_1x_1 + p_2x_2 \text{ subject to } \alpha \ln x_1 + x_2 = u. \tag{10B.1.ii}$$

Exercise 10B.2

Verify the solutions given in equations (10.8).

Answer: Solving the maximization problem

$$\min_{x_1,x_2} p_1x_1 + p_2x_2 \text{ subject to } x_1^{\alpha}x_2^{(1-\alpha)} = u, \tag{10B.2.i}$$

we get first order conditions

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial x_1} &= p_1 - \alpha x_1^{\alpha-1}x_2^{(1-\alpha)} = 0, \\ \frac{\partial \mathscr{L}}{\partial x_2} &= p_2 - (1-\alpha)x_1^{\alpha}x_2^{-\alpha} = 0. \end{aligned} \tag{10B.2.ii}$$

Solving these for x_1, we get

$$x_1 = \frac{\alpha p_2 x_2}{(1-\alpha)p_1}. \tag{10B.2.iii}$$

Plugging this into the constraint $u = x_1^{\alpha}x_2^{(1-\alpha)}$, we get

$$u = \left(\frac{\alpha p_2 x_2}{(1-\alpha)p_1}\right)^{\alpha} x_2^{(1-\alpha)} = \left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{\alpha} x_2. \tag{10B.2.iv}$$

Solving this equation for x_2, we then get

$$x_2 = \left(\frac{(1-\alpha)p_1}{\alpha p_2}\right)^{\alpha} u \tag{10B.2.v}$$

which is then equal to $h_2(p_1, p_2, u)$ as derived in the text. Substituting this back into (10B.2.iii), we also get

$$x_1 = \left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{(1-\alpha)} u = h_1(p_1, p_2, u). \tag{10B.2.vi}$$

Exercise 10B.3

Verify the solutions given in equations (10.8) and (10.9).

Answer: Plugging the demand functions for x_1 and x_2 into the Cobb-Douglas utility function $u(x_1, x_2) = x_1^{\alpha} x_2^{(1-\alpha)}$, we get

$$\left(\frac{\alpha I}{p_1}\right)^{\alpha} \left(\frac{(1-\alpha)I}{p_2}\right)^{(1-\alpha)} = \frac{I\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}{p_1^{\alpha} p_2^{(1-\alpha)}} \tag{10B.3.i}$$

which is equal to the indirect utility function $V(p_1, p_2, I)$.

Similarly, multiplying the compensated demand functions by prices and adding them, we get

$$p_1\left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{(1-\alpha)} u + p_2\left(\frac{(1-\alpha)p_1}{\alpha p_2}\right)^{\alpha} u = \frac{u p_1^{\alpha} p_2^{(1-\alpha)}}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}} \tag{10B.3.ii}$$

which is equal to the expenditure function $E(p_1, p_2, u)$.

Exercise 10B.4

Verify that (10.11) and (10.12) are true for the functions that emerge from utility maximization and expenditure minimization when tastes can be modeled by the Cobb-Douglas function $u(x_1, x_2) = x_1^{\alpha} x_2^{(1-\alpha)}$.

Answer: Plugging the expenditure function into the (uncompensated) demand function for x_1, we get

$$\begin{aligned} x_1(p_1, p_2, E(p_1, p_2, u)) &= \frac{\alpha}{p_1}\left(\frac{u p_1^{\alpha} p_2^{(1-\alpha)}}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}\right) \\ &= \left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{(1-\alpha)} u = h_1(p_1, p_2, u). \end{aligned} \tag{10B.4.i}$$

The same holds if we substitute the expenditure function into the (uncompensated) demand function for x_2.

Also, if we substitute the indirect utility function into the compensated demand function for x_1, we get

$$h_1(p_1,p_2,V(p_1,p_2,I)) = \left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{(1-\alpha)}\left(\frac{I\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}{p_1^{\alpha}p_2^{(1-\alpha)}}\right) = \frac{\alpha I}{p_1} = x_1(p_1,p_2,I) \quad (10B.4.ii)$$

and again the same holds if we substitute the indirect utility function into the compensated demand function for x_2.

Exercise 10B.5

Verify that the equations in (10.19) are correct for the Cobb-Douglas utility function $u(x_1,x_2) = x_1^{\alpha}x_2^{(1-\alpha)}$.

Answer: We previously derived the demand function for x_1 under these tastes as $x_1(p_1,p_2,I) = \alpha I/p_1$. From this, we can derive

$$T = tx_1(p_1+t,p_2,I) = \frac{t\alpha I}{p_1+t}. \quad (10B.5.i)$$

We also previously derived the indirect utility and expenditure functions for these tastes as

$$V(p_1,p_2,I) = \frac{I\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}{p_1^{\alpha}p_2^{(1-\alpha)}} \text{ and } E(p_1,p_2,u) = \frac{up_1^{\alpha}p_2^{(1-\alpha)}}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}. \quad (10B.5.ii)$$

The lump sum tax L is, as just derived in the text, $L = I - E(p_1,p_2,V(p_1+t,p_2,I))$. Put in terms of the expressions derived previously, this implies

$$L = I - \left(\frac{p_1^{\alpha}p_2^{(1-\alpha)}}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}\right)\left(\frac{I\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}{(p_1+t)^{\alpha}p_2^{(1-\alpha)}}\right) = I - I\left(\frac{p_1}{p_1+t}\right)^{\alpha}. \quad (10B.5.iii)$$

Exercise 10B.6

Verify that the numbers calculated in the previous paragraph are correct.

Answer: Plugging in the appropriate values, we get

$$T = \frac{t\alpha I}{p_1+t} = \frac{(2.5)(0.25)(100,000)}{10+2.5} = 5,000 \quad (10B.6.i)$$

and

$$L = I - I\left(\frac{p_1}{p_1+t}\right)^{\alpha} = 100,000 - 100,000\left(\frac{10}{10+2.5}\right)^{0.25} = 5,425.84. \quad (10B.6.ii)$$

Subtracting T from L gives us a dead weight loss of approximately \$426.

Exercise 10B.7

What shape must the indifference curves have in order for the second derivative of the expenditure function with respect to price to be equal to zero (and for the "slice" of the expenditure function in panel (b) of Graph 10.14 to be equal to the blue line)?

Answer: The indifference curves would have a sharp kink — as those for perfect complements. This is because it is the substitution effect that is creating the concavity of the expenditure function slice in the graph — and that strict concavity goes away only when the substitution effect goes away. And the substitution effect only goes away if the curvature of the indifference curve goes away. Put differently, the "naive" expenditure function that says that consumer will choose the same bundle to reach the same indifference curve when prices change is literally correct if the consumer does not substitute — i.e. if the goods are perfect complements.

Exercise 10B.8

In a 2-panel graph with the top panel containing an indifference curve and the lower panel containing a compensated demand curve for x_1 derived from that indifference curve, illustrate the case when the inequality in equation (10.39) becomes an equality? (*Hint*: Remember that our graphs of compensated demand curves are graphs of the *inverse* of a slice of the compensated demand functions, with a slope of 0 turning into a slope of infinity.)

Answer: The compensated demand curve will be perfectly vertical (which is equivalent to saying $\partial h_i(p_1, p_2, u)/\partial p_i = 0$) if and only if there are no substitution effects — i.e. if the indifference curve has a sharp kink as in the case of perfect complements.

10C Solutions to Odd Numbered End-of-Chapter Exercises

Exercise 10.1

Consider a good x_1 in a model where a consumer chooses between x_1 and a composite good x_2.

A: *Explain why the following either cannot happen or, if you think it can happen, how:*

(a) *Own price demand for a good is perfectly vertical but taxing the good produces a dead weight loss.*

Answer: Panel (a) of Exercise Graph 10.1 illustrates two different prices — p_1 and p_1' — for x_1, with $p_1' > p_1$. (Thus, p_1' would be the tax-inclusive price.) The optimal consumption of x_1 is x_1^A when price is p_1' as well as when price is p_1 — thus, price has no impact on the demand for x_1. In the lower graph of panel (a), this translates into a perfectly vertical demand curve. However, there is still a substitution effect that gives rise to the deadweight loss. This deadweight loss can be either seen as the vertical distance labeled DWL in the upper graph — or as the shaded triangle in the lower graph. Thus, we have an example where the demand is vertical but there is still a deadweight loss from taxation.

(b) *Own price demand is downward sloping (not vertical) and there is no deadweight loss from taxing the good.*

Answer: In panel (b) of Exercise Graph 10.1, we again illustrate two different prices for x_1 but this time model the tastes as those for perfect complements. The demand for x_1 is higher at the lower price — where x_1^C is chosen — than at the higher price — where x_1^A is chosen. This translates into a downward sloping demand curve in the lower graph. But now there is no substitution effect — which leads to a vertical $MWTP$ curve and the disappearance of the deadweight loss triangle. Put differently, the tax revenues raised through this tax are exactly the same that we could raise through a lump sum tax that makes the consumer equally well off. We therefore have an example of a downward sloping demand curve with no deadweight loss from taxation.

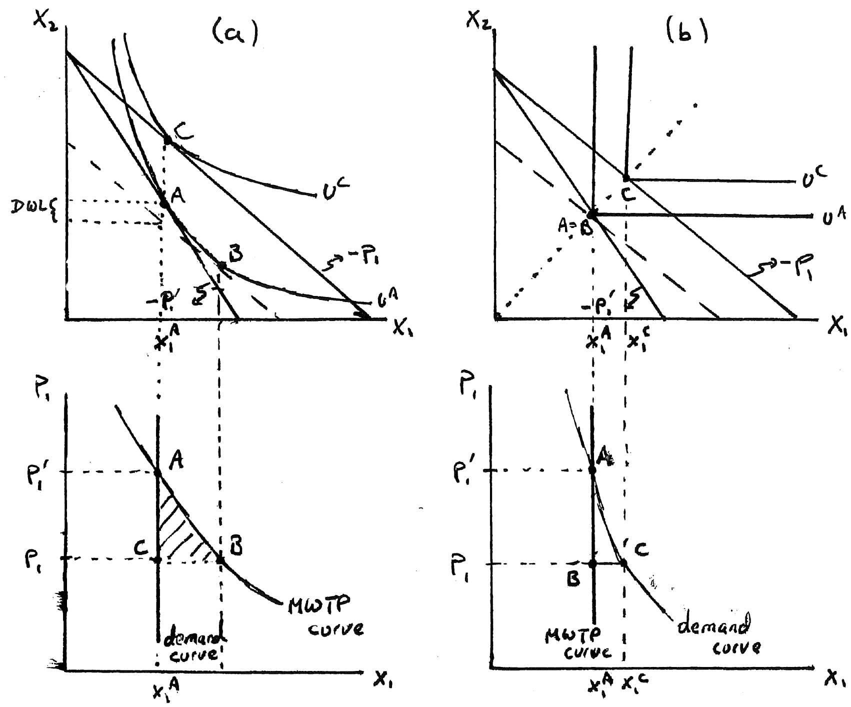
B: *Now suppose that the consumer's tastes can be summarized by the CES utility function $u(x_1, x_2) = (0.5x_1^{-\rho} + 0.5x_2^{-\rho})^{-1/\rho}$.*

(a) *Are there values for ρ that would result in the scenario described in A(a)?*

Answer: To solve for the demand for x_1, we have to solve

$$\max_{x_1,x_2} (0.5x_1^{-\rho} + 0.5x_2^{-\rho})^{-1/\rho} \text{ subject to } I = p_1x_1 + x_2. \qquad (10.1.i)$$

This gives us



Exercise Graph 10.1 : Demand and Deadweight Loss from Taxation

$$x_1 = \frac{I}{p_1 + p_1^{1/(\rho+1)}} \text{ and } x_2 = \frac{p_1^{1/(\rho+1)I}}{p_1 + p_1^{1/(\rho+1)}} = \frac{I}{p_1^{\rho/(\rho+1)} + 1}. \tag{10.1.ii}$$

Regardless of ρ, x_1 is an inverse function of p_1, implying a downward sloping demand curve. Since there is no way for CES demands to be perfectly vertical, there are no values for ρ that would result in the scenario in A(a).

(b) *Are there values for ρ that would result in the scenario described in A(b)?*

Answer: In order for there to be no deadweight loss, there cannot be a substitution effect. The only way there is no substitution effect is if the tastes are for perfect complements — i.e. the elasticity of substitution is 0. And CES utility functions have elasticity of substitution of zero only when $\rho = \infty$. Thus, in order for the scenario to work, $\rho = \infty$. (In that case, $x_1 = I/(p_1 + 1)$ — which implies the demand for x_1 falls as price increases; i.e. demand for x_1 is downward sloping.)

(c) *Would either of the scenarios work with tastes that are quasilinear in x_1?*

Answer: No. The scenario in A(a) would not work because quasilinearity in x_1 implies no income effect — which would put C in panel (a) of Exercise Graph 10.1 directly above B (rather than above A). The substitution effect then implies that A lies to the left of C — which implies a downward sloping (and not a vertical) demand curve. The scenario in A(b) won't work because quasilinear tastes have substitution effects — and thus give rise to deadweight losses from taxation.

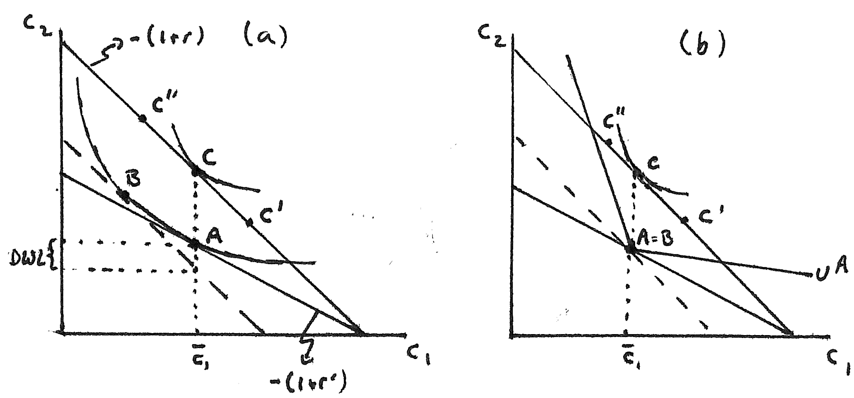
Exercise 10.3

Suppose that consumption takes place this period and next period, and consumption is always a normal good. Suppose further that income now is positive and income next period is zero.

A: *Explain why the following either cannot happen or, if you think it can happen, how:*

(a) *Savings behavior is immune to changes in the interest rate, but taxing interest income causes a dead weight loss.*

Answer: In panel (a) of Exercise Graph 10.3, we graph two budgets, one with a high interest rate r and one with a low (after-tax) interest rate r'. If tastes are such that the optimal level of consumption in period 1 (i.e. c_1) is unaffected by the interest rate, the optimal bundles A and C lie at the same level $\bar{c}_1$ (as drawn in panel (a)). Thus, a tax on interest income (which lowers the effective interest rate) causes no change in savings. At the same time, there is still a substitution effect that gives rise to a dead weight loss which is indicated as DWL on the vertical axis.



Exercise Graph 10.3 : Savings Behavior and Deadweight Loss from Interest Taxation

(b) *Savings behavior is immune to changes in the interest rate, and taxing interest income causes no dead weight loss.*

Answer: This is also in principle possible — but the indifference curve that is "tangent" at A must now have a kink sufficiently large to eliminate the substitution effect with $B = A$. This is illustrated in panel (b) of the graph. The kink cannot be as sharp as a kink for perfect complements but would need to be sufficiently sharp to eliminate the substitution effect.

(c) *Savings decreases with increases in the interest rate and there is a deadweight loss from taxation of interest.*

Answer: Savings decreases with an increase in the interest rate if A lies to the left of C — i.e. if consumption now (c_1) increases as the interest rate increases. This can be accomplished in panel (a) of Exercise Graph 10.3 by simply re-drawing the indifference curve on the higher interest rate r budget to be tangent at C'. It does not require us to change anything about the indifference curve that is tangent at A — which implies we will continue to have the same substitution effect that gives rise to the deadweight loss.

(d) *Savings increases with increases in the interest rate and there is a deadweight loss from taxation of interest.*

Answer: Savings increases with an increase in the interest rate if A lies to the right of C — i.e. if consumption now (c_1) decreases as the interest rate increases. This can be accomplished in panel (a) of Exercise Graph 10.3 by simply re-drawing the indifference curve on the higher interest rate r budget to be tangent at C''. It does not require us to change anything about the indifference curve that is tangent at A — which implies we will continue to have the same substitution effect that gives rise to the deadweight loss.

(e) *Savings decreases with an increase in the interest rate and there is no deadweight loss.*

Answer: We now have to change the optimal bundle on the high interest rate r budget in panel (b) of Exercise Graph 10.3. If savings decreases with an increase in the interest rate, A has to lie to the left of C — so that consumption now (c_1) increases as the interest rate goes up. Thus, if we change the higher indifference curve to be tangent at C', we have the desired savings response. Since we did not change the lower (kinked) indifference curve, there is still no substitution effect — and thus no deadweight loss from the tax on interest. This would also work for tastes that treat c_1 and c_2 as perfect complements.

(f) *Savings increases with an increase in the interest rate and there is no deadweight loss.*

Answer: We again have to change the optimal bundle on the high interest rate r budget in panel (b) of Exercise Graph 10.3. If savings increases with an increase in the interest rate, A has to lie to the right of C — so that consumption now (c_1) decreases as the interest rate goes up. Thus, if we change the higher indifference curve to be tangent at C'', we have the desired savings response. Since we did not change the lower (kinked) indifference curve, there is still no substitution effect — and thus no deadweight loss from the tax on interest. (This would not work for perfect complements, however — because the sharp kink required for perfect complements would not permit A to lie to the right of C.) So the statement is in fact possible — except for the fact that the problem assumed at the outset that consumption now and in the future is normal. This is not the case at C'' — which implies the statement can be true only if consumption now is an inferior good.

B: *Now suppose that tastes can be summarized by the CES utility function* $u(c_1, c_2) = (0.5c_1^{-\rho} + 0.5c_2^{-\rho})^{-1/\rho}$*, where* c_1 *is consumption in the first period and* c_2 *is consumption in the second period.*

(a) *Are there values for* ρ *that would result in the scenario in A(a) and A(b)?*

Answer: In order for savings behavior to be immune to changes in the interest rate, it must be that consumption now (c_1) is immune to changes in the interest rate. We can calculate the optimal level of c_1 by solving the problem

$$\max_{c_1,c_2}(0.5c_1^{-\rho} + 0.5c_2^{-\rho})^{-1/\rho} \text{ subject to } (1+r)(I-c_1) = c_2 \quad (10.3.i)$$

where I is current income. This gives us

$$c_1 = \frac{I}{1+(1+r)^{-\rho/(\rho+1)}}. \quad (10.3.ii)$$

When ρ is set to zero — i.e. when tastes become Cobb-Douglas — we then get that $c_1 = I/2$ and current consumption is therefore immune to the interest rate. This then implies that savings is immune to the interest rate if and only if $\rho = 0$. Since the elasticity of substitution is positive when $\rho = 0$, this implies there are substitution effects that result in a dead weight loss from a tax on interest. The scenario in A(a) is therefore possible, but the scenario in A(b) is not possible with these CES tastes.

(b) *Are there values for* ρ *that would result in the scenario in A(c)?*

Answer: In order for savings to decrease with an increase in the interest rate, it must be the case that consumption now (c_1) increases with an increase in the interest rate. Equation (10.3.ii) describes how c_1 changes with the interest rate — and by taking the derivative with respect to r, we can tell whether c_1 increases or decreases as r increases. This derivative is

$$\frac{\partial c_1}{\partial r} = \frac{\rho}{(\rho+1)}\left(\frac{I}{\left(1+(1+r)^{-\rho/(\rho+1)}\right)^2(1+r)^{(2\rho+1)/(\rho+1)}}\right). \quad (10.3.iii)$$

Consumption now will increase (and savings will decrease) with the interest rate if this derivative is positive. And, since the term in parenthesis is positive (given that I and r are positive), the derivative is positive if and only if $\rho > 0$. Furthermore, so long as $\rho \neq \infty$, the elasticity of substitution is positive — which implies the existence of substitution effects that create dead weight losses from taxing interest income. Therefore, the scenario in A(c) arises whenever $0 < \rho < \infty$.

(c) *Are there values for ρ that would result in the scenario in A(d)?*

Answer: Savings increases with an increase in the interest rate if and only if consumption now (c_1) falls with an increase in the interest rate. This occurs only if the derivative from equation (10.3.iii) is negative, which in turn occurs only so long as $-1 < \rho < 0$. Since the elasticity of substitution is positive for all such values of ρ, this further implies the emergence of dead weight losses from taxation of interest income. Thus, the scenario in A(d) emerges whenever $-1 < \rho < 0$.

(d) *Are there values for ρ that would result in the scenario in A(e) or A(f)?*

Answer: The only way that there are no substitution effects that give rise to dead weight losses is if the elasticity of substitution is zero — which only happens if $\rho = \infty$. As ρ approaches infinity, the optimal level of current consumption approaches to

$$c_1 = \frac{I}{1 + (1+r)^{-1}}. \tag{10.3.iv}$$

The derivative of c_1 with respect to r is positive. Thus, when $\rho = \infty$ (which is necessary for there to be no deadweight loss from taxation of interest), c_1 will increase (and savings will therefore decrease) with an increase in the interest rate. Scenario A(e) therefore arises when $\rho = \infty$ and scenario A(f) is not possible under this CES specification of tastes.

Exercise 10.5

Everyday Application: *Teacher Pay and Pro-Basketball Salaries: Do we have our priorities in order? We trust our school aged children to be taught by dedicated teachers in our schools, but we pay those teachers only about $50,000 per year. At the same time, we watch pro-basketball games as entertainment — and we pay some of the players 400 times as much!*

A: *When confronted with these facts, many people throw their hands up in the air and conclude we are just hopelessly messed up as a society — that we place more value on our entertainment than on the future of our children.*

(a) *Suppose we treat our society as a single individual. What is our marginal willingness to pay for a teacher? What is our marginal willingness to pay for a star basketball player?*

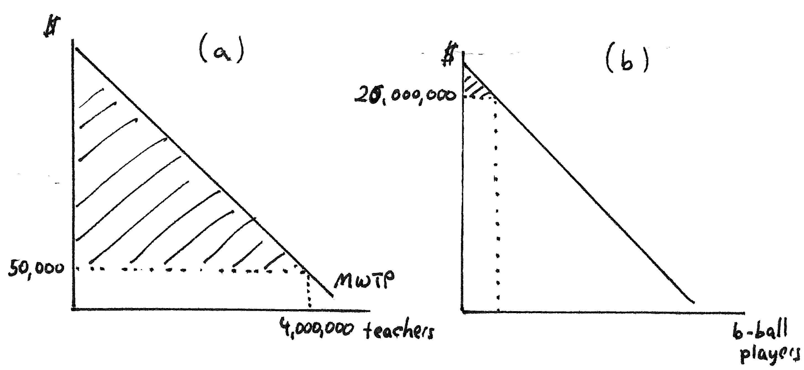
Answer: Our marginal willingness to pay for a teacher is $50,000 per year, and our marginal willingness to pay for a star basketball player is $20,000,000 per year.

(b) *There are about 4 million teachers that work in primary and secondary schools in the United States. What is the smallest dollar figure that could represent our total willingness to pay for teachers?*

Answer: By paying $50,000 per year to 4 million teachers, we are paying a total of $200 billion (i.e. $200,000,000,000). Given we are choosing to pay this amount, that is the least that our total willingness to pay for teachers could be.

(c) *Do you think our* actual *total willingness to pay for teachers is likely to be much greater than that minimum figure? Why or why not?*

Answer: $200 billion would be our total willingness to pay if our marginal willingness to pay curve were perfectly horizontal at $50,000; i.e. if our marginal willingness to pay for the first teacher were the same as our marginal willingness to pay for each additional teacher. But that is almost certainly not the case — rather, our marginal willingness to pay for the first teacher is likely to be very high — with decreasing marginal willingness to pay for each additional teacher. Only with the 4 millionth teacher does our marginal willingness to pay reach $50,000. Since total willingness to pay is the entire area under the marginal willingness to pay curve (and not just the amount we actually pay), our total willingness to pay for our teachers is likely much higher than $200 billion per year. In panel (a) of Exercise Graph 10.5, the *additional* amount we are willing to pay for teachers (above the $200 billion we actually pay) is depicted as the large shaded area.



Exercise Graph 10.5 : Marginal and Total Willingness to Pay

(d) *For purposes of this problem, assume there are 10 star basketball players at any given time. What is the least our total willingness to pay for star basketball players could be?*

Answer: Since we are paying $20 million to each, our total willingness to pay for the 10 star basketball players is at least $200 million.

(e) *Is our actual total willingness to pay for basketball players likely to be much higher than this minimum?*

Answer: Our actual total willingness to pay is somewhat higher than that since our $MWTP$ curve for basketball players is downward sloping (just as it is for teachers). But since there are only 10 such players, the $MWTP$ for the 10th player is probably not nearly as much lower than the $MWTP$ of the first player as the $MWTP$ of the 4 millionth teacher is lower than the $MWTP$ for the first teacher. Thus, \$200 million is a closer approximation to our total willingness to pay for basketball players than \$200 billion is to our total willingness to pay for teachers. This can be seen graphically by comparing the shaded areas in panels (a) and (b) of Exercise Graph 10.5 — where the shaded area in each graph depicts the amount we are willing to pay *in addition* to what we had to pay.

(f) *Do the facts cited at the beginning of this question really warrant the conclusion that we place more value on our entertainment than on the future of our children?*

Answer: We concluded that our total willingness to pay for star basketball players is approximately \$200 million — and our total willingness to pay for teachers is substantially larger than \$200 billion. While it is true that we are willing to pay more for a star basketball player *on the margin*, the total value we place on the services from the basketball players is therefore substantially less than what we are willing to pay for the services of teachers. Put differently, not only do we actually pay more for all the teachers than we do for basketball players, but the consumer surplus we get from teachers is many orders of magnitude larger than what we get from basketball players.

(g) *Adam Smith puzzled over an analogous dilemma: He observed that people were willing to pay exorbitant amounts for diamonds but virtually nothing for water. With water essential for sustaining life and diamonds just an item that appeals to our vanity, how could we value diamonds so much more than water? This became known as the* diamond-water paradox. *Can you explain the paradox to Smith?*

Answer: Exactly the same reasoning holds as does for teachers and star basketball players. The two panels of Exercise Graph 10.5 could simply be re-labeled, with the first representing water and the second diamonds. The only difference is that the example is even more extreme — our $MWTP$ for the last gallon of water is close to zero but we consume a lot of water — causing our total willingness to pay to be represented by a very large triangle under the $MWTP$ curve. But most of us consume very few diamonds. On the margin, we value a diamond more than a gallon of water, but we place much more value on our water consumption than on our diamond consumption.

B: *Suppose our marginal willingness to pay for teachers (x_1) is given by $MWTP = A - \alpha x_1$ and our marginal willingness to pay for star basketball players (x_2) is given by $MWTP = B - \beta x_2$.*

(a) *Given the facts cited above, what is the lowest that A and B could be?*

Answer: If $MWTP$ curves were perfectly horizontal, then $A = 50,000$ and $B = 20,000,000$.

(b) *If A and B were as you just concluded, what would α and β be?*

Answer: Since $MWTP$ curves can't slope up, it would have to be that they are perfectly flat — with slope of zero. Thus, if $A = 50,000$ and $B = 20,000,000$, then $\alpha = \beta = 0$.

(c) *What would be our marginal and total willingness to pay for teachers and star basketball players?*

Answer: Under these parameters, our total willingness to pay for teachers would be $200 billion and our total willingness to pay for star basketball players would be $200 million.

(d) *Suppose* $A = B = \$100$ *million. Can you tell what α and β must be?*

Answer: In order for the marginal willingness to pay of the 4 millionth teacher to be $50,000, the slope term α would have to be $99,950,000/4,000,000 = 24.9875$. In order for the marginal willingness to pay for the 10th basketball player to be $20,000,000, the slope term β must be $80,000,000/10 = 8,000,000$.

(e) *Using the parameter values you just derived (with* $A = B = \$100$ *million), what is our total willingness to pay for teachers and star basketball players?*

Answer: The triangle in panel (a) of the graph would then be $(99,950,000(4,000,000))/2 = 199,900,000,000,000$ = $199,900 billion. To this, we add what we actually pay for teachers — $200 billion — and we get a total willingness to pay for teachers of $200,100 billion or approximately $200 *trillion*! The analogous triangle in panel (b) is $80,000,000(10)/2 = 400,000,000$. Adding what we actually pay for star basketball players, we get a total willingness to pay of $600,000,000 or $600 million. (That would mean we value our teachers over 300,000 times as much as our star basketball players.)

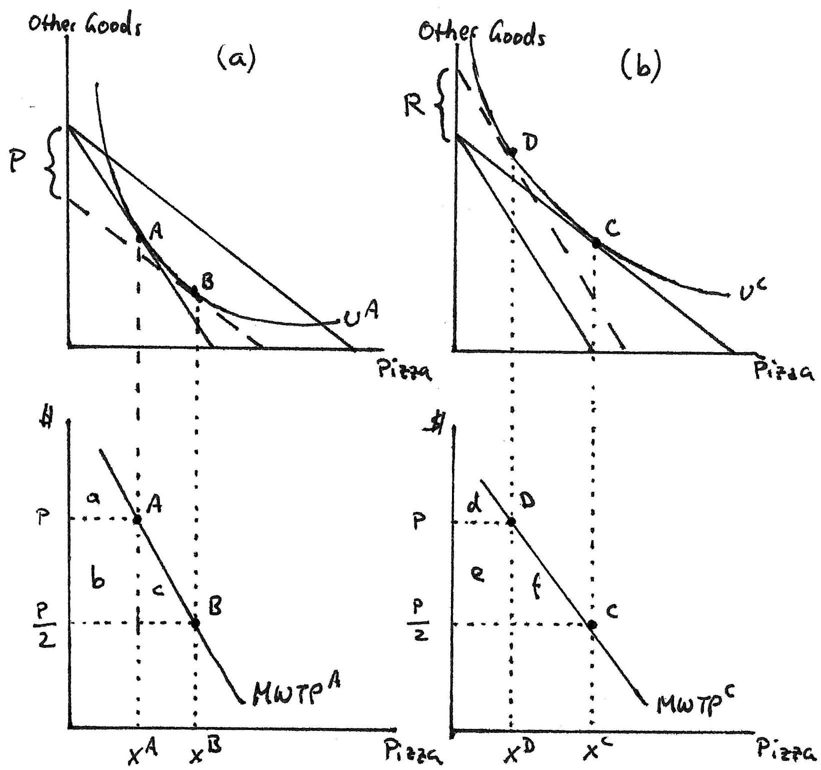
Exercise 10.7

Everyday Application: *To Trade or Not to Trade Pizza Coupons: Exploring the Difference between Willingness to Pay and Willingness to Accept: Suppose you and I are identical in every way — same exogenous income, same tastes over pizza and "other goods". The only difference between us is that I have a coupon that allows the owner of the coupon to buy as much pizza as he/she wants at 50% off.*

A: *Now suppose you approach me to see if there was any way we could make a deal under which I would sell you my coupon. Below you will explore under what conditions such a deal is possible.*

(a) *On a graph with pizza on the horizontal axis and "other goods" on the vertical, illustrate (as a vertical distance) the most you are willing to pay me for my coupon. Call this amount* P.

Answer: This is illustrated in the top graph in panel (a) of Exercise Graph 10.7.



Exercise Graph 10.7 : Trading Pizza Coupons

The steeper budget is your (without-coupon) budget, while the shallower budget is my (with coupon) budget that has half the price for pizza. Without a coupon, your optimal bundle is A — and you reach utility level u^A. Getting the coupon means getting the shallower slope for yourself — but buying it means that you are giving up income. In deciding the most you are willing to pay for a coupon, you therefore have to decide how much you are willing to shift the shallower budget in — and the most you are willing to shift it is an amount that will get you the same utility you can get without the coupon. Thus, the most you are willing to shift the coupon budget in is the amount that creates the tangency at B with u^A and the dashed budget (that has the same slope as the with-coupon budget. The vertical distance between my (with-coupon) budget and the dashed budget is the most you are willing to pay me for the coupon — a distance that can be measured anywhere between the two parallel lines. It is indicated as distance P in the graph.

(b) *On a separate but similar graph, illustrate (as a vertical distance) the least I would be willing to accept in cash to give up my coupon. Call this amount R.*

Answer: The top graph in panel (b) illustrates this. By just using the coupon, I will optimize at bundle C — and will reach the indifference curve u^C. Since I can get to that utility level without selling you the coupon, I will not be willing to make a deal that gets me less utility. By selling you the coupon, I will face a steeper budget but I will have received cash from you — i.e. I will face a budget with the steeper (no-coupon) slope but further out than your initial no-coupon budget. The least I am willing to accept for the coupon is an amount that will make me just as well off as I am with the coupon. I can determine that amount by taking your initial budget and shifting it out until it is tangent to my original optimal indifference curve u^C — which would land me at bundle D in the graph. The amount you have to give me in cash to get me to D is the vertical distance between your original (no-coupon) budget and the dashed budget in the graph. That distance, labeled R, is the least I am willing to accept for the coupon.

(c) *Below each of the graphs you have drawn in (a) and (b), illustrate the same amounts P and R (as areas) along the appropriate marginal willingness to pay curves.*

Answer: In the lower graph of panel (a), the marginal willingness to pay curve is derived from the indifference curve u^A. In the absence of a coupon, you will buy x^A in pizza at the no-coupon price p. This gives you consumer surplus of a. If you end up buying the coupon from me at the maximum price you are willing to pay (P), you will buy x^B in pizza and attain consumer surplus of $a+b+c$. Since A and B lie on the same indifference curve, you are equally happy attaining consumer surplus a without having paid me anything for the coupon or consumer surplus $a+b+c$ after paying me P for the coupon. In order for you to be truly indifferent between these two options, it must therefore be the case that $P = b+c$.

In the lower graph of panel (b), the marginal willingness to pay curve is derived from my indifference curve u^C. In the absence of selling my coupon, I buy x^C pizza — and get consumer surplus of $d+e+f$. If I sell the coupon at the lowest price R that I am willing to accept, I end up buying x^D pizza and get consumer surplus of just d. Since I am equally happy in both cases, it must be that I am indifferent between getting consumer surplus of $d+e+f$ without receiving any cash from you or getting consumer surplus d and getting R in cash. Thus, $R = e+f$.

(d) *Is P larger or smaller than R? What does your answer depend on? (*Hint*: By overlaying your lower graphs that illustrate P and R as areas along marginal willingness to pay curves, you should be able to tell whether one is bigger than the other or whether they are the same size depending on what kind of good pizza is.)*

Answer: Asking if P is larger or smaller than R is then the same as asking if $b+c$ is larger or smaller than $e+f$. Suppose first that pizza is a quasilinear good for us. Then if I transferred the indifference curve u^C onto the top graph of panel (a), the tangency C would lie vertically above B — because a move from the dashed budget in panel (a) to my original (with-coupon) budget is simply an increase in income without a price change. Such an increase in income would not change consumption of pizza when pizza is quasilinear. Similarly, if we transferred the indifference curve u^A onto panel (b), the tangency at A would lie vertically below D. This is because the move from the dashed budget in panel (b) to your (no-coupon) budget is a simple decrease in income without a price change — which causes to change in consumption of pizza when pizza is quasilinear. This implies that, in the lower graphs, A lies at exactly the same place as D and B lies at exactly the same place as C. Put differently, when pizza is a quasilinear good, $MWTP^A$ lies exactly on top of $MWTP^C$ — which implies $b+c=e+f$ or $P=R$. The most you are willing to pay me for the coupon is then exactly equal to the least I am willing to accept.

Now suppose that pizza is an inferior good. Then the same logic we just went through implies that D will lie to the left of A and C will lie to the left of B — which implies that $b+c>e+f$ or $P>R$. Thus, when pizza is an inferior good, the most you are willing to pay is greater than the least I am willing to accept for the coupon. If, on the other hand, pizza is a normal good, then the same logic implies that D lies to the right of A and C lies to the right of B — which further implies that $b+c<e+f$ of $P<R$. Thus, when pizza is a normal good for us, then the most you are willing to pay is less than the least I am willing to accept for the coupon.

(e) True or False*: You and I will be able to make a deal so long as pizza is not a normal good. Explain your answer intuitively.*

Answer: This is true. We have just concluded that when pizza is an inferior good, you are willing to pay me more than the least I am willing to accept — so there is room for us to make a deal and both become better off. When pizza is quasilinear (i.e. borderline between normal and inferior), then the least I am willing to accept is exactly the most you are willing to pay — so in principle we can make a deal but neither one of us will be better or worse off for it. But when pizza is a normal good for us, the least I am willing to accept is more than them most you are willing to pay — so there is no way we will be able to strike a deal.

Intuitively, this makes sense in the following way: We began by saying that you and I are identical in every way — except there is one way in which we are not identical: I have a coupon and you do not. Thus, I am in essence richer than you are to begin with. If pizza is a normal good, then richer people will buy more pizza than poorer people — and so the coupon has more value to richer people because they would use it more. It is for this reason that we can't make a deal if pizza is normal for us. But if pizza is an inferior good, then being richer means I will want less pizza — and so I have less use for the coupon than you do. As a result, you will be willing to pay more than the least I am willing to accept. And if pizza is quasilinear, rich and poor buy the same amount of pizza — and thus make the same use of the coupon. Thus, if pizza is quasilinear, the coupon is worth the same to us.

B: *Suppose your and my tastes can be represented by the Cobb-Douglas utility function* $u(x_1, x_2) = x_1^{0.5}x_2^{0.5}$*, and suppose we both have income* $I = 100$*. Let pizza be denoted by* x_1 *and "other goods" by* x_2*, and let the price of pizza be denoted by* p*. (Since "other goods" are denominated in dollars, the price of* x_2 *is implicitly set to 1.)*

(a) *Calculate our demand functions for pizza and other goods as a function of* p*.*

Answer: Solving the problem

$$\max_{x_1,x_2} x_1^{1/2}x_2^{1/2} \text{ subject to } px_1 + x_2 = 100, \tag{10.7.i}$$

we get $x_1 = 50/p$ and $x_2 = 50$.

(b) *Calculate our compensated demand for pizza* (x_1) *and other goods* (x_2) *as a function of* p *(ignoring for now the existence of a coupon).*

Answer: Solving the problem

$$\min_{x_1,x_2} px_1 + x_2 \text{ subject to } u = x_1^{1/2}x_2^{1/2}, \tag{10.7.ii}$$

we get $x_1 = u/(p^{1/2})$ and $x_2 = p^{1/2}u$.

(c) *Suppose* $p = 10$ *and the coupon reduces this price by half (to 5). Assume again that I have a coupon but you do not. How much utility do you and I get when we make optimal decisions?*

Answer: Using our (uncompensated) demand functions, we can calculate that your pizza consumption is $x_1 = 50/10 = 5$ while mine is $x_1 = 50/5 = 10$. This corresponds to x_A and x_D in Exercise Graph 10.7. Both of us consume 50 in other goods. Thus, your utility is $u(5,50) = 5^{1/2}50^{1/2} \approx 15.81$ and my utility is $u(10,50) = 10^{1/2}50^{1/2} \approx 22.36$.

(d) *How much pizza will you consume if you pay me the most you are willing to pay for the coupon? How much will I consume if you pay me the least I am willing to accept?*

Answer: If you pay me the most you are willing to pay, you will remain at the same utility level but will pay only half as much (i.e. \$5 instead of \$10). The compensated demand function therefore tells us that you will consume $x_1 = 15.81/(5^{1/2}) \approx 7.07$. If you pay me the least I am willing to accept, I will end up with the same utility as before but with a price that is twice as high (i.e. \$10 instead of \$5). Thus, the compensated demand function tells me that $x_1 = 22.36/(10^{1/2}) \approx 7.07$. In terms of the graphs in Exercise Graph 10.7, this implies that $x_B = 7.07 = x_D$.

(e) *Calculate the expenditure function for me and you.*

Answer: To get the expenditure function, we substitute the compensated demands back into the objective function of the minimization problem — i.e. we substitute $x_1 = u/(p^{1/2})$ and $x_2 = p^{1/2}u$ into $px_1 + x_2$ to get

$$E(p,u) = p\left(\frac{u}{p^{1/2}}\right) + p^{1/2}u = 2p^{1/2}u. \qquad (10.7.iii)$$

(f) *Using your answers so far, determine R — the least I am willing to accept to give up my coupon. Then determine P — the most you are willing to pay to get a coupon.(*Hint*: Use your graphs from A(a) to determine the appropriate values to plug into the expenditure function to determine how much income I would have to have to give up my coupon. Once you have done this, you can subtract my actual income $I = 100$ to determine how much you have to give me to be willing to let go of the coupon. Then do the analogous to determine how much you'd be willing to pay, this time using your graph from A(b).)*

Answer: The budget required for me to be just as happy without the coupon (i.e. when $p = 10$) is the expenditure necessary for me to reach utility level 22.36 (which is u^C in our graph) at $p = 10$ — i.e. $E(10, 22.36) = 2(10^{1/2})22.36 \approx 141.42$. Since I started out with an income of \$100, this implies that you would have to give me approximately \$41.42 for the coupon in order for me to be just as happy; i.e. $R = 41.42$. The budget required for you to be just as happy with the coupon as you were without is the expenditure necessary to get you to utility level 15.81 (which is u^A in our graph) at the with-coupon price of 5 — i.e. $E(5, 15.81) = 2(5^{1/2})15.81 \approx 70.71$. Since you started with an income of \$100, this means you would be willing to pay me as much as $100 - 70.71 = 29.29$ to get the coupon; i.e. $P = 29.29$.

(g) *Are we able to make a deal under which I sell you my coupon? Make sense of this given what you found intuitively in part A and given what you know about Cobb-Douglas tastes.*

Answer: No, we are not able to make a deal since the most you are willing to pay me (\$29.29) is less than the least I am willing to accept (\$41.42). This is consistent with what we concluded in part A where we said that we would not be able to strike a deal if pizza is a normal good for us. Cobb-Douglas tastes are tastes over normal goods — so under the tastes represented by the utility function we have been working with, pizza is in fact a normal good.

(h) *Now suppose our tastes could instead be represented by the utility function $u(x_1, x_2) = 50\ln x_1 + x_2$. Using steps similar to what you have just done, calculate again the least I am willing to accept and the most you are willing to pay for the coupon. Explain the intuition behind your answer given what you know about quasilinear tastes.*

Answer: Solving the problem

$$\max_{x_1,x_2} 50\ln x_1 + x_2 \text{ subject to } px_1 + x_2 = 100, \qquad (10.7.iv)$$

we get the (uncompensated) demands $x_1 = 50/p$ and $x_2 = 50$. Thus, both you and I consume 50 in other goods, but I consume 10 pizzas while you only consume 5 because I face a with-coupon price of \$5 per pizza while you face a without-coupon price of \$10 per pizza.

Plugging $(x_1, x_2) = (10, 50)$ into the utility function, we get my utility (equivalent to u^C in the graph) of 165.13. Plugging $(x_1, x_2) = (5, 50)$ into the utility function for you, we get your utility (equivalent to u^A in our graph) as 130.47.

Solving the minimization problem

$$\min_{x_1,x_2} px_1 + x_2 \text{ subject to } u = 50\ln x_1 + x_2, \qquad (10.7.v)$$

we can derive the compensated demands $x_1 = 50/p$ and $x_2 = u - 50\ln(50/p)$. (Note that the compensated and uncompensated demands for the quasilinear good x_1 are the same — which makes sense since there are no income effects to make the two demands different.) Next, we can find the expenditure function by just plugging the compensated demands into the objective function of the minimization problem to get

$$E(p,u) = p\left(\frac{50}{p}\right) + u - 50\ln\left(\frac{50}{p}\right) = u + 50\left(1 - \ln\left(\frac{50}{p}\right)\right). \qquad (10.7.vi)$$

To determine the expenditure necessary for me to get to my current utility level in the absence of the coupon (i.e. when $p = 10$ instead of $p = 5$), we calculate $E(10, 165.13) \approx 134.66$. Since I start with an income of \$100, that means the least I am willing to accept for the coupon is $R = 134.66 - 100 =$ \$34.66. To calculate the expenditure necessary to get you to your current utility level in the presence of a coupon (i.e. when $p = 5$ instead of $p = 10$), we calculate $E(5, 130.47) \approx 65.34$. Since you also start with an income of \$100, this means that the most you are willing to pay for the coupon is $P = 100 - 65.34 =$ \$34.66. Thus $P = R$ as we concluded it has to be when pizza is a quasilinear good.

(i) *Can you demonstrate, using the compensated demand functions you calculated for the two types of tastes, that the values for P and R are in fact areas under these functions (as you described in your answer to A(c)? (*Note*: This part requires you to use integral calculus.)*

Answer: The compensated demand function for pizza is $x_1 = 50/p$ (as calculated in the previous part). The area in the graph is the integral under this function between the with-coupon and the without-coupon prices. (Note: In our graphs, we are graphing the inverse of the compensated demand functions — which is why the area appears as an area to the left of the curve rather than an area under the curve.) Thus, the least I am willing to accept (R) is

$$R = \int_5^{10} \frac{50}{p} dp = 50 \ln p|_5^{10} = 50 \ln 10 - 50 \ln 5 = 34.66. \quad (10.7.vii)$$

Since, because of the quasilinearity of pizza, our compensated demand functions are the same (because u does not appear in the functions), the same holds for P. Thus, $P = R = 34.66$, exactly as we concluded before.

Exercise 10.9

Everyday Application: *To Take, or not to Take, the Bus*: *After you graduate, you get a job in a small city where you have taken your sister's offer of living in her apartment. Your job pays you $20 per hour and you have up to 60 hours per week available. The problem is you also have to get to work.*

A: *Your sister's place is actually pretty close to work — so you could lease a car and pay a total (including insurance and gas) of $100 per week to get to work, spending essentially no time commuting. Alternatively, you could use the city's sparse bus system — but unfortunately there is no direct bus line to your place of work and you would have to change buses a few times to get there. This would take approximately 5 hours per week.*

(a) *Now suppose that you do not consider time spent commuting as "leisure" — and you don't consider money spent on transportation as "consumption". On a graph with "leisure net of commuting time" on the horizontal axis and "consumption dollars net of commuting costs" on the vertical, illustrate your budget constraint if you choose the bus and a separate budget constraint if you choose to lease the car.*

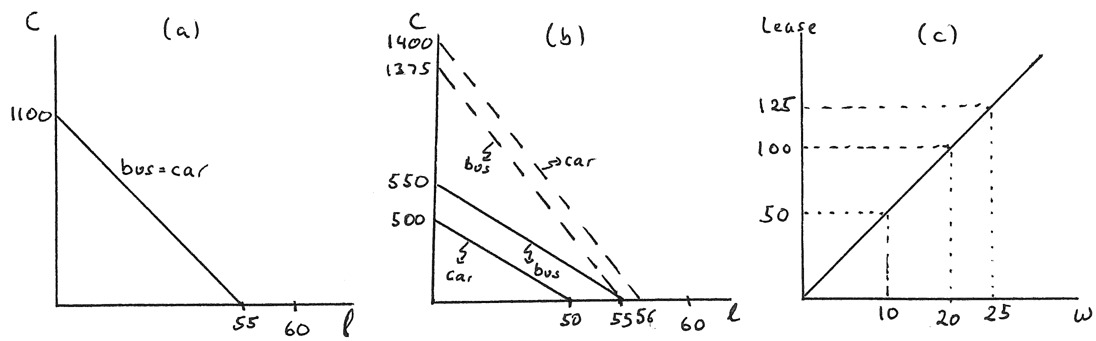
Answer: If you choose to take the bus, you reduce your leisure time from 60 to 55 hours per week — and therefore can earn up to $20(55)=$1,100 per week. If you choose to lease a car, you can work up to 60 hours per week — but you have to pay $100 no matter how much you work. If you work for 60 hours, you earn $25(60)=$1,200, but since you have to pay $100 for the lease, the most you can consume is $1,100. In order to pay for the lease, you have to work at least 5 hours. Thus, the two budget constraints are exactly the same. They are illustrated in panel (a) of Exercise Graph 10.9.

(b) *Do you prefer the bus to the car?*

Answer: Since the two choices result in the same effective budget constraint, you are indifferent between taking the bus and leasing the car.

(c) *Suppose that before you get to town you find out that a typo had been made in your offer letter and your actual wage is $10 per hour instead of $20 per hour. How does your answer change?*

Answer: If you choose the bus, the loss of leisure remains unchanged — leaving you again with 55 hours per week. At the lower wage, that allows you to consume at most $10(55)=$550 per week. If you choose the lease, it will take you 10 hours just to come up with the payment for the lease. Even if you work all 60 hours, you will therefore be left with consumption



Exercise Graph 10.9 : To Take, or not to Take, the Bus

corresponding to only 50 hours of work — or $500. The lease budget is therefore strictly lower than the bus budget, as illustrated by the two solid budget lines in panel (b) of Exercise Graph 10.9. You would therefore choose the bus.

(d) *After a few weeks, your employer discovers just how good you are and gives you a raise to $25 per hour. What mode of transportation do you take now?*

Answer: If you choose the bus, you again will take the same 5 hour reduction in your leisure — leaving you with at most $25(55)=$1,375 in consumption. If you choose the lease, you can pay for the lease with just 4 hours of work — leaving you with up to 56 hours that you can devote to generating consumption. Now the lease budget therefore strictly dominates the bus budget, as illustrated by the dashed budgets in panel (b) of the graph. You will therefore go by car.

(e) *Illustrate in a graph (not directly derived from what you have done so far) the relationship between wage on the horizontal axis and the most you'd be willing to pay for the leased car.*

Answer: We know from (a) that, when $w = 20$, a lease payment of $100 per week makes you exactly indifferent. This is because, at a wage of $20 per hour, it takes 5 hours to make enough money to pay for the lease — which is exactly the same number of hours as it takes to ride the bus. Thus, to get the maximum lease payment Y that you would be willing to pay at wage w, you simply multiply the number of hours required for the bus by the wage — i.e. $Y = 5w$. Put differently, the value of your time on the bus determines the maximum lease you are willing to pay. This is illustrated in panel (c) of GExercise Graph 10.9.

(f) *If the government taxes gasoline and thus increases the cost of driving a leased cars (while keeping buses running for free), predict what will happen to the demand for bus service and indicate what types of workers will be the source of the change in demand.*

Answer: Demand for bus service will increase, with the increase in demand determined by the lowest wage workers that previously chose to lease cars.

(g) *What happens if the government improves bus service by reducing the time one needs to spend to get from one place to the other?*

Answer: If the government improves bus service, a person with a wage that previously made him indifferent between the bus and leasing a car will now take the bus because the most he would be willing to pay for a lease will fall. Thus, lower wage workers will switch to the improved bus system.

B: *Now suppose your tastes were given by $u(c,\ell) = c^{\alpha}\ell^{(1-\alpha)}$, where c is consumption dollars net of commuting expenses and ℓ is leisure consumption net of time spent commuting. Suppose your leisure endowment is L and your wage is w.*

(a) *Derive consumption and leisure demand assuming you lease a car that costs you \$$Y$ per week which therefore implies no commuting time.*

Answer: We need to solve the problem

$$\max_{c,\ell} c^{\alpha}\ell^{(1-\alpha)} \text{ subject to } w(L-\ell) = Y + c. \tag{10.9.i}$$

This gives us

$$c = \alpha(wL - Y) \text{ and } \ell = \frac{(1-\alpha)(wL-Y}{w}. \tag{10.9.ii}$$

(b) *Next, derive your demand for consumption and leisure assuming you take the bus instead, with the bus costing no money but taking T hours per week from your leisure.*

Answer: Now we need to solve the problem

$$\max_{c,\ell} c^{\alpha}\ell^{(1-\alpha)} \text{ subject to } w(L-T-\ell) = c. \tag{10.9.iii}$$

This gives us

$$c = \alpha w(L-T) \text{ and } \ell = (1-\alpha)(L-T). \tag{10.9.iv}$$

(c) *Express the indirect utility of leasing the car as a function of Y.*

Answer: To get the indirect utility from leasing a car, we simply have to plug the optimal values for c and ℓ from equation (10.9.ii) into the utility function — which gives

$$V(Y) = \frac{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}(wL-Y)}{w^{(1-\alpha)}}. \tag{10.9.v}$$

(d) *Express your indirect utility of taking the bus as a function of T.*

Answer: Now we need to take the optimal levels of c and ℓ when taking the bus (from equation (10.9.iv)) into the utility function to get

$$V(T) = \alpha^{\alpha}(1-\alpha)^{(1-\alpha)} w^{\alpha}(L-T). \tag{10.9.vi}$$

(e) *Using the indirect utility functions, determine the relationship between Y and T that would keep you indifferent between taking the bus and leasing the car. Is your answer consistent with the relationship you illustrated in A(e) and your conclusions in A(f) and A(g)?*

Answer: Setting $V(Y)$ equal to $V(T)$, i.e.

$$\frac{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}(wL-Y)}{w^{(1-\alpha)}} = \alpha^{\alpha}(1-\alpha)^{(1-\alpha)} w^{\alpha}(L-T), \tag{10.9.vii}$$

we get, after canceling some terms, $Y = wT$. This is consistent with what we found in part A where we said that $Y = 5w$ when the time to take the bus is 5 hours per week. It is also consistent with our conclusion that the lowest wage workers who previously leased cars will switch to buses if the government increases the cost of leasing a car by taxing gasoline — because as Y increases, the equation $Y = wT$ implies that w will also have to increase to keep someone indifferent between leasing a car and taking the bus. It is similarly consistent with the conclusion that a more efficient bus system — i.e. a lower T — will cause the same types of workers to switch to the bus.

(f) *Could you have skipped all these steps and derived this relationship directly from the budget constraints? Why or why not?*

Answer: Yes, you could have. The two budget constraints can be expressed as

$$c = w(L-\ell) - Y \text{ and } c = w(L-T-\ell). \tag{10.9.viii}$$

Setting these equal to each other, you get $Y = wT$. This works because the only factors operating in this problem are wealth effects — there are no substitution effects. It is analogous to what we did in part A where we simply focused on the relationship between the budget constraints to determine what the consumer would prefer — we did not have to actually draw any indifference curves.

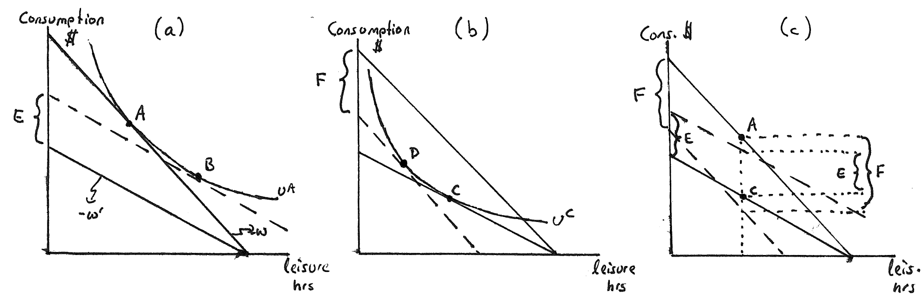
Exercise 10.11

Business Application: *Negotiating an Endorsement Deal and a Bribe*: *Suppose you are an amateur athlete and your uncle owns the cereal company "Wheaties." Your uncle offers you a job working for his company at a wage of w per hour. After looking around for other jobs, you find that the most you could make elsewhere is w', where $w' < w$. You have a weekly leisure endowment of L and can allocate any amount of that to work. Given the higher wage at Wheaties, you accept your uncle's job offer.*

A: *Then you win a gold medal in the Olympics. "Greeties", the makers of grits, ask you for an endorsement. As part of the deal, they will pay you some fixed weekly amount to appear on their boxes of grits. Unfortunately, your uncle (who hates his competitor "Greeties" with the white hot intensity of a thousand suns) will fire you if you accept the deal offered by "Greeties". Therefore, if you accept the deal, your wage falls to w'.*

(a) *On a graph with Consumption on the vertical and Leisure on the horizontal axis, graph your budget constraint before the "Greeties" offer.*

Answer: This is graphed in panel (a) of Exercise Graph 10.11(1) as the steeper of the two budget lines.



Exercise Graph 10.11(1) : Endorsement Deals and Bribes

(b) *On the same graph, illustrate your budget if you worked for someone other than your uncle prior to your success in the Olympics.*

Answer: This is illustrated as the shallower of the two budget lines in panel (a) of the graph.

(c) *Illustrate the minimum amount that "Greeties" would have to pay you (weekly) for your endorsement in order for you to accept the deal. Call this amount E.*

Answer: The optimal bundle at Wheaties is indicated as bundle A — yielding utility level u^A. You would not be willing to enter any endorsement deal that does not at least get you to that same utility level. The Greeties endorsement check must therefore be sufficient to get you to u^A. Such a deal does not change your wage at Greeties — it just causes your Greeties budget to shift out in a parallel way. If the shift is sufficient to get you to bundle B, it is the lowest possible amount you would accept. The is indicated in dollar terms on the vertical axis as E.

(d) *How does this amount E compare to the amount necessary to get you to be able to consume bundle A under a Greeties endorsement deal?*

Answer: The minimum amount E that is acceptable to you is smaller than what is necessary to get to bundle A. You can tell it is less because A lies outside the budget constraint that emerges when Greeties offers E. The difference emerges because of a substitution effect.

(e) *Now suppose that you accepted the endorsement deal from "Greeties" but, unfortunately, the check for the endorsement bounces because "Greeties" goes bankrupt. Therefore the deal is off, but your angry uncle has already fired you. Deep down inside your uncle still cares about you and will give you back your old job if you come back and ask him for it. The problem is that you have to get past his greedy secretary who has full control over who gets to see your uncle. When you get to the "Wheaties" office, she informs you that you have to commit to pay her a weekly bribe if you want access to your uncle. On a new graph, illustrate the largest possible (weekly) payment you would be willing to make. Call this F.*

Answer: Once the endorsement deal falls through, you will optimize at bundle C in panel (b) of Exercise Graph 10.11(1) — along the budget formed by the lower market wage w' in the absence of any lump sum payments. This would give you utility level u^C. Getting employment at Wheaties implies getting the higher wage w (and thus the steeper budget), but paying the weekly bribe means that this budget shifts inward in a parallel way. The most you would be willing to pay to get your old job back is an amount that makes you just as well off as you are without getting back to Wheaties — i.e. an amount that gets you to utility level u^C. This is equivalent to a shift in the steeper budget that gets you to the new optimal bundle D. The amount of the weekly bribe is then indicated in dollar terms on the vertical axis as F.

(f) *If your uncle's secretary just asks you for a weekly bribe that gets you to the bundle C that you would consume in the absence of returning to Wheaties, would you pay her such a bribe?*

Answer: Yes, you would — because the shift in the steeper budget required to get you to C is less than the shift induced by F. You can tell that this is so by the fact that C lies outside the budget that is created by the highest possible bribe F you'd be willing to pay. The reason you are willing to pay more than the amount that would make C affordable is a substitution effect.

(g) *Suppose your tastes are such that the wealth effect from a wage change is exactly offset by the substitution effect — i.e. no matter what the wage, you will always work the same amount (in the absence of receiving endorsement checks or paying bribes). In this case, can you tell whether the amount E (i.e. the minimum endorsement check) is greater than or equal to the amount F (i.e. the maximum bribe)?*

Answer: Yes — we will conclude that $F > E$. Here is how we can tell: If wealth and substitution effects exactly offset as described, then A (in panel (a) of Exercise Graph 10.11(1)) lies exactly above C (from panel (b) of Exercise Graph 10.11(1)) — because A is optimal at the high wage (in the absence of a bribe) and C is optimal at the low wage (in the absence of a bribe). This is illustrated in panel (c) of the graph where A and C are plotted at the same level of leisure. We then transfer the dashed lines from panel (a) and (b) — to give us the vertical distances E and F. The key is that we concluded that the dashed line in panel (a) must lie *below* A, and the dashed line in panel (b) must lie *below* C. But it is then shown in panel (c) of the graph that this implies F — the vertical distance between the steeper lines — must be larger than E — the vertical distance between the shallower lines. You would therefore be willing to bribe more to get back your job than you required in an endorsement to give it up.

B: *Suppose that your tastes over weekly consumption c and weekly leisure ℓ can be represented by the utility function $u(c,\ell) = c^{0.5}\ell^{0.5}$ and your weekly leisure endowment is $L = 60$.*

(a) *If you accept the initial job with Wheaties, how much will you work?*

Answer: To answer this, we need to solve the problem

$$\max_{c,\ell} c^{0.5}\ell^{0.5} \text{ subject to } w(60-\ell) = c. \tag{10.11.i}$$

This results in

$$c = 30w \text{ and } \ell = 30. \tag{10.11.ii}$$

Thus, you will work for 30 hours at Wheaties.

(b) *Suppose you accept a deal from Greeties that pays you a weekly amount $\overline{E}$. How much will you work then? Can you tell whether this is more or less than you would work at Wheaties?*

Answer: We now have to solve the problem

$$\max_{c,\ell} c^{0.5}\ell^{0.5} \text{ subject to } \overline{E} + w'(60-\ell) = c. \tag{10.11.iii}$$

This solves to

$$c = 0.5\overline{E} + 30w' \text{ and } \ell = \frac{0.5\overline{E} + 30w'}{w'} = \frac{0.5\overline{E}}{w'} + 30. \tag{10.11.iv}$$

You will therefore take more than the 30 hours of leisure you took at Wheaties — which implies you will work fewer hours.

(c) *Suppose that the wage w at Wheaties is \$50 per hour and the wage w' at Greeties (or any other potential employer other than Wheaties) is \$25 per hour. What is the lowest possible value for E — the weekly endorsement money from Greeties — that might get you to accept the endorsement deal?*

Answer: First, we have to figure out how much utility you can assure yourself of by just working at Wheaties — because you would not accept an endorsement deal from Greeties that results in less utility than that. At Wheaties, we calculate that $c = 30w$ and $\ell = 30$ — which implies you will consume a bundle $A = (c,\ell) = (1500,30)$ at $w = 50$. This gives utility of $u^A = 1500^{0.5}30^{0.5} \approx 212.13$. From the results in equation (10.11.iv), we know that your consumption bundle at Greeties with an endorsement deal $\overline{E}$ results in the bundle $B = (c,\ell) = (0.5\overline{E} + 750, (0.5\overline{E} + 750)/25)$ when

$w' = 25$. Since E would result in the same utility at B as at A, we could then simply set the utility at the consumption bundle with the endorsement deal equal to 212.13 and solve for E — i.e. we could solve

$$(0.5E+750)^{0.5}\left(\frac{(0.5E+750)}{25}\right)^{0.5} = 212.13. \tag{10.11.v}$$

This gives us $E = 612.30$.

Alternatively, we could use the expenditure function approach. To do so, we need to solve the expenditure minimization problem

$$\min_{c,\ell} w\ell + c \text{ subject to } u = c^{0.5}\ell^{0.5} \tag{10.11.vi}$$

to solve for the compensated demands

$$c = w^{0.5}u \text{ and } \ell = w^{-0.5}u. \tag{10.11.vii}$$

Plugging these back into the expenditure equation $w\ell + c$, we get that the expenditure necessary to get to utility level u at wage w is

$$E(w,u) = 2w^{0.5}u. \tag{10.11.viii}$$

The value of your endowment at a wage of $25 is 25(60)=1500. The expenditure necessary to get to utility level $u^A = 212.13$ at $w' = 25$ is $E(25,212.13) = 2(25^{0.5})(212.13) = 2121.30$. The minimum necessary weekly endorsement check to get you to accept the Greeties offer is then the difference between these; i.e. $E = 2121.30 - 1500 = 612.30$. (Note that the E that refers to this minimum endorsement check is not the same as the *expenditure function* $E(w,u)$.)

(d) *How much will you work if you accept this endorsement deal E?*

Answer: Your optimal leisure (from the equation (10.11.iv)) is then

$$\ell = \frac{0.5\overline{E}}{w'} + 30 = \frac{0.5(612.30)}{25} + 30 \approx 42.25. \tag{10.11.ix}$$

Thus, you will work approximately $60 - 42.25 = 17.75$ hours per week at Greeties.

(e) *Suppose you have accepted this deal but Greeties now goes out of business. What is the highest possible weekly bribe F you'd be willing to pay your uncle's secretary in order to get your job at Wheaties back?*

Answer: First, we have to calculate the utility level you would get if you do not get your old job at Wheaties back. With $E = 0$, our results from equation (10.11.iv) become identical to those from equation (10.11.ii) — $c = 30w'$ and $\ell = 30$. When $w' = 25$, this implies an optimal consumption bundle $C = (c,\ell) = (750,30)$ — which gives utility $u^C = 750^{0.5}30^{0.5} = 150$.

We can then solve for F in one of two ways. Using the expenditure function $E(w,u) = 2w^{0.5}u$, we can calculate the minimum expenditure necessary at the Wheaties wage $w = 50$ to get to the utility level u^C — i.e. $E(50,150) = 2(50^{0.5})(150) = 2121.32$. Your leisure endowment at the Wheaties wage of $w = 50$ is worth $50(60) = 3000$. Thus, you would be willing to pay $3000 - 2121.32 = 878.68$ in a weekly bribe to get your Wheaties job back.

You can also calculate this by directly setting the utility from paying the bribe at the higher wage equal to $u^C = 150$ (in a way analogous to what we did in the previous part). This would give you the equation

$$(0.5F+1500)^{0.5}\left(\frac{(0.5F+1500)}{50}\right)^{0.5} = 150. \tag{10.11.x}$$

When solved for F, this gives $F = -878.68$; i.e. you would be willing to pay this amount which is the same as calculated through the expenditure function approach.

(f) *How much would you work assuming that the secretary has successfully extracted the maximum amount you are willing to pay to get your Wheaties job back?*

Answer: We can use our answer in equation (10.11.iv) since F is just like a negative endorsement deal that gets to the higher wage.

$$\begin{aligned} c &= 0.5(-F) + 30w = 0.5(-878.68) + 30(50) \approx 1060.66 \text{ and} \\ \ell &= \frac{0.5(-F)}{w} + 30 = \frac{0.5(-878.68)}{50} + 30 \approx 21.21. \end{aligned} \tag{10.11.xi}$$

Thus, you will take 21.21 hours of leisure — which means you will work for 60-21.21=38.79 hours.

(g) *Re-draw your graphs from part A but now label all the points and intercepts in accordance with your calculations. Does your prediction from A(g) about the size of the maximum bribe relative to the size of the minimum endorsement hold true?*

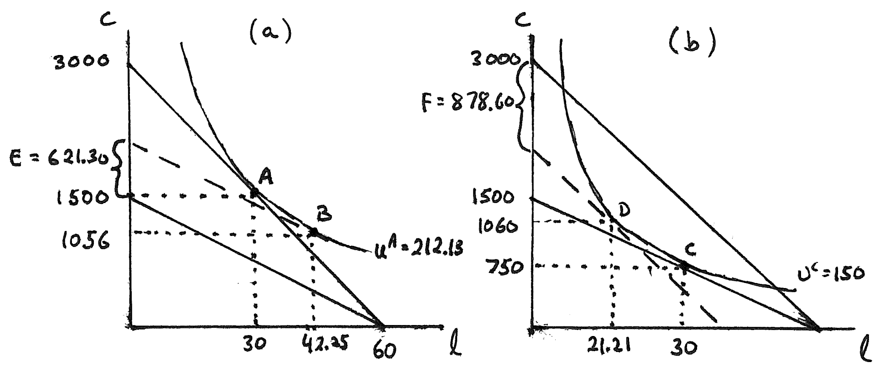
Answer: This is done in Exercise Graph 10.11(2).

The prediction from part A(g) holds true. Here we have a case where the wealth and substitution effects exactly offset one another in the absence of bribes or endorsements — which results in leisure of 30 hours at both the high wage (point A) and the low wage (point C). We predicted in part A(g) that this should lead to $F > E$ — which holds for this numerical example where $F = 1060.66 > 612.30 = E$.

Exercise 10.13

Policy Application: *Price Subsidies*: *Suppose the government decides to subsidize (rather than tax) consumption of grits.*

A: *Consider a consumer that consumes boxes of grits and "other goods".*



Exercise Graph 10.11(2) : Endorsement Deals and Bribes: Part 2

(a) *Begin by drawing a budget constraint (assuming some exogenous income) with grits on the horizontal axis and "other consumption" on the vertical. Then illustrate a new budget constraint with the subsidy — reflecting that each box of grits now costs the consumer less than it did before.*

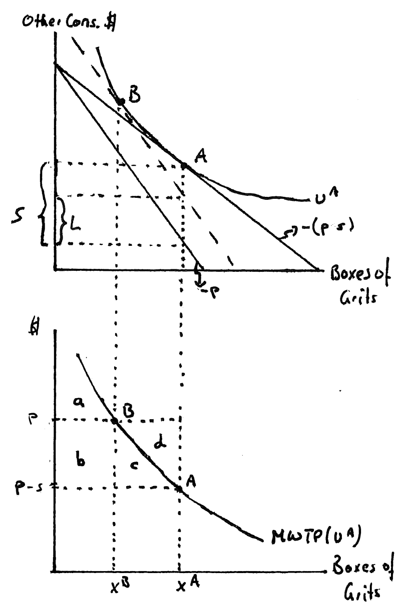
Answer: This is illustrated in the top graph of Exercise Graph 10.13(1), with $(p - s)$ indicating the price with the subsidy and p indicating the price without.

(b) *Illustrate the optimal consumption of grits with an indifference curve tangent to the after-subsidy budget. Then illustrate in your graph the amount that the government spends on the subsidy for you. Call this amount S.*

Answer: The optimal consumption bundle is illustrated as bundle A. The vertical intercept of that bundle indicates how much in other consumption the consumer is able to afford given that grits are subsidized. Had they not been subsidized, a much lower amount (read off the no-subsidy budget) would be available for other consumption. The difference is S — the amount the government paid for this consumer under the price subsidy policy.

(c) *Next, illustrate how much the government could have given you in a lump sum cash payment instead and made you just as happy as you are under the subsidy policy. Call this amount L.*

Answer: The government could have chosen not to alter the price of grits (and thus not alter the slope of the no-subsidy budget line) but instead simply shift that budget out in a parallel way by giving a cash subsidy. The amount in cash the government could have given to make the consumer just as happy as she is under the price subsidy is then an amount that creates the dashed budget which is tangent to the post-subsidy indifference curve u^A. This tangency occurs at bundle B, and the cost of this cash subsidy is simply the vertical difference between the dashed bud-



Exercise Graph 10.13(1) : Subsidizing Grits

get and the parallel no-subsidy budget. That distance can be measured anywhere (since the lines are parallel) and is indicated as the distance L.

(d) *Which is bigger — S or L?*

Answer: S is bigger than L because of the substitution effect from A to B. You don't have to give someone as much in unrestricted cash as you would have to spend in a subsidy that is restricted to the purchase of one good.

(e) *On a graph below the one you have drawn, illustrate the relevant MWTP curve and show where S and L can be found on that graph.*

Answer: The graph below the top graph derives the $MWTP$ or compensated demand curve that corresponds to utility level u^A. Under the price subsidy, the consumer consumes at A — which gives consumer surplus of $a+b+c$. The government is paying the difference between p and $(p-s)$ for each of the x^A boxes of grits the consumer buys — which means that the cost of the price subsidy is $S = b + c + d$.

Under the cash subsidy, the consumer faces the higher price p (rather than $(p-s)$) and buys x^B rather than x^A assuming she receives the cash subsidy L. This leaves her with consumer surplus of a in the grits market — but she is equally happy since both A and B lie on the same indifference curve. The only way she can be equally happy is if the cash subsidy was enough to make up for the loss in consumer surplus in the grits market — i.e. $L = b + c$.

(f) *What would your tastes have to be like in order for S to be equal to L.*

Answer: The substitution effect that creates the difference between S and L would have to disappear — which happens only if there is a sharp kink in the indifference curve at A (such as if grits and other goods are perfect complements). In that case, $A = B$ and $L = S$. In the lower graph, this implies that B lies directly above A — leading to a perfectly vertical $MWTP$ curve and the disappearance of the area d.

(g) *True or False: For almost all tastes, price subsidies are inefficient.*

Answer: This is true — so long as there aren't sharp kinks in just the right places of indifference curves — i.e. so long as goods are somewhat substitutable at the margin, $S > L$ which leaves the difference as a deadweight loss. If the substitutability goes away, so does the deadweight loss triangle d as the $MWTP$ curve becomes vertical.

B: *Suppose the consumer's tastes are Cobb-Douglas and take the form $u(x_1, x_2) = x_1^{\alpha} x_2^{(1-\alpha)}$ where x_1 is boxes of grits and x_2 is a composite good with price normalized to 1. The consumer's exogenous income is I.*

(a) *Suppose the government price subsidy lowers the price of grits from p to $(p-s)$. How much S will the government have to pay to fund this price subsidy for this consumer?*

Answer: We need to solve the problem

$$\max_{x_1,x_2} x_1^{\alpha} x_2^{(1-\alpha)} \text{ subject to } px_1 + x_2. \tag{10.13.i}$$

This solves to

$$x_1 = \frac{\alpha I}{p} \text{ and } x_2 = (1-\alpha)I. \tag{10.13.ii}$$

When the government lowers the price to $(p-s)$, demand is $x_1 = \alpha I/(p-s)$. For each box of grits, the consumer pays $(p-s)$ while the government pays s. Thus, the government's expense is

$$S = \frac{s\alpha I}{p-s}. \tag{10.13.iii}$$

(b) *How much utility does the consumer attain under this price subsidy?*

Answer: Under the price subsidy, the consumer chooses the bundle $(x_1, x_2) = (\alpha I/(p-2), (1-\alpha)I)$. Substituting this into the utility function, we get the indirect utility function

$$V(p,s) = \left(\frac{\alpha I}{p-s}\right)^{\alpha} ((1-\alpha)I)^{(1-\alpha)} = \frac{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}{(p-s)^{\alpha}} I. \tag{10.13.iv}$$

(c) *How much L would the government have had to pay this consumer in cash to make the consumer equally happy as she is under the price subsidy?*

Answer: Given that we know how much utility the consumer gets under the price subsidy, we now have to ask what expenditure (in cash) would have been necessary to get to the same utility level at the non-subsidized price p. We can derive the expenditure function by solving the expenditure minimization problem

$$\min_{x_1,x_2} px_1 + x_2 \text{ subject to } u = x_1^{\alpha} x_2^{(1-\alpha)}, \tag{10.13.v}$$

and then plug the compensated demands into the objective $px_1 + x_2$, or we can simply invert the indirect utility function. Either way, we get

$$E(p,u) = \frac{p^{\alpha}}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}. \tag{10.13.vi}$$

We are interested in knowing the expenditure necessary at p to get to utility level $V(p,s)$ from equation (10.13.iv); i.e. we are interested in

$$E(p,V(p,s)) = \left(\frac{p^{\alpha}}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}\right)\left(\frac{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}{(p-s)^{\alpha}}I\right) = \left(\frac{p}{p-s}\right)^{\alpha} I. \tag{10.13.vii}$$

This is the total expenditure necessary to get to the price-subsidy utility level $V(p,s)$. Since the consumer starts with an income I, the amount of cash L the consumer would have to get to be equally happy as under the price subsidy would therefore be

$$L = E(p,V(p,s)) - I = \left[\left(\frac{p}{p-s}\right)^{\alpha} - 1\right] I. \tag{10.13.viii}$$

(d) *What is the deadweight loss from the price subsidy?*

Answer: The deadweight loss is then just the difference between what the government spends under the price subsidy (S) and what the government could have spent in a lump sum way (L) to make the consumer just as well off. This is

$$DWL = S - L = \left[1 + \frac{s\alpha}{p-s} - \left(\frac{p}{p-s}\right)^{\alpha}\right] I. \tag{10.13.ix}$$

(e) *Suppose $I = 1000$, $p = 2$, $s = 1$ and $\alpha = 0.5$. How much grits does the consumer buy before any subsidy, under the price subsidy and under the utility-equivalent cash subsidy? What is the deadweight loss from the price subsidy?*

Answer: We have calculated that the demand for grits is $x_1 = \alpha I/p$. Thus, when the price is unsubsidized originally, the consumer buys $x_1 = 0.5(1000)/2 = 250$. The price subsidy lowers the effective price for the consumer to 1 — which implies the new quantity demanded is $x_1 = 0.5(1000)/1 = 500$. To calculate the equivalent cash subsidy, we can use equation (10.13.viii) to get

$$L = \left[\left(\frac{2}{2-1}\right)^{0.5} - 1\right](1000) \approx 414.21. \tag{10.13.x}$$

The consumer's income under the cash subsidy would therefore rise to \$1,414.21 but the price would remain at $p = 2$. The consumer's demand would therefore be $x_1 = 0.5(1414.21)/2$ which is approximately 354.

Finally, the deadweight loss is simply $(S - L)$. The cost of the price subsidy, given that the consumer will demand 500 units of x_1 and the cost of the subsidy is \$1 per unit, is \$500. The deadweight loss is therefore 500-414.21=\$85.79. You can also get this by simply plugging the relevant values into the DWL equation we calculated in equation (10.13.ix); i.e.

$$DWL = \left[1 + \frac{1(0.5)}{2-1} - \left(\frac{2}{2-1}\right)^{0.5}\right](1000) \approx 85.79. \tag{10.13.xi}$$

(f) *Continue with the values from he previous part. Can you calculate the compensated demand curve you illustrated in A(e) and verify that the area you identified as the deadweight loss is equal to what you have calculated? (*Hint*: You need to take an integral and use some of the material from the appendix to answer this.)*

Answer: The compensated demand curve arises from the expenditure minimization problem

$$\min_{x_1,x_2} px_1 + x_2 \text{ subject to } u = x_1^{0.5}x_2^{0.5}. \tag{10.13.xii}$$

From this, we get compensated demands

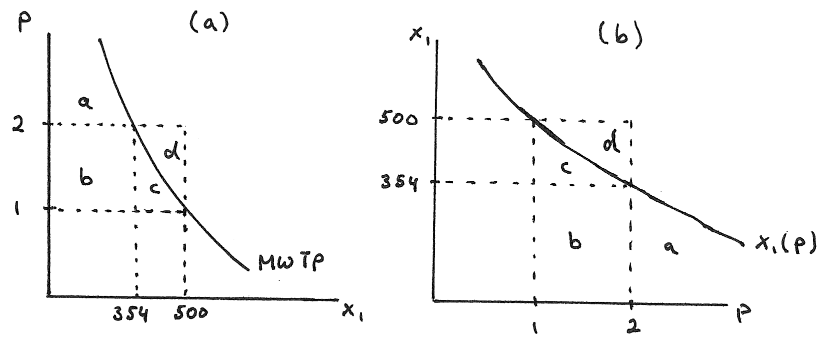
$$x_1 = \frac{u}{p^{0.5}} \text{ and } x_2 = p^{0.5}u. \tag{10.13.xiii}$$

Using the indirect utility function $V(p,s)$ from equation (10.13.iv), we can determine the utility the consumer gets under the price subsidy as

$$V(2,1) = \left(\frac{0.5^{0.5}0.5^{0.5}}{(2-1)^{0.5}}\right)(1000) = 500. \tag{10.13.xiv}$$

The appropriate compensated (or $MWTP$) curve is then $x_1 = 500/(p^{0.5})$. The inverse of this function is sketched in panel (a) of Exercise Graph 10.13(2).

It is similar to the lower graph in Exercise Graph 10.13(1) where we indicated that $S = b + c + d$, $L = b + c$ and $DWL = d$. Panel (b) of Exercise Graph 10.13(2) then simply inverts panel (a), placing x_1 on the vertical (rather than the horizontal) and p on the horizontal (rather than the vertical) axes.



Exercise Graph 10.13(2) : Subsidizing Grits: Part 2

The area $L = c + b$ is then simply the integral under the function $x_1 = 500/(p^{0.5})$ evaluated from $p = 1$ to $p = 2$. This is

$$\int_1^2 \frac{500}{p^{0.5}} dp = 2(500)p^{0.5}|_1^2 = 1000(2^{0.5} - 1) \approx 414.21. \tag{10.13.xv}$$

This is exactly what we calculated for L in the previous part. The area $S = b + c + d$ is simply 500. Thus, the deadweight loss is $DWL = 500 - 414.21 = 85.79$, again exactly as we calculated before.

Exercise 10.15

Policy Application: *International Trade and Child Labor: Consider again the end-of-chapter problem 8.9 about the impact of international trade on child labor in the developing world.*

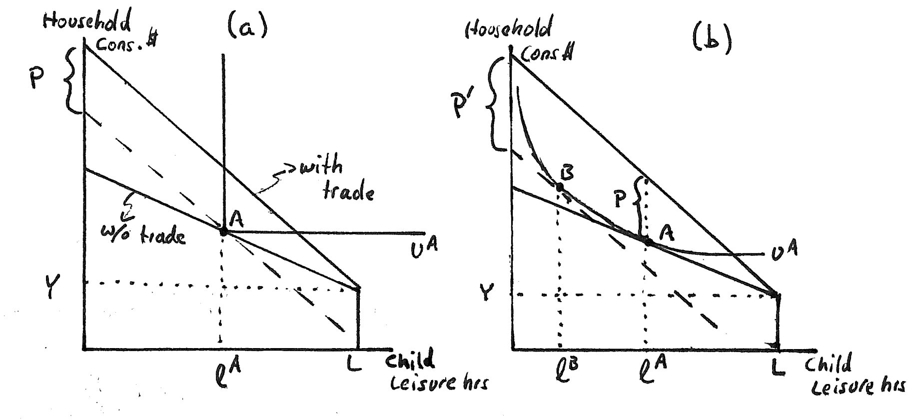
A: *Suppose again that households have non-child income Y, that children have a certain weekly time endowment L, and that child wages are w in the absence of trade and $w' > w$ with trade.*

(a) *On a graph with child leisure hours on the horizontal axis and household consumption on the vertical, illustrate the before and after trade household budget constraints.*

Answer: This is illustrated in panel (a) of Exercise Graph 10.15.

(b) *Suppose that tastes over consumption and child leisure were those of perfect complements. Illustrate in your graph how much a household would be willing to pay to permit trade — i.e. how much would a household be willing to pay to increase the child wage from w to w'?*

Answer: This is also illustrated in panel (a) of Exercise Graph 10.15. The household would be willing to give up an amount P that makes it just as well off as it would be in the absence of trade. In the absence of trade, the household optimizes at A — leaving the child with leisure of ℓ^A. Paying



Exercise Graph 10.15 : Child Leisure and Trade

a lump sum amount to get the higher wage is equivalent to shifting the after-trade (steeper) budget inward in a parallel way — and to figure out the highest payment the household is willing to make, we shift this budget until it just barely reaches the original indifference curve u^A. Given the shape of the perfect complements indifference curve, this means we shift the after-trade budget until it intersects at A. We can then read the size of the payment P as the vertical distance between the parallel lines. This distance is indicated on the vertical axis of the graph.

(c) *If the household paid the maximum it was willing to pay to cause the child wage to increase, will the child work more or less than before the wage increase?*

Answer: The child will continue to work the same amount — i.e. it will receive leisure of ℓ^A which implies that it's labor supply has not changed.

(d) *Re-draw your graph, assume that the same bundle (as at the beginning of part (b)) is optimal, but now assume that consumption and leisure are quite (though not perfectly) substitutable. Illustrate again how much the household would be willing to pay to cause the wage to increase.*

Answer: This is illustrated in panel (b) of Exercise Graph 10.15. We can now shift the after-trade budget inward until it is tangent to B — which puts it below A. The amount that the household is willing to pay is indicated as the vertical distance between the parallel lines on the vertical axis.

(e) *If the household actually had to pay this amount to get the wage to increase, will the child end up working more or less than before trade?*

Answer: The child will now work more — because it's leisure time has fallen from ℓ^A to ℓ^B — which means it's labor supply has increased by the same amount. (This is due to the emergence of the substitution effect that was absent in panel (a)).

(f) *Does your prediction of whether the child will work more or less if the household pays the maximum bribe to get the higher wage depend on how substitutable consumption and child leisure are?*

Answer: No — it does not depend on how substitutable. Even the slightest degree of substitutability in the indifference curve will cause B to lie to the left of A — which implies an increase in the child's labor supply. This goes away only if the substitutability is assumed away completely — and it never goes in the other direction (because substitution effects always point in the same direction).

(g) *Can you make a prediction about the relative size of the payment the household is willing to make to get the higher child wage as it relates to the degree of substitutability of consumption and child leisure? Are "good" parents willing to pay more or less?*

Answer: The size of the payment a household is willing to make to get the higher wages for the child increases with the degree of substitutability of household consumption with child leisure. You can see this by comparing P to P' in the graphs. In panel (b) of the graph, P is illustrated as the vertical difference between A and the after-trade budget (which is the same vertical distance found in (a) under perfect complementarity). This is unambiguously smaller than P' because B unambiguously falls to the left of A when there is some substitutability between consumption and child leisure. We concluded in the chapter 8 exercise that "good parents" (i.e. those that reduce child labor as child wages increase in the absence of a required payment to make that increase happen) are those who view consumption and child leisure as not very substitutable — parents that find it difficult to think of the household being better off if the child is not also working less. We are now finding that such "good parents" (i.e. those more like what is graphed in panel (a)) are not willing to pay as much of a bribe to get child wages to increase as "bad parents" are.

B: *Suppose that the household's tastes over consumption and leisure can be represented by the CES utility function $u(c,\ell) = (\alpha c^{-\rho} + (1-\alpha)\ell^{-\rho})^{-1/\rho}$.*

(a) *Derive the optimal household consumption and child leisure levels assuming the household has non-child weekly income Y, the child has a weekly time endowment of L, and the child wage is w.*

Answer: We have to solve the problem

$$\max_{c,\ell} \left(\alpha c^{-\rho} + (1-\alpha)\ell^{-\rho}\right)^{-1/\rho} \text{ subject to } Y + w(L-\ell) = c. \qquad (10.15.i)$$

This gives us

$$c = \left(\frac{\alpha w}{(1-\alpha)}\right)^{1/(\rho+1)} (Y + wL) \left[w + \left(\frac{\alpha w}{(1-\alpha)}\right)^{1/(\rho+1)} \right]^{-1} \qquad (10.15.ii)$$

and

$$\ell = (Y + wL)\left[w + \left(\frac{\alpha w}{(1-\alpha)}\right)^{1/(\rho+1)}\right]^{-1}. \qquad (10.15.iii)$$

(b) *Verify your conclusion from end-of-chapter problem 8.9 that parents are neither "good" nor "bad" when $Y = 0$ and $\rho = 0$; i.e. parents will neither increase nor decrease child labor when w increases.*

Answer: When $Y = 0$ and $\rho = 0$, the child leisure expression becomes

$$\ell = wL\left[w + \left(\frac{\alpha w}{(1-\alpha)}\right)\right]^{-1} = wL\left[\frac{w}{(1-\alpha)}\right]^{-1} = (1-\alpha)L. \qquad (10.15.iv)$$

Child leisure and thus child labor therefore do not depend on w, which implies an increase in child wages will result in neither an increase nor a decrease in child labor.

(c) *If international trade raises household income Y, what will happen to child labor in the absence of any change in child wages? Does your answer depend on how substitutable c and ℓ are?*

Answer: Since Y enters the leisure demand function positively, chid leisure will increase — i.e. child labor will fall. The answer does not qualitatively depend on the value of ρ — and thus does not depend on the elasticity of substitution.

(d) *When $\alpha = 0.5$ and $w = 1$, does your answer depend on the household elasticity of substitution between consumption and child leisure?*

Answer: When $\alpha = 0.5$ and $w = 1$, the expressions for optimal consumption and leisure reduce to

$$c = \frac{Y+L}{2} \text{ and } \ell = \frac{Y+L}{2}. \qquad (10.15.v)$$

The answer therefore does not depend on the household elasticity of substitution.

(e) *How much utility will the household get when $\alpha = 0.5$ and $w = 1$?*

Answer: The utility is then given by

$$u\left(\frac{Y+L}{2}, \frac{Y+L}{2}\right) = \left[0.5\left(\frac{Y+L}{2}\right)^{-\rho} + 0.5\left(\frac{Y+L}{2}\right)^{-\rho}\right]^{-1/\rho} = \frac{Y+L}{2}. \qquad (10.15.vi)$$

(f) *Derive the expenditure function for this household as a function of w and u. What does this reduce to when $\alpha = 0.5$? (Hint: You can assume $Y = 0$ for this part.)*

Answer: We now need to solve the problem

$$\min_{c,\ell} \; w\ell + c \text{ subject to } u = (\alpha c^{-\rho} + (1-\alpha)\ell^{-\rho})^{-1/\rho}. \qquad (10.15.vii)$$

This gives compensated demands

$$c = \left(\frac{\alpha w}{(1-\alpha)}\right)^{1/(\rho+1)} \left[\alpha\left(\frac{\alpha w}{(1-\alpha)}\right)^{-\rho/(\rho+1)} + (1-\alpha)\right]^{1/\rho} u \qquad (10.15.\text{viii})$$

and

$$\ell = \left[\alpha\left(\frac{\alpha w}{(1-\alpha)}\right)^{-\rho/(\rho+1)} + (1-\alpha)\right]^{1/\rho} u. \qquad (10.15.\text{ix})$$

Plugging these into the expenditure expression $w\ell + c$, we get the expenditure function

$$E(w,u) = \left[w + \left(\frac{\alpha w}{(1-\alpha)}\right)^{1/(\rho+1)}\right] \left[\alpha\left(\frac{\alpha w}{(1-\alpha)}\right)^{-\rho/(\rho+1)} + (1-\alpha)\right]^{1/\rho} u. \qquad (10.15.\text{x})$$

When $\alpha = 0.5$, this reduces to

$$E(w,u) = \left(w + w^{1/(\rho+1)}\right)\left(0.5w^{-\rho/(\rho+1)} + 0.5\right)^{1/\rho} u. \qquad (10.15.\text{xi})$$

(g) *Suppose non-child income $Y = 0$, child time is $L = 100$, $\alpha = 0.5$, $\rho = 1$ and w is initially 1. Then international trade raises w to 2. How does the household respond in its allocation of child leisure?*

Answer: We calculated in (d) that, when $\alpha = 0.5$ and $w = 1$, consumption and leisure are both $(Y+L)/2$ regardless of ρ. When $Y = 0$ and $L = 100$, this implies that initial household consumption and child leisure will both be equal to 50, leaving the child to work 50 hours per week. When $\alpha = 0.5$, $Y = 0$, $L = 100$ and $\rho = 1$, the optimal household consumption from equation (10.15.ii) and the optimal child leisure from equation (10.15.iii) become

$$c = \frac{100w^{3/2}}{w + w^{1/2}} \text{ and } \ell = \frac{100w}{w + w^{1/2}}. \qquad (10.15.\text{xii})$$

Plugging in $w = 2$, we then get $c \approx 82.84$ and $\ell \approx 58.58$. Household allocation of child leisure therefore increased by 8.58 hours, and child labor falls from 50 to 41.42.

(h) *Using your expenditure function, can you determine how much the household would be willing to pay to cause child wages to increase from 1 to 2? If it did in fact pay this amount, how would it change the amount of child labor?*

Answer: To determine how much the household would be willing to pay to induce an increase in child wages from 1 to 2, we first have to know household utility at the original wage $w = 1$. From our answer to (e), we see that household utility when $\alpha = 0.5$ and $w = 1$ is independent of ρ and equal to $(Y+L)/2$ which reduces to 50 when $Y=0$ and $L=100$. The most the household would be willing to pay to cause an increase in child wages from 1 to 2 is therefore an amount that would, once the wages increase, leave the household with utility of 50.

The expenditure necessary to attain utility of 50 at $w=2$ is given by the expenditure function. In our answer to (f), we calculated this for the case when $\alpha=0.5$ in equation (10.15.xi). When $\rho=1$, this reduces to $E(w,u)=(0.5w+w^{0.5}+0.5)u$. Evaluating this at $w=2$ and $u=50$, we get $E(2,50)\approx 145.71$. The value of the household's endowment — which is just the value of the child's leisure since $Y=0$ — at $w=2$ is 2(100)=200. The most the household is willing to pay for the child wage to increase from 1 to 2 is therefore 200-145.71=54.29.

If the household did in fact pay 54.29 to cause child wages to increase from 1 to 2, we can again use equation (10.15.iii) to determine the new optimal level of child leisure. When $\alpha=0.5$ and $\rho=1$, this equation becomes

$$\ell=(Y+wL)\left(w+w^{1/2}\right)^{-1}. \qquad (10.15.xiii)$$

The payment of 54.29 is then a lump sum negative amount that can be inserted in place of Y. Substituting this, and substituting $w=2$ and $L=100$, we get

$$\ell=(-54.29+2(100))\left(2+2^{1/2}\right)^{-1}\approx 42.68. \qquad (10.15.xiv)$$

This implies that child labor would increase to 100-42.68=57.32 hours.

(i) *Repeat the two previous steps for the case when $\rho=-0.5$ instead of 1.*

Answer: None of our answers for $w=1$ change when ρ changes as illustrated in previous parts of the question. Thus, initially the household will again choose $c=50$, $\ell=50$ and attain utility $u=50$, with the child working 50 hours per week.

To calculate how the household decision will change when wage increases to 2, we can again use equations (10.15.ii) and (10.15.iii). When $\alpha=0.5$, $L=100$, $Y=0$ and $\rho=-0.5$, these reduce to

$$c=\frac{100w^2}{1+w} \text{ and } \ell=\frac{100}{1+w}, \qquad (10.15.xv)$$

and when $w=2$, this implies $c=133.33$ and $\ell=33.33$. Thus, child labor *increases* from 50 when $w=1$ to $100-33.33=66.66$ when $w=2$.

To determine how much the household would be willing to pay to cause wages to increase in this way, we can again use the expenditure function. When $\alpha=0.5$ and $\rho=-0.5$, this reduces to

$$E(w,u)=\left(w+w^2\right)(0.5w+0.5)^{-2}u. \qquad (10.15.xvi)$$

Since the household's utility at $w = 1$ is 50, we need to calculate

$$E(2,50) = \left(2+2^2\right)(0.5(2)+0.5)(50) = 133.33. \quad (10.15.xvii)$$

Since the value of the household endowment is 2(100)=200 when $w = 2$, the household would therefore be willing to pay up to $200-133.33 = 66.67$ in order to get the wage to increase from 1 to 2.

Finally, we can determine the amount of child labor (if the household actually paid 66.67 to cause an increase in the wage) by returning to equation (10.15.iii) which, when $\alpha = 0.5$ and $\rho = -0.5$, reduces to

$$\ell = \frac{Y + wL}{w + w^2}. \quad (10.15.xviii)$$

Since the payment of 66.67 is equivalent to a negative household income, we can set Y to -66.67. Substituting this, and letting $L = 100$, we then get $\ell = 22.22$. Thus, child labor would increase to $100 - 22.22 = 77.78$ hours per week.

(j) *Are your calculations consistent with your predictions in (f) and (g) of part A of the question?*

Answer: Yes. In (f), we predicted the following: When parents pay the most they are willing to pay to open trade and raise child wages, children will work more than they did before — and the amount they will work more increases the more substitutable consumption and child leisure are. When ρ changes from 1 to -0.5, it causes the elasticity of substitution to increase (from 0.5 to 2). We have shown that at $\rho = 1$ (when the elasticity of substitution is low), child labor increases from 50 to 57.32 when parents have to pay the maximum bribe to get the higher child wage; and at $\rho = -0.5$ (when the elasticity of substitution is high), child labor increases from 50 to 77.78.

In (g), we predicted that the size of the payment a household is willing to make to get higher child wages increases as consumption and child leisure become more substitutable. We have shown that this payment increases from 54.29 to 66.67 when ρ falls from 1 to -0.5 — i.e. when the elasticity of substitution increases from 0.5 to 2.

Conclusion: Potentially Helpful Reminders

1. The regular (or uncompensated) demand curve is always the one you want to use if you are trying to predict what consumers will actually do as a result of a price change (regardless of what causes that price change).

2. The compensated demand (or marginal willingness to pay) curve is always the one you want to use when assessing changes in consumer surplus that result from price changes.

3. Since deadweight loss is a loss of consumer surplus, it is always measured on the compensated demand curve.
4. If the underlying good is quasilinear, the compensated and uncompensated demand curves are the same curves — which implies that this is the one case where you can measure consumer surplus changes along regular (uncompensated) demand curves.
5. Since marginal willingness to pay curves arise from substitution effects, they will be steep for small substitution effects and shallow for large substitution effects.
6. For students who do the B-part of the chapter, note the multiple ways we have found to calculate the various functions in the duality picture. Note further that this allows us multiple ways of calculating things like changes in consumer surplus or deadweight losses. In particular, we can either use the expenditure function — or we can take integrals on the compensated demand curve. A number of the end-of-chapter exercises illustrate the equivalence of these two methods.
7. Again for students who do the B-part of the chapter: Be sure to understand the duality picture well. You might be given one of the functions in the picture and then asked to tackle a problem — which is hard unless you know how to get from one part of the duality picture to another. For instance, if you already have the indirect utility function and you need the expenditure function, it is much easier to simply invert the indirect utility function rather than setting up the expenditure minimization problem and solving it. But be sure not to memorize but rather try to understand why the functions are related to one another as the picture shows.