

## CHAPTER

# 11

## One Input and One Output: A Short-Run Producer Model

In this chapter, we begin to look at the theory of the firm as a price taker. We do this in the simplest possible setting — where producers use a single input to produce a single output — because it is in this setting that we can develop the basics of profit maximization. As will become clearer in upcoming chapters, this one-input/one-output model can be thought of as a *short run* model of producer choice because we can think of inputs (other than the one input in the model) as being fixed in the short run. So, for instance, when we say “labor is the only input”, one interpretation of this statement is that it really means “labor is the only input that can be varied in the short run” because other inputs like factory size can only be changed in the long run.

### Chapter Highlights

The main points of the chapter are:

1. A **production plan** specifies a bundle of inputs and outputs just as a consumption bundle in consumer theory represents a bundle of different goods. Just as consumers choose consumption bundles that maximize utility, **producers choose production plans that maximize profit**. Both try “to do the best they can given their circumstances,” but — unlike in the consumer case — what is “best” has a more concrete meaning because we can quantify profit as the difference between economic revenues and costs.
2. As a result, “indifference curves” for producers — or **isoprofit curves** — are not a matter of “tastes” but emerge from the fact that producers are indifferent between production plans that result in the same level of profit. Since the profit of any production plan depends on the input and output prices, this implies that these producer “indifference curves” emerge from prices. This is unlike the consumer case where prices have nothing to do with indifference curves.

3. The fundamental constraint the producers face is a **technological constraint** that limits which production plans are in fact technologically feasible. Unlike the consumer case where constraints are shaped by prices, the technological constraint — modeled through the **production frontier** or the **production function** — has nothing to do with prices.
4. Any profit maximizing production plan (that involves positive production) has the characteristic that the **marginal revenue product of the input** equals the marginal cost of the input represented by the input's price. From this insight we can derive both the (short run) **output supply** curve (or function) as well as the (short run) **input demand** curve (or function), each sloping in the expected direction because of the **law of diminishing marginal product**.
5. There are **two ways to think of profit maximization**: (1) As a single problem in which we equate marginal revenue products with input prices, or (2) as a two-step problem in which the firm first derives its **cost curve** (or function) and then asks where the difference between revenues and costs is maximized. The latter allows us to state the profit maximizing condition as  $MC = p$  for price taking firms.
6. Finally, **economic profit** is the difference between *economic* revenues and costs. This implies that, when a firm makes zero economic profit, it is doing as well in this industry as it could be in the next best alternative industry — it does not mean that “the firm isn’t making money.” Thus, it makes sense for firms to produce so long as economic profit is not negative.

## 11A Solutions to Within-Chapter Exercises for Part A

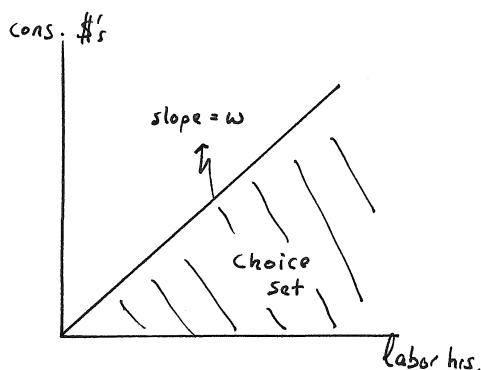
### Exercise 11A.1

Can you model a worker as a “producer of consumption” and interpret his choice set within the context of the single input, single output producer model?

Answer: A worker uses the input “labor hours” to produce the output “dollars for consumption” by selling labor hours at the market wage  $w$ . This can be graphed with “labor hours” on the horizontal and “dollars of consumption” on the vertical, as in Exercise Graph 11A.1. For every input hour, the worker gets  $w$  — thus implying a slope  $w$  of the production frontier.

### Exercise 11A.2

Which of the producer choice sets in Graph 11.1 is non-convex? What makes it non-convex?



Exercise Graph 11A.1 : Worker as a “Producer of Consumption”

**Answer:** The producer choice set in panel (b) is non-convex because you can choose two production plans (such as  $A'$  and  $C'$ ) such that the line connecting the two plans lies outside the set.

#### Exercise 11A.3

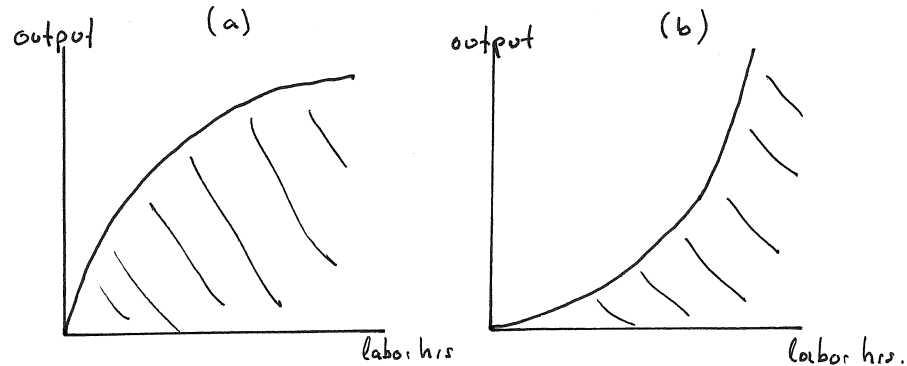
Suppose my technology was such that each additional worker hour, beginning with the second one, is less productive than the previous. Would my producer choice set be convex? What if my technology was such that each additional worker hour, beginning with the second one, is more productive than the previous.

**Answer:** The first of these is graphed in panel (a) of Exercise Graph 11A.3 while the second is graphed in panel (b). Each worker becoming less productive than the previous leads to the increasingly shallow slope in (a) while each worker becoming more productive than the previous leads to the increasingly steep slope in (b). The first producer choice set is convex; the second is not.

#### Exercise 11A.4

Under the production technology in Graph 11.1b, what is the approximate marginal benefit of hiring an additional labor hour when I already have 95 labor hours employed?

**Answer:** We know from Graph 11.1(b) in the text that I can produce 390 cards per day with 90 labor hours and 400 cards per day with 100 labor hours. Thus, the additional 10 labor hours increase total output by 10 cards — which gives me a marginal benefit of workers in that range (including the 95th worker) of about 1 card per worker hour. This is also reflected in the marginal product graph (Graph 11.2(b) in the text) where the marginal product for the 95th worker hour is indicated as 1.



Exercise Graph 11A.3 : Two different types of producer choice sets

**Exercise 11A.5**

Relate your answer from exercise 11A.4 to a point on the  $MP_\ell$  curve plotted in Graph 11.2b.

Answer: The  $MP_\ell$  curve tells me that the 95th worker hour results in an increase in total output of 1 card per day — which is the same as what we derived in the previous exercise from the production frontier. This is because the marginal product curve is just the slope of the production frontier, and between 90 and 100 labor hours, the slope of the production frontier is approximately 1.

**Exercise 11A.6**

What would the  $MP_\ell$  curves look like for the technologies described in within-chapter exercise 11A.3?

Answer: For the first technology described in exercise 11A.3, the  $MP_\ell$  curve would be downward sloping, and for the second it would be upward sloping.

**Exercise 11A.7**

*True or False:* The Law of Diminishing Marginal Product implies that producer choice sets in single input models must be convex beginning at some input level.

Answer: This is true. At the input level at which the law of diminishing marginal product sets in, the production frontier begins to become shallower and shallower — resulting in a choice set underneath that is convex from that point forward.

**Exercise 11A.8**

*True or False:* If the Law of Diminishing Marginal Product did not hold in the dairy industry, I could feed the entire world milk from a single cow. (*Hint:* Think

of the cow as a fixed input and feed for the cow as the variable input for which you consider the marginal product in terms of milk produced per day.)

Answer: Take a given cow and begin feeding it. Each ounce of feed results in more milk per day from the cow than each previous ounce of feed. If the law of diminishing marginal product of feed does not at some point decrease, this means I can keep feeding the cow and I will get the cow to produce ever increasing amounts of milk (per day) from each additional ounce of feed. At some point, I will have given the cow so much feed that we will have produced enough milk for the world from that single cow. Obviously this is absurd, which illustrates why the assumption of diminishing marginal product of feed must hold.

#### Exercise 11A.9

Without knowing what prices and wages are in the economy, can you tell by looking at a single isoprofit curve whether profits for production plans along this curve are positive or negative? What has to be true about an isoprofit curve in order for profit to be zero?

Answer: The intercept of an isoprofit curve on the vertical axis is Profit divided by price. You can therefore tell whether profits along an isoprofit curve are positive by checking whether or not the isoprofit curve has positive intercept. If it has positive intercept, then profit for all the production plans along that isoprofit curve is positive (assuming price is positive); if it has negative intercept, then profit for all production plans along the curve is negative. If the isoprofit curve intersects at the origin, then profit is zero.

#### Exercise 11A.10

What would have to be true in order for an isoprofit curve to have a negative slope?

Answer: The slope of an isoprofit curve is  $w/p$ . In order for this slope to be negative, either wage or output price would therefore have to be negative.

#### Exercise 11A.11

How would the blue isoprofit curve in Graph 11.3a change if the wage rises to \$30? What if instead the output price falls to \$2?

Answer: Along the original blue isoprofit curve, wage is \$20 and output price is \$5 — resulting in a slope of  $w/p = 20/5 = 4$ . If the wage increases to \$30, the slope increases to  $30/5 = 6$ . Since the intercept is given by Profit/ $p$  (and  $w$  therefore does not enter the intercept term), the intercept remains at 40.

Now suppose that instead the price fell from \$5 to \$2. Originally, the intercept of the blue isoprofit is Profit/ $p$  = Profit/5 = 40 — which implies that Profit along the isoprofit curve is \$200. In order for profit to remain unchanged at the intercept when price falls to \$2, it would have to be the case that the new intercept is  $200/2 = 100$ . Thus, the intercept of the isoprofit would shift up from 40 to 100. The slope

of the isoprofit is  $w/p$  or  $20/p$  when  $w = 20$ . Thus, when price falls to \$2, the new slope of the isoprofit curve must be  $20/2 = 10$ ; i.e. the slope increases from 4 to 10.

#### Exercise 11A.12

It appears from panel (f) of Graph 11.5 that profits are smallest (i.e. most negative) when I stop hiring at 22 labor hours per day. What can you conclude about the slope of the production frontier in panel (c) of the graph at 22 daily labor hours? Explain.

Answer: At 22 daily labor hours, the slope of the production frontier must be equal to  $20/5=4$  which is equal to the slope of isoprofit curves ( $w/p = 20/5$ ). Notice that the production frontier has two points with this slope — the profit maximizing point  $A$  where the highest possible isoprofit curve is obtained a second point on the initially non-convex portion of the production frontier. At both of these points,  $w/p = MP_\ell$  because the slope of the isoprofit curve is  $(w/p)$  and the slope of the production frontier is  $MP_\ell$ . Simply rearranging the terms in this equation, we can see that this is equivalent to saying that at both of these points  $w = pMP_\ell$ . In panel (f), we show that this holds at labor hours of 22 and 78 — so it must equivalently hold in panel (c) for those labor hours. Put differently, at both these labor hours there exists an isoprofit curve that is just tangent to the production frontier — but only the second of these tangencies — point  $A$  in panel (c) — is profit maximizing (because the other tangency lies on an isoprofit which is lower and thus has less profit associated with each of its production plans.)

#### Exercise 11A.13

Suppose I have already signed a contract with my former student who is providing me with the factory space, machinery and raw materials for my business, and suppose that I agreed in that contract to pay my former student \$100 per month for the coming year. Is this an economic cost with respect to my decision of whether and how much to produce this year?

Answer: No, it is not an economic cost of producing for this year because I am legally obligated to pay it regardless of whether or how much I produce. Thus, my decision of whether or how much to produce has no impact on this “cost”.

#### Exercise 11A.14

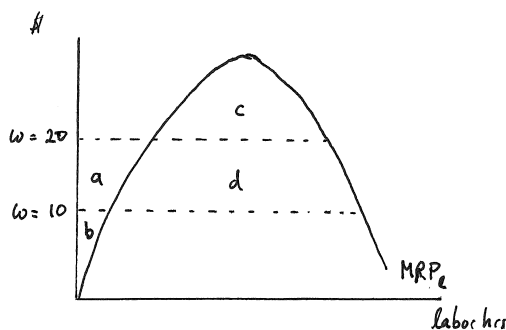
There are also production plans to the left of  $A$  where the slope of the production frontier is shallower. Why are we not considering these?

Answer: We are not considering these because all of them would lie on isoprofit curves that fall below the magenta curve that is tangent at  $B$  — and thus all of these production plans would result in less profit than  $B$ .

**Exercise 11A.15**

Which areas in the lower panel of Graph 11.6a add up to the \$200 profit I made before wages fell? Which areas add up to the \$1045 profit I make after wages fall?

Answer: The graph is replicated in Exercise Graph 11A.15 with the critical areas labeled. The \$200 profit at the original wage of \$20 is equal to  $c - a - b$ . The \$1045 profit at the new wage of \$10 is equal to  $c + d - b$  — which is equal to the shaded green minus the shaded magenta areas in the textbook graph.



Exercise Graph 11A.15 : Changing Profit as  $w$  falls

**Exercise 11A.16**

Given an intercept of  $-100$  of this isoprofit curve, what is the value of profit indicated by the shaded green minus the shaded magenta area in the lower panel of Graph 11.6b?

Answer: The shaded green minus the shaded magenta areas in panel (b) is equal to the profit at production plan C. The intercept of the isoprofit curve tangent at C in the top panel is  $-100$  and is equal to  $\text{Profit}/p$ . Since the output price  $p$  is \$5, we therefore know that  $\text{Profit}/5 = -100$  — which implies that  $\text{Profit} = -\$500$ . Thus, the shaded green minus the shaded magenta areas in the lower part of panel (b) is  $-\$500$ .

**Exercise 11A.17**

Had the increase in the market wage been less dramatic, would my best course of action still necessarily have been to shut down production?

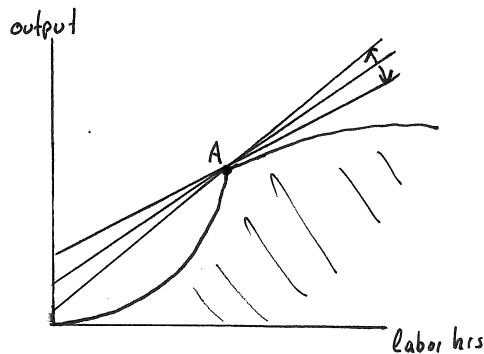
Answer: No. For smaller increases in the wage, the slope of the isoprofit curves would increase less dramatically. For sufficiently small increases in the wage, that would result in a tangency of one of the new isoprofit curves with the production frontier such that the isoprofit curve would still have positive intercept on the vertical axis — and thus the production plan at that tangency would still give positive

profit. In that case, this is the optimal production plan and the firm should not shut down.

#### Exercise 11A.18

What would have to be true of the production frontier in order for the original optimal production plan  $A$  to remain optimal as wages either rise somewhat or fall somewhat? (*Hint: Consider what role kinks in the producer choice set might play.*)

Answer: In Exercise Graph 11A.18, we illustrate a production frontier with a kink at  $A$ . At this kink, a number of different slopes are “tangent”. Thus, beginning with wages and prices such that  $A$  is profit maximizing, we can change wages up or down within some range and still have the steeper or shallower isoprofits that result be “tangent” at  $A$  with positive intercepts on the vertical axis. In such cases, the producer would not change production plans as wages change.



Exercise Graph 11A.18 : No change in profit maximizing behavior with changes in wages

#### Exercise 11A.19

Why would it be economically rational for me to still stay open for business when  $w = w^*$  where my profit is zero?

Answer: When economic profit is zero, this means that the producer does as well in this activity as she could in the next best alternative activity. If she spends time on her business, this means that her labor hours are being compensated — so she is indeed “making money”. But she is making just as much money as she could doing the next best thing. So, when her profit is zero, she does as well as she could anywhere else — which makes it economically rational to continue producing. (It would also in this case be economically rational to do the next best thing — which would also result in zero profit.)



**Exercise 11A.20**

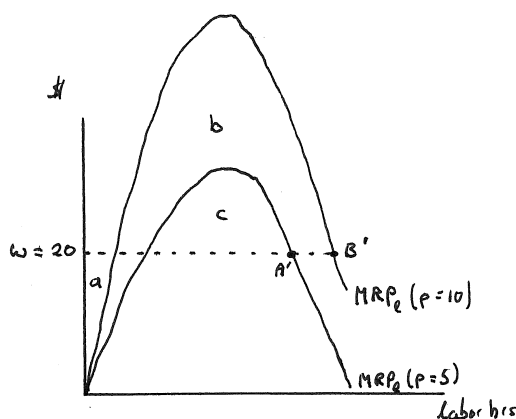
If I had signed a contract and agreed to make monthly payments for the next year to my former students who provided me with my factory space, would  $w^*$  — the highest wage at which I will still produce — be any different?

Answer: No,  $w^*$  would be no different because I have to make the monthly payments whether I produce this year or not — which means that they are not an economic cost that is relevant for my current economic decisions.

**Exercise 11A.21**

What areas in the lower panel of Graph 11.8 add up to my new profit? What is the dollar value of this new profit (which you can calculate from the intercept of the isoprofit curve in the top panel of the graph)?

Answer: The new profit areas on the lower panel of the graph must be read off the new (magenta)  $MRP_\ell$  curve at the unchanged wage of  $w = 20$ . The graph is replicated here as Exercise Graph 11A.21 with some relevant areas labeled with lower case letters. The new profit is then  $b + c - a$ . The dollar value for this area can be calculated from the top panel of the graph in the text. The intercept of the new optimal (magenta) isoprofit (tangent at  $B$ ) is Profit/ $p=209$ . Since  $p = 10$ , this implies Profit/ $10=209$  or Profit=\$2,090.



Exercise Graph 11A.21 : New Profit when  $p$  increases to \$10

**Exercise 11A.22**

Can you tell from Graph 11.8 how the labor demand curve will change when  $p$  changes?

Answer: At the lower price  $p = 5$ , the labor demand curve lies on the downward sloping portion of the original (blue)  $MRP_\ell(p = 5)$  curve. At the new price  $p = 10$ , the labor demand curve lies on the downward sloping portion of the new (magenta)  $MRP_\ell(p = 10)$  curve. Thus, the labor demand curve shifts out as  $p$  increases. Similarly, it would shift in when  $p$  decreases.

#### Exercise 11A.23

What value would  $p$  have to take in order for isoprofits to have the same slope as when wages increased to \$30 per hour (as in Graph 11.6b)? What would be my optimal course of action in that case?

Answer: When  $p = 5$  and  $w$  increases from \$20 to \$30, the slope of the isoprofit, which is always  $w/p$ , increases from 4 to 6. If  $w$  remained unchanged at \$20, the slope  $w/p$  would change to 6 if  $20/p = 6$  — i.e. if  $p = 10/3 = 3.33$ . The optimal course of action depends only on the slope of the isoprofit — and thus, when the slope is unchanged, the optimal action remains unchanged.

#### Exercise 11A.24

In Graph 11.9 we implicitly held wage fixed. What happens to the supply curve when wage decreases?

Answer: In panel (a), the isoprofits for any given  $p$  become shallower as  $w$  decreases — causing the tangency with the production frontier to shift up and to the right. Thus, as  $w$  decreases, output increases for any  $p$ . This then implies that the supply curve shifts to the right.

#### Exercise 11A.25

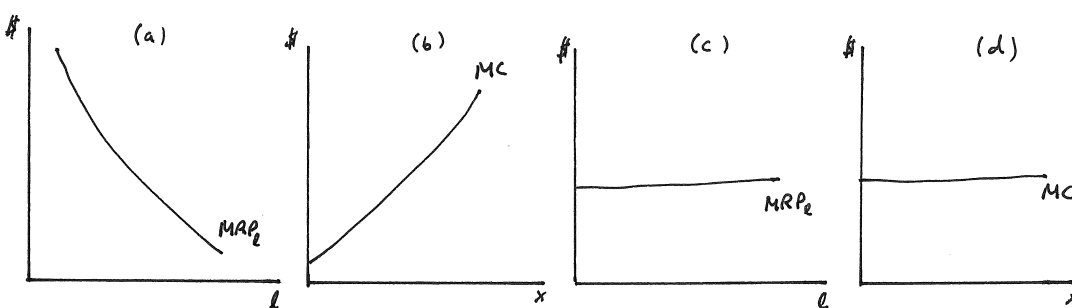
If the wage rate used to construct the panels on the right of Graph 11.10 is \$20, can you conclude what the slope of the production frontier in panel (a) at 10 units of output is? Can you conclude what labor input is required to produce 10 units of output, and then what the vertical values of the curves in panels (b) and (c) are for that level of labor input?

Answer: The slope of the production frontier at 10 units of output must be 1. This is because we know from panel (e) that the marginal cost is 20 *in terms of dollars* that are used to buy the input labor — and we know labor costs \$20 per unit. Thus the additional labor required to produce one more unit of  $x$  must be 1. Similarly, the total labor input required to produce 10 units of output must be  $300/20=15$ . Since we just concluded that, when we are at output level of 10, it takes one additional unit of labor to produce 1 additional unit of output. Thus, at  $\ell = 15$  (which is how much labor it takes to produce 10 units of  $x$ ), the marginal product of labor must be 1 — which is then the vertical value of the  $MP_\ell$  curve at  $\ell = 15$ . The vertical value for  $\ell = 15$  in panel (c) is then just  $p$ .

**Exercise 11A.26**

What would be the shape of the  $MRP_\ell$  and  $MC$  curves if the entire producer choice set was strictly convex? What would the shape be for the production frontier graphed in Graph 11.1(a)?

Answer: These are illustrated in Exercise Graph 11A.26 — with panels (a) and (b) illustrating the curves for strictly convex choice sets (where production becomes increasingly difficult throughout) and panels (c) and (d) illustrating the curves for the linear production constraint in Graph 11.1.



Exercise Graph 11A.26 :  $MRP$  and  $MC$  Curves

**Exercise 11A.27**

*True or False:* On a graph with output on the horizontal and dollars on the vertical, the “marginal revenue” curve must always be a flat line so long as the producer is a price taker.

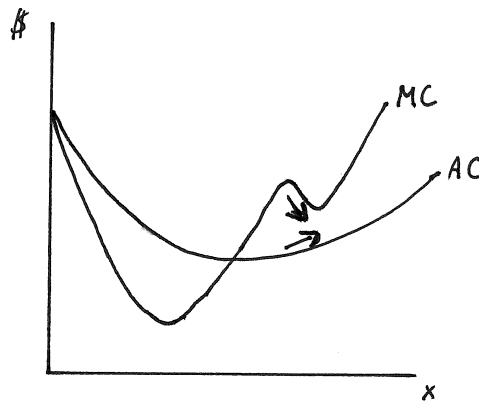
Answer: This is true. The marginal revenue is the additional revenue the producer can get from producing one more unit of output. If the producer is a price taker, she can always sell additional units of output at the market price. Thus, the marginal revenue of producing more is always equal to the market price.

**Exercise 11A.28**

Can  $MC$  fall while  $AC$  rises? (*Hint:* The answer is yes.) Can you give an analogous example of marginal test grades falling while the average grade rises at the same time?

Answer: Exercise Graph 11A.28 illustrates a case where this is the case. Note that all that is required for  $AC$  to increase is for  $MC$  to lie above  $AC$ . It does not necessarily require  $MC$  to be increasing as well. So long as  $MC$  lies above  $AC$ , it drags up the  $AC$  even if, for some interval, the  $MC$  is actually falling. You can again think of this in terms of your grades. Suppose that, after the second exam, you have

a course average of 55. Then you score a 100 on the third exam. This marginal grade of 100 raises your course average to 70. Now suppose you make a 90 on the fourth exam. Your marginal grade has fallen from 100 — but because it lies above your average (of 70) going into the exam, it will still raise your average (to 75). So, your average grade is increasing from 70 to 75 despite the fact that your marginal grade fell from 100 to 90.

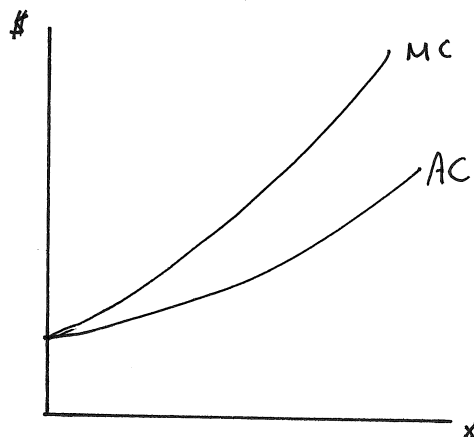


Exercise Graph 11A.28 : Falling  $MC$  with increasing  $AC$

#### Exercise 11A.29

How do the marginal and average cost curves look if the producer choice set is convex?

Answer: This is illustrated in Exercise Graph 11A.29. The shapes should make intuitive sense — the curves must be upward sloping because a convex producer choice set implies it gets increasingly difficult to produce additional output starting from the first unit. Thus, it must mean that each additional unit is more expensive than the last. And, as always,  $MC$  and  $AC$  must start at the same point — and  $MC$  must lie above  $AC$  if  $AC$  is upward sloping.

Exercise Graph 11A.29 :  $MC$  and  $AC$  when the producer choice set is Convex

## 11B Solutions to Within-Chapter Exercises for Part B

### Exercise 11B.1

How would this production function look differently if we did not specify that output levels off at  $2\alpha$ ?

Answer: The production function would have negative slope beginning at  $\ell = \pi/\beta$  — and would then oscillate between positive and negative slopes as labor increases.

### Exercise 11B.2

Define the production function generating the production frontier in Graph 11.1a and define the corresponding producer choice set formally.

Answer: The production function has constant slope and is given by  $f(\ell) = 4\ell$ . The producer choice set is then simply given by

$$C(f: \mathbb{R}_+^1 \rightarrow \mathbb{R}_+^1) = \{(x, \ell) \in \mathbb{R}^2 \mid x \leq 4\ell\}. \quad (11B.2)$$

### Exercise 11B.3

Given that  $f$  is really defined as in equation (11.3), how should equation (11.5) be modified to accurately reflect the marginal product of labor for labor hours above 100?

Answer: Since the marginal product of labor above 100 is zero, the full definition of the marginal product function is

$$MP_{\ell} = \begin{cases} 6.2832 \sin(0.031416\ell) & \text{for } 0 \leq \ell \leq 100 \text{ and} \\ 0 & \text{for } \ell > 100. \end{cases} \quad (11B.3)$$

#### Exercise 11B.4

Derive the marginal product of labor from the production function you derived in exercise 11B.2 above. Compare this to the graphical derivation in Graph 11.2a.

Answer: Taking the derivative of the production function  $f(\ell) = 4\ell$ , we get  $MP_{\ell} = 4$ . This tells us that the marginal product of an additional worker is always 4 units of output — i.e. the  $MP$  curve is horizontal as derived graphically in part A of the text.

#### Exercise 11B.5

Check to see that the Law of Diminishing Marginal Product (of labor) is satisfied for the production function in equation (11.3).

Answer: The Law of Diminishing Marginal Product says that, at some point, the marginal product of labor decreases as more labor is hired — i.e. at some point, the derivative of  $MP_{\ell}$  becomes negative. Recalling that the derivative of  $(\sin x)$  is  $(\cos x)$ , we get that the derivative of  $MP_{\ell}$  for this production function is

$$\frac{dMP_{\ell}}{dx} = 0.031416(6.2832) \cos(0.031416\ell) \approx 0.1974 \cos(0.031416\ell) \quad (11B.5)$$

for  $\ell < 100$  (since we defined the production function to flatten out at  $\ell = 100$ .) You can now check to see whether this derivative of the  $MP_{\ell}$  is positive or negative for different values of  $\ell$  by plugging in such different values between 0 and 100. You will find that the derivative is positive up to approximately  $\ell = 50$  — indicating that the  $MP_{\ell}$  curve initially slopes up (as in our graphs). However, as  $\ell$  increases above 50, the derivative of  $MP_{\ell}$  becomes negative — indicating that at approximately  $\ell = 50$ , the  $MP_{\ell}$  curve slopes down. Thus, the law of diminishing marginal product holds.

#### Exercise 11B.6

Without doing the math, can you tell if the curve  $\ell(p, 20)$  slopes up or down? How does it relate to  $\ell(p, 10)$ ?

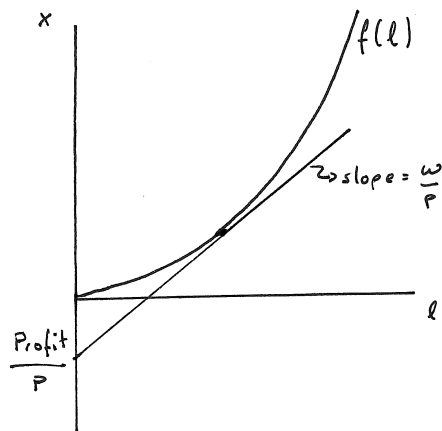
Answer: We concluded in part A that, as  $p$  increases, the profit maximizing producer increases output (assuming it was optimal for her to produce to being with). With labor as the only input, this also implies that she must hire more labor. Thus,  $\ell(p, 20)$  is increasing in  $p$  — i.e. it slopes up. We also concluded that, as wages

fall, the producer will increase output (assuming it was optimal for her to produce to begin with). Thus, when wages fall from  $w = 20$  to  $w = 10$ , output increases and, given there is only one input available to the producer, this implies she will hire more workers. Thus,  $\ell(p, 20) < \ell(p, 10)$  — which means that the labor demand curve shifts outward as  $w$  falls.

### Exercise 11B.7

Consider a production function that gives rise to increasing marginal product of labor throughout (beginning with the first labor hour). *True or False:* In this case, the mathematical optimization problem will unambiguously lead to a “solution” for which profit is negative.

Answer: This is true. This is illustrated in Exercise Graph 11B.7. The shape of the production function  $f(\ell)$  arises from the assumption of increasing marginal product of labor throughout. This shape implies a single tangency for isoprofits with slope  $w/p$  — and this “optimal” isoprofit must then necessarily have a negative vertical intercept. Since the vertical intercept of isoprofit curves is  $\text{Profit}/p$ , and since  $p$  is always assumed to be positive, it must be that the “solution” to the optimization problem gives us a production plan that results in negative profit. The true solution is therefore not the one given by the tangency.



Exercise Graph 11B.7 : Mathematical “solution” when  $MP_\ell$  is increasing throughout

### Exercise 11B.8

Consider a production function that gives rise to increasing marginal product of labor throughout (beginning with the first labor hour). *True or False:* In this case, the mathematical optimization problem will give a single solution — albeit one that minimizes rather than maximizes profit.

Answer: It is again apparent from Exercise Graph 11B.7 that, with the shape of the production function that emerges from increasing marginal product throughout, there is a single tangency for every isoprofit slope  $w/p$ . As we argued in the previous answer, this single tangency represents a production plan that gives rise to negative profit. In fact, this production plan gives the least profit (i.e. the most negative profit) of any of the possible production plans that lie on the production frontier. You can see this by simply noting that every other production plan on the frontier lies on a higher isoprofit curve — i.e. on an isoprofit parallel to the one that gives rise to the tangency but lying to the northwest. In this sense, the mathematical solution that gives the tangency production plan actually gives us the lowest possible profit conditional on producing along the production frontier (rather than inside the producer choice set.)

#### Exercise 11B.9

Give an example of a producer choice set and economic conditions such that infinite production would be “optimal”.

Answer: The producer choice set and the economic conditions (i.e. wages and prices) depicted in Exercise Graph 11B.7 represent such an example. Pick any point on the producer choice set and draw the isoprofit curve that goes through that point. You will be able to draw a parallel isoprofit curve that lies to the northwest (i.e. with more profit) that also intersects the production frontier — with the resulting production plan using more labor and producing more output. Since you can do this for any point on the production frontier, you can keep doing it no matter how large your production already is. Thus, you do not reach a profit maximizing production plan until you produce an infinite amount of the output.

#### Exercise 11B.10

Do you think the scenario you outlined in the previous within-chapter exercise makes sense under the assumption of “price taking” behavior by producers?

Answer: No, it does not. The price taking assumption is based on an assumption that the market contains many small producers, none of which is large enough to have any impact on price. If any producer ever became as large as the example implies, this price taking assumption no longer makes sense — the fact that the producer keeps increasing production must eventually drive up wages (as the producer is hiring more and more scarce workers) and drive down the price that the producer can charge (as consumers won't be willing to keep paying the same price no matter how much of the good is dumped on the market). The price taking assumption is therefore inconsistent with producer choice sets that give rise to increasing marginal product throughout or even increasing marginal product for large amounts of the input.



**Exercise 11B.11**

What has to be true about  $\alpha$  in order for this production function to exhibit diminishing marginal product of labor?

Answer: The marginal product of labor is

$$MP_\ell = \frac{\partial f(\ell)}{\partial \ell} = \alpha A \ell^{\alpha-1}. \quad (11B.11.i)$$

This is declining in labor if its derivative is less than 0; i.e. if

$$\frac{\partial MP_\ell}{\partial \ell} = (\alpha - 1)(\alpha) A \ell^{\alpha-2} < 0 \quad (11B.11.ii)$$

which holds only if  $\alpha < 1$ .

**Exercise 11B.12**

Suppose  $0 < \alpha < 1$  and  $A > 0$ . Holding price fixed, is the labor demand function downward sloping in the wage? Holding wage fixed, is it upward or downward sloping in price? Can you graphically illustrate why your answers hold?

Answer: To determine whether the labor demand function is downward sloping in the wage, we simply need to take the derivative of the demand function with respect to wage and check whether this derivative is negative. We then get

$$\frac{\partial \ell(p, w)}{\partial w} = \left( \frac{1}{\alpha - 1} \right) \left( \frac{1}{\alpha A p} \right)^{1/\alpha-1} w^{(2-\alpha)/(\alpha-1)}. \quad (11B.12.i)$$

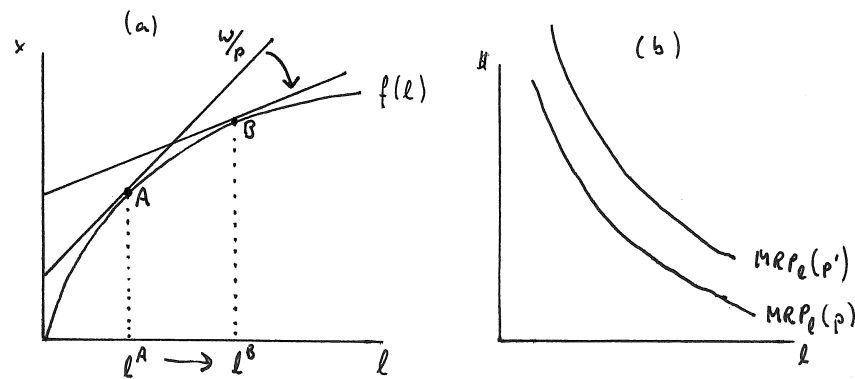
Given  $0 < \alpha < 1$  and  $A > 0$ , the second and third terms in this equation are positive. The first term  $(1/(\alpha - 1))$ , however, is negative. Thus, the derivative of the labor demand function with respect to  $w$  is negative — i.e. the labor demand function slopes down in  $w$ .

We can similarly determine which way the labor demand function slopes in  $p$ . Now we take the partial derivative of the labor demand with respect to  $p$  and check whether we can determine whether this derivative is positive or negative. The derivative is

$$\frac{\partial \ell(p, w)}{\partial p} = - \left( \frac{1}{\alpha - 1} \right) \left( \frac{w}{\alpha A} \right)^{1/(\alpha-1)} p^{-\alpha/(\alpha-1)}. \quad (11B.12.ii)$$

As before, the second and third terms in this equation are positive (given  $0 < \alpha < 1$  and  $A > 0$ ). The first term  $1/(\alpha - 1)$  is negative, but it is preceded by a negative sign which makes it positive. Thus, the partial derivative of  $\ell(p, w)$  with respect to  $p$  is positive — which implies that the labor demand function is increasing (or upward sloping) in  $p$ .

Both these results make intuitive sense. The first says that an increase in the cost of inputs (i.e. labor) will cause me to hire fewer workers; while the second says that an increase in the price at which I can sell the output will cause me to hire more workers. We can illustrate this graphically as we did in part A of the text. In panel

Exercise Graph 11B.12 : Changes in labor demanded as  $w$  and  $p$  change

(a) of Exercise Graph 11B.12, we illustrate the shape of the production function  $f(\ell)$  which gives rise to a convex producer choice set. Suppose that  $w$  and  $p$  were initially such that the steeper isoprofit arises — giving rise to the optimal production plan  $A$  at which  $\ell^A$  in labor hours is hired. Now suppose that  $w$  falls — causing the slope of isoprofits to fall. This results in a new profit maximizing production plan  $B$  — with  $\ell^B$  in labor hours hired. Thus, as  $w$  falls,  $\ell$  increases, which is the same as saying that labor demand increases as wages fall or, put differently, the labor demand function is downward sloping in  $w$ . (The labor demand function, holding  $p$  fixed, is the inverse of the labor supply curve we graph in panel (b) — but the downward slope remains as we flip the axes.) Similarly, if  $p$  increases, the isoprofit curves become shallower — again leading to an increase in labor hours hired from production plan  $A$  to  $B$ . Thus, as price increases, labor demanded goes up — i.e. the labor demand function is upward sloping in  $p$ .

The same can also be seen in the marginal revenue product graph of panel (b). Given the shape of the production function, we know that marginal product is diminishing throughout. This implies that the marginal revenue product curve is downward sloping — giving rise to the downward sloping labor demand curve and the fact that the labor demand function is downward sloping in  $w$ . At the same time, if  $p$  increases (from  $p$  to  $p'$ ), then the marginal revenue product curve shifts up. Thus, for any given wage, more labor is hired as  $p$  increases — i.e. labor demand is upward sloping in  $p$ .

### Exercise 11B.13

Suppose  $0 < \alpha < 1$  and  $A > 0$ . Holding wage fixed, is the supply function upward sloping in price? Holding price fixed, is the supply function upward sloping in wage? Can you graphically illustrate why your answers hold?

Answer: To see whether the supply function is upward or downward sloping in

price, we have to take the derivative of  $x(p, w)$  with respect to  $p$  and check whether this derivative is positive or negative. This derivative is

$$\frac{\partial x(p, w)}{\partial p} = -\left(\frac{\alpha}{\alpha-1}\right) \left[ A \left( \frac{w}{\alpha A} \right)^{\alpha/(\alpha-1)} \right] p^{(1-2\alpha)/(\alpha-1)}. \quad (11B.13.i)$$

Since  $0 < \alpha < 1$  and  $A > 0$ , the bracketed term is positive, as is the last term in the equation. This leaves the first term  $\alpha/(\alpha-1)$  which is negative. However, it is preceded by a negative sign — so the whole term is therefore positive. Thus,  $x(p, w)$  is increasing in  $p$  — which is to say that the supply function is upward sloping in  $p$ .

We can similarly check whether the supply function is upward or downward sloping in  $w$  by considering the derivative

$$\frac{\partial x(p, w)}{\partial w} = \frac{\alpha}{\alpha-1} \left[ A \left( \frac{1}{\alpha A p} \right)^{\alpha/(\alpha-1)} \right] w^{1/(\alpha-1)}. \quad (11B.13.ii)$$

Again, the bracketed and the last term are positive, but the first term  $\alpha/(\alpha-1)$  is negative since  $0 < \alpha < 1$ . As a result, the whole derivative is negative, which implies that the output supply function is decreasing, or downward sloping, in  $w$ .

Both these results again make intuitive sense. The first says that, as output price increases, the profit maximizing supply quantity increases as well. The second says that, as input price (wage) decreases, the profit maximizing output quantity increases. This can also be seen in panel (a) of Exercise Graph 11B.12. Suppose we start at prices and wages such that  $A$  is the profit maximizing production plan. An increase in  $p$  and a decrease in  $w$  both have the effect of decreasing  $w/p$  — the slope of the isoprofit curves. Thus, both result in increased output to a new profit maximizing production plan such as  $B$ .

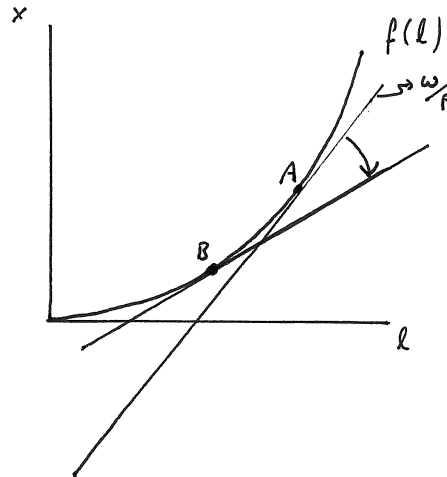
#### Exercise 11B.14

How do your answers to the previous two exercises change when  $\alpha > 1$ ? Can you make sense of what is going on? (*Hint:* Graph the production function and illustrate the tangencies of isoprofits for different wages and prices.)

Answer: The mathematical answers would reverse entirely — with labor demand now increasing in  $w$  and decreasing in  $p$ , and with output supply increasing in  $w$  and decreasing in  $p$ . This does not make intuitive sense. An increase in output price should cause producers to increase production and hire more labor, not the other way around; and an increase in input price (wage) should cause a decrease in production and less labor demanded. The reason that the math is now giving the wrong answer is because the “right” answer would be a corner solution. This is because, when  $\alpha > 1$ , the production function has increasing slope — implying increasing (rather than diminishing) marginal product of labor.

To see what is going on, consider Exercise Graph 11B.14.

Here we graph a production function with the slope that  $\alpha > 1$  implies. Suppose we initially face wage  $w$  and price  $p$ , giving rise to the isoprofit that is tangent at  $A$ . This is the production plan that the math gives us as the “solution”. The problem, of course, is that profit is negative under this plan (as indicated by the negative vertical

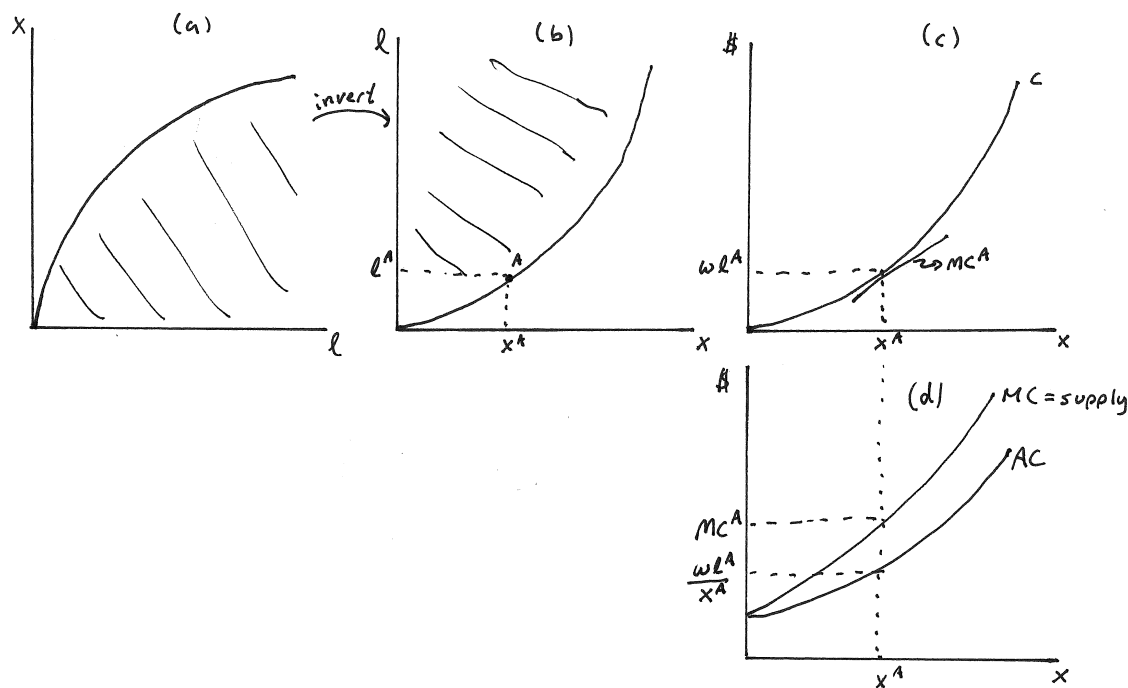
Exercise Graph 11B.14 : Nonsensical “solutions” when  $\alpha > 1$ 

intercept of the isoprofit curve). Now suppose that  $w$  falls — which causes isoprofits to become shallower (given that their slope is  $w/p$ ). The new production plan found by the mathematical optimization method we used is now  $B$  — indicating less labor input and less output supply. Thus, the math suggests that a drop in the input price (wage) results in less labor demanded and less output supplied, exactly counter to what should be true. Again, of course, profit at  $B$  is negative — so neither  $A$  nor  $B$  are truly profit maximizing production plans, which leads to the nonsensical result. Similarly, suppose  $p$  increases from the original. This similarly causes  $w/p$  to fall — making isoprofits shallower. Again we move from the “solution”  $A$  to the “solution”  $B$  — suggesting that an increase in output price causes the producer to produce less and hire fewer workers. And again the result is nonsensical because neither the original plan  $A$  nor the new plan  $B$  are actually profit maximizing. In fact, in both cases the producer should produce an infinitely large quantity.

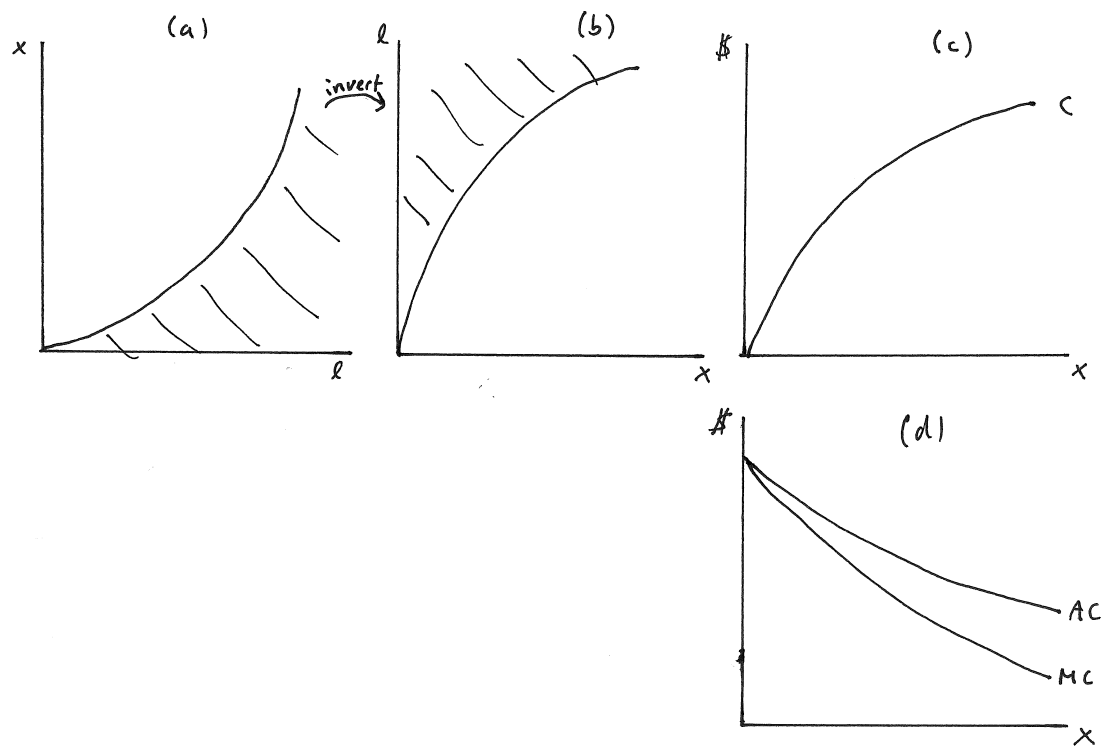
#### Exercise 11B.15

Graphically illustrate the way we have just derived the output supply function assuming  $\alpha$  lies between 0 and 1. What changes when  $\alpha > 1$ ?

**Answer:** In Exercise Graph 11B.15(1), we start with a convex production set in panel (a) (which emerges when  $\alpha < 1$ ). This is inverted in panel (b) — which corresponds to the step where we invert the production function and write it in terms of  $\ell(x)$ . We then multiply this by  $w$  to get the cost function in panel (c) (graphed with  $w$  held fixed) — then take the derivative to get  $MC$  in (d) and we divide the cost function by  $x$  to get the  $AC$  in panel (d).

Exercise Graph 11B.15(1) : Solution when  $\alpha > 1$ 

When  $\alpha > 1$ , the producer choice set is non-convex as illustrated in panel (a) of Exercise Graph 11B.15(2). We can again invert the production to get it into the form  $\ell(x)$  that tells us how much labor is required for each level of output — then multiply by  $w$  to get the cost function that is illustrated in panel (c) with wage held fixed. Then we get the  $MC$  and  $AC$  the same way as before in panel (d). The difference is that we cannot now claim that the  $MC$  function is the supply curve — because the supply curve has to lie above  $AC$ .



Exercise Graph 11B.15(2) : Nonsensical "solutions" when  $\alpha > 1$

## 11C Solutions to Odd Numbered End-of-Chapter Exercises

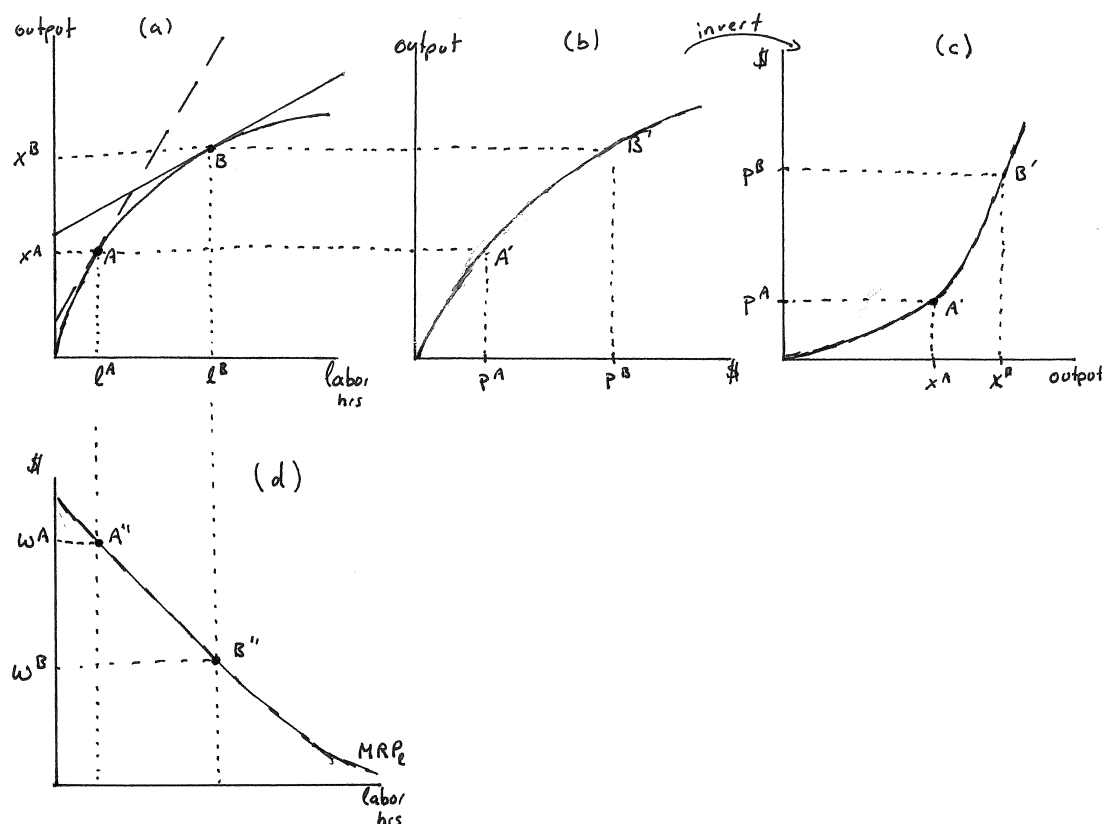
### Exercise 11.1

Throughout part A of the text, we used the technology we called more “realistic” in panel (b) of Graph 11.1.

**A:** Suppose now that the producer choice set was instead strictly convex everywhere.

- (a) Illustrate what such a technology would look like in terms of a production frontier.

Answer: This is illustrated in panel (a) of Exercise Graph 11.1(1).



Exercise Graph 11.1(1) : Production when Choice Set is Convex

- (b) *Derive the output supply curve with price on the vertical and output on the horizontal axis (in graphs analogous to those in Graph 11.9) for this technology.*

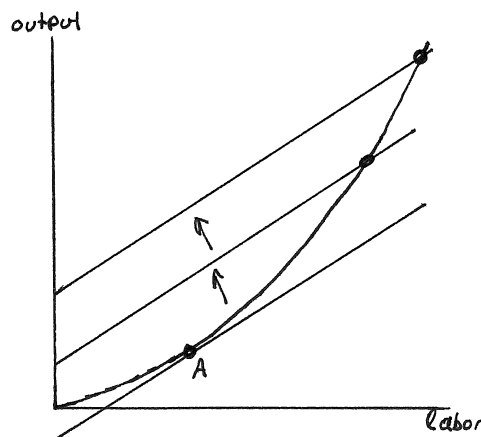
Answer: In panel (a) of Exercise Graph 11.1(1), two isoprofits corresponding to a low and a high price level are illustrated tangent to the producer choice set. The dashed one corresponds to the low output price  $p^A$  while the solid one corresponds to the higher output price  $p^B$ . The profit maximizing production plans at these two prices are then given by A and B. Panel (b) of the graph then plots the output levels  $x^A$  and  $x^B$  from these optimal production plans for the two price levels  $p^A$  and  $p^B$  on the horizontal axis. Panel (c) then simply flips the axes — giving us an upward sloping output supply curve.

- (c) *Derive the labor demand curve for such a technology.*

Answer: This is done in panel (d) of Exercise Graph 11.1(1) where the marginal revenue product — which is derived from the slope of the production frontier — is downward sloping throughout (because the slope of the production frontier is becoming shallower throughout).

- (d) *Now suppose the technology were instead such that the marginal product of labor is always increasing. What does this imply for the shape of the producer choice set?*

Answer: This implies that the production frontier gets steeper throughout — which in turn implies that the producer choice set is non-convex as drawn in Exercise Graph 11.1(2).



Exercise Graph 11.1(2) : Increasing Marginal Product of Labor

- (e) *How much should the firm produce if it is maximizing its profits in such a case? (Hint: Consider corner solutions.)*



Answer: If the firm is a price taker, then it should produce an infinite amount. This is illustrated in Exercise Graph 11.1(2) where three parallel isoprofits are drawn. The lowest is tangent to the production frontier, but this is not a profit maximizing production plan because we can still go to higher isoprofits that contain production plans which are technologically feasible. In fact, we can keep going to higher and higher isoprofits that continue to intersect the production frontier — and thus can keep going up.

**B:** Suppose that the production function a firm faces is  $x = f(\ell) = 100\ell^\alpha$ .

- (a) For what values of  $\alpha$  is the producer choice set strictly convex? For what values is it non-convex?

Answer: The producer choice set is strictly convex as long as  $\alpha < 1$  and non-convex if  $\alpha > 1$ .

- (b) Suppose  $\alpha = 0.5$ . Derive the firm's output supply and labor demand function.

Answer: The production function in this problem takes the same form as the one used in the last part of the chapter within the text. Using the results from there, and plugging 100 in for  $A$  and 0.5 for  $\alpha$ , we get

$$\ell(p, w) = \left(\frac{w}{50p}\right)^{-2} = \frac{2500p^2}{w^2} \quad \text{and} \quad x(p, w) = 100\left(\frac{w}{50p}\right)^{-1} = \frac{5000p}{w} \quad (11.1.i)$$

- (c) How much labor will the firm hire and how much will it produce if  $p = 10$  and  $w = 20$ ?

Answer: Plugging these into our expressions for labor demand and output supply, we get  $\ell = 625$  and  $x = 2,500$ .

- (d) How does labor demand and output supply respond to changes in  $w$  and  $p$ ?

Answer: The derivative of the labor demand function is negative with respect to  $w$  and positive with respect to  $p$  — which implies the firm will hire less labor as wage increases and more as output price increases. Similarly, the derivative of the output supply function is positive with respect to price and negative with respect to wage — which implies that supply increases in price and decreases in wage.

- (e) Suppose that  $\alpha = 1.5$ . How do your answers change?

Answer: If we used the functions that arise from our mathematical optimization problem in the chapter, we would get the incorrect answer that

$$\ell(p, w) = \left(\frac{w}{150p}\right)^2 \quad \text{and} \quad x(p, w) = 100\left(\frac{w}{150p}\right)^3. \quad (11.1.ii)$$

The answer is wrong because it is actually a local minimum, not a maximum — equivalent to the tangency at  $A$  in Exercise Graph 11.1(2). To see

this, you can check to see whether profit is positive by plugging the “optimal” labor and output values into the equation  $\pi = xp - w\ell$  for profit; i.e.

$$\pi = 100 \left( \frac{w}{150p} \right)^3 p - w \left( \frac{w}{150p} \right)^2 = \frac{2}{3} \left( \frac{w^3}{150^2 p^2} \right) - \frac{w^3}{150^2 p^2} = -\frac{1}{3} \left( \frac{w^3}{150^2 p^2} \right) < 0. \quad (11.1.iii)$$

You can also tell that the “answer” can’t possibly be right by checking how labor demand and output supply would respond to changes in  $w$  and  $p$  — both would increase with increases in wages and decreases in output prices. The correct answer, of course, is that a price taking firm under these conditions would produce an infinite amount of output. (Of course that is not realistic — such a firm would have market power and would thus not be a price taker.)

### Exercise 11.3

*Consider a profit maximizing firm.*

**A:** *Explain whether the following statements are true or false:*

- (a) *For price-taking, profit maximizing producers, the “constraint” is determined by the technological environment in which the producer finds herself while the “tastes” are formed by the economic environment in which the producer operates.*

Answer: The physical constraint that a producer cannot get around arises from the simple laws of physics — you can only get so much  $x$  out of the inputs you use. The more sophisticated the technology available to the producer, the more  $x$  she can squeeze out — and thus the technology creates the production constraint that tells the producer which production plans are feasible and which are not. In terms of tastes, we typically assume that producers simply care about profit — and that more profit is better than less. To form indifference curves for producers who simply care about profit, we have to find production plans that all result in the same amount of profit. And how easy it is to make profit depends on how high the output price is and how low the input prices are relative to the output price. Thus, the production plans that produce the same level of profit will differ depending on the economic environment — depending on how much the producer can get for her output and how much she has to pay for the inputs in her production plans.

- (b) *Every profit maximizing producer is automatically cost-minimizing.*

Answer: This is true. If you are maximizing profit, you must be producing whatever you are producing without wasting any inputs — i.e. you must be producing the output at the minimum cost possible.

- (c) *Every cost-minimizing producer is automatically profit maximizing.*

Answer: This is not true. You could be producing some arbitrary quantity without wasting any inputs — i.e. in the least cost way. But that does not

mean you are producing the “right” quantity. Cost minimization is only part of profit maximization — it only takes into account input prices as they relate to the cost of production. Only when you profit maximize do you also take into account the output price and what that implies for how much you should produce in a cost minimizing way.

- (d) *Price taking behavior makes sense only when marginal product diminishes at least at some point.*

Answer: This is true. If marginal product always increases, then it is getting cheaper and cheaper to produce additional units of output. And if I can sell all my output at the same price (i.e. if I am a price taker), then I should keep producing and drive down my average cost.

**B:** Consider the production function  $x = f(\ell) = \alpha \ln(\ell + 1)$ .

- (a) *Does this production function have increasing or decreasing marginal product of labor.*

Answer: Marginal product for this production function is given by

$$MP_\ell = \frac{\alpha}{\ell + 1}. \quad (11.3.i)$$

Thus, as  $\ell$  increases,  $MP_\ell$  clearly decreases. You can also see this by simply taking the derivative of marginal product of labor with respect to  $\ell$

$$\frac{\partial MP_\ell}{\partial \ell} = \frac{-\alpha}{(\ell + 1)^2}. \quad (11.3.ii)$$

- (b) *Set up the profit maximization problem and solve for the labor input demand and output supply functions.*

Answer: The profit maximization problem is

$$\max_{\ell, x} px - w\ell \quad \text{subject to } x = \alpha \ln(\ell + 1) \quad (11.3.iii)$$

which can also be written as the unconstrained maximization problem

$$\max_{\ell} p\alpha \ln(\ell + 1) - w\ell. \quad (11.3.iv)$$

The first order condition for this problem is

$$\frac{p\alpha}{\ell + 1} - w = 0 \quad (11.3.v)$$

which can be solved to get the labor demand function

$$\ell(w, p) = \frac{\alpha p - w}{w}. \quad (11.3.vi)$$

Substituting this into the production function, we then get the output supply function

$$x(w, p) = \alpha \ln \left( \frac{p\alpha - w}{w} + 1 \right) = \alpha \ln \left( \frac{\alpha p}{w} \right). \quad (11.3.vii)$$

- (c) *Recalling that  $\ln x = y$  implies  $e^y = x$  (where  $e \approx 2.7183$  is the base of the natural log), invert the production function and derive from this the cost function  $c(w, x)$ .*

Answer: To invert the production function  $x = \alpha \ln(\ell + 1)$ , we first note that this implies

$$e^{x/\alpha} = \ell + 1 \quad (11.3.viii)$$

which can then be solved in terms of  $\ell$  to give us the amount of labor required for each level of output; i.e.

$$\ell = e^{x/\alpha} - 1. \quad (11.3.ix)$$

When multiplied by  $w$ , we then get the cost function

$$c(w, x) = w(e^{x/\alpha} - 1). \quad (11.3.x)$$

- (d) *Determine the marginal and average cost functions.*

Answer: The marginal cost is

$$MC(w, x) = \frac{\partial c(w, x)}{\partial x} = \frac{w}{\alpha} e^{x/\alpha}, \quad (11.3.xi)$$

and the average cost is

$$AC(w, x) = \frac{we^{x/\alpha}}{x} - \frac{w}{x}. \quad (11.3.xii)$$

- (e) *Derive from this the output supply and labor demand functions. Compare them to what you derived directly from the profit maximization problem in part (b).*

Answer: To derive the output supply function, we begin by setting  $p$  equal to  $MC$ ; i.e.

$$p = \frac{w}{\alpha} e^{x/\alpha}. \quad (11.3.xiii)$$

Multiplying both sides by  $\alpha/w$ , taking natural logs and then multiplying both sides by  $\alpha$ , we then get

$$x(p, w) = \alpha \ln \frac{\alpha p}{w}. \quad (11.3.xiv)$$

Plugging this back into equation (11.3.ix), we get the labor input demand

$$\ell(w, p) = e^{\ln(\alpha p/w)} - 1 = \frac{\alpha p}{w} - 1 = \frac{\alpha p - w}{w}. \quad (11.3.xv)$$

Note that the output supply and labor demand equations are identical to those derived directly through profit maximization earlier in the problem. The equations are correct, however, only for prices above the lowest point of  $AC$ .

- (f) *In your mathematical derivations, what is required for a producer to be cost minimizing? What, in addition, is required for her to be profit maximizing?*

Answer: The only requirement for the producer to be cost minimizing is that she not waste any input — i.e. that she hire labor in accordance with equation (11.3.ix). There is no requirement imposed by cost minimization on *how much* to produce. The additional requirement imposed by profit maximization is that output quantity be chosen such that  $p = MC$ . Alternatively, we could phrase this same additional requirement as the requirement that  $pMP_\ell = w$ .

#### Exercise 11.5

*When we discussed optimal behavior for consumers in Chapter 6, we illustrated that there may be two optimal solutions for consumers whenever there are non-convexities in either tastes or choice sets. We can now explore conditions under which multiple optimal production plans might appear in our producer model.*

**A:** *Consider only profit maximizing firms whose tastes (or isoprofits) are shaped by prices.*

- (a) *Consider first the standard production frontier that has initially increasing marginal product of labor and eventually decreasing marginal product of labor. True or False: If there are two points at which isoprofits are tangent to the production frontier in this model, the lower output quantity cannot possibly be part of a truly optimal production plan.*

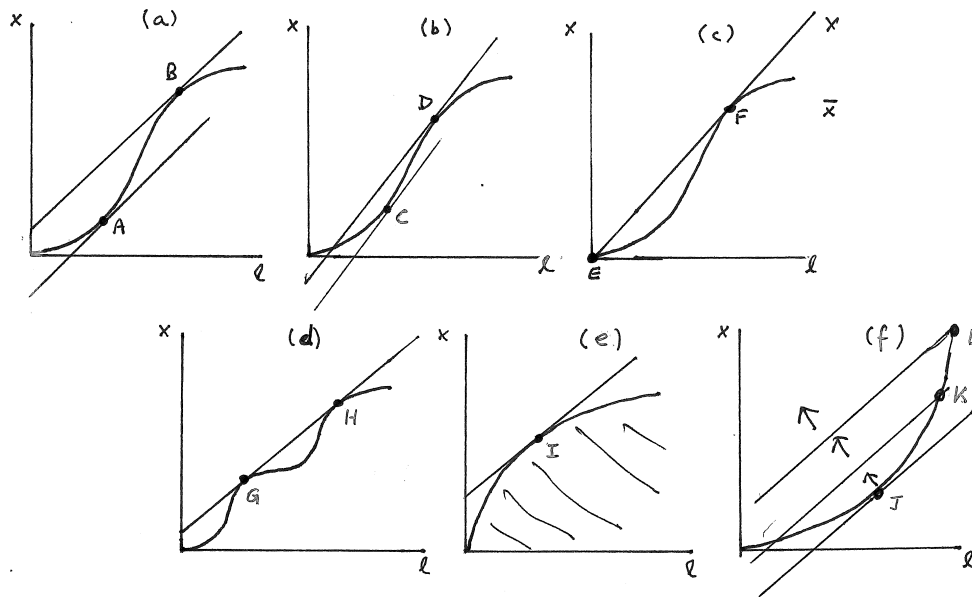
Answer: This is true. Panel (a) of Exercise Graph 11.5 illustrates this with two tangencies at  $A$  and  $B$ . Since the isoprofit tangent at  $A$  has a negative intercept, profit at the production plan  $A$  is negative.

- (b) *Could it be that neither of the tangencies represents a truly optimal production plan?*

Answer: Yes. This is illustrated in panel (b) of Exercise Graph 11.5 where the two tangencies occur at production plans  $C$  and  $D$ . In this case, both isoprofit curves have negative intercepts — and both therefore involve negative profit. In this case, the profit maximizing production plan is  $(0,0)$  — i.e. no labor input is bought and no output is produced.

- (c) *Illustrate a case where there are two truly optimal solutions where one of these does not occur at a tangency.*

Answer: This is illustrated in panel (c) of Exercise Graph 11.5. The production plan  $F$  occurs at a tangency and involves zero profit because the isoprofit curve intersects at the origin. Thus, profit at  $F$  is the same as profit at  $E = (0,0)$  where no labor is purchased and no output is produced. Thus both  $E$  and  $F$  are truly profit maximizing production plans.



Exercise Graph 11.5 : Profit Maximizing under Different Production Sets

- (d) What would a production frontier have to look like in order for there to be two truly optimal production plans which both involve positive levels of output? (Hint: Consider technologies that involve multiple switches between increasing and decreasing marginal product of labor.)

Answer: This is illustrated in panel (d) of Exercise Graph 11.5 where both G and H are tangencies that lie on the same isoprofit curve (and thus result in the same amount of positive profit).

- (e) True or False: If the producer choice set is convex, there can only be one optimal production plan.

Answer: This is true. In panel (e) of Exercise Graph 11.5, a strictly convex production frontier is illustrated — with only I emerging as a profit maximizing production plan.

- (f) Where does the optimal production plan lie if the production frontier is such that the marginal product of labor is always increasing?

Answer: This is illustrated in panel (f) of Exercise Graph 11.5 where the increasing marginal product of labor results in an ever steepening production frontier. The only tangency occurs at J — but J actually results in negative profit since the isoprofit has negative intercept. The producer can do better by moving to higher isoprofits that intersect the production frontier — as those that intersect at K and L. But it's always possible to go to an even higher isoprofit and move higher on the production frontier.

Thus, the profit maximizing quantity is infinite (which, of course, does not make sense in a world of scarcity — but neither does an ever increasing marginal product of labor.)

- (g) *Finally, suppose that the marginal product of labor is constant throughout. What production plans might be optimal in this case?*

Answer: This would result in a production frontier that is simply a straight line. If the ratio  $w/p$  happens to be the same as the slope of this production frontier, then every production plan on the production frontier lies on the same isoprofit curve which in turn intersects at the origin. Thus, all production plans on the production frontier yield zero profit — and all are therefore optimal. If the isoprofits are steeper than the production frontier, then all production plans on the frontier other than (0,0) result in negative profit — and only (0,0) is optimal. If, on the other hand, the isoprofits are shallower than the production frontier, the optimal output quantity is infinite for reasons similar to what we described in part (f) where we considered a production frontier with increasing marginal product of labor throughout.

**B:** *In the text, we used a cosine function to illustrate a production process that has initially increasing and then decreasing marginal product of labor. In many of the end-of-chapter exercises, we will instead use a function of the form  $x = f(\ell) = \beta\ell^2 - \gamma\ell^3$  where  $\beta$  and  $\gamma$  are both greater than zero.*

- (a) *Illustrate how the profit maximization problem results in two “solutions”. (Use the quadratic formula to solve for these.)*

Answer: The profit maximization problem is then

$$\max_{\ell, x} px - w\ell \quad \text{subject to} \quad x = \beta\ell^2 - \gamma\ell^3 \quad (11.5.i)$$

which can be written in the form of an unconstrained maximization problem by substituting the constraint into the objective to give us

$$\max_{\ell} p(\beta\ell^2 - \gamma\ell^3) - w\ell. \quad (11.5.ii)$$

The first order condition for this problem is then

$$-3\gamma p\ell^2 + 2\beta p\ell - w = 0. \quad (11.5.iii)$$

For an equation of the form  $ax^2 + bx + c = 0$ , the quadratic formula gives the solutions

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \quad (11.5.iv)$$

We can therefore solve equation (11.5.iii) for  $\ell$  by applying this formula, letting  $a = -3\gamma p$ ,  $b = 2\beta p$  and  $c = -w$ , to give us

$$\ell = \frac{-2\beta p + \sqrt{4\beta^2 p^2 - 12\gamma p w}}{-6\gamma p} \text{ and } \ell = \frac{-2\beta p - \sqrt{4\beta^2 p^2 - 12\gamma p w}}{-6\gamma p}. \quad (11.5.v)$$

- (b) Which of your two “solutions” is unambiguously not the actual profit maximizing solution?

Answer: Our two “solutions” can be rewritten as

$$\ell = \frac{\beta}{3\gamma} - \frac{\sqrt{\beta^2 p^2 - 3\gamma p w}}{3\gamma p} \text{ and } \ell = \frac{\beta}{3\gamma} + \frac{\sqrt{\beta^2 p^2 - 3\gamma p w}}{3\gamma p}, \quad (11.5.vi)$$

where the latter is clearly greater than the former. We know from our intuitive graphs that only the higher of the two production plans could be optimal — thus the first solution is definitely not a profit maximizing solution.

- (c) What else would you have to check to be sure that the other “solution” is profit maximizing?

Answer: You would have to check that the larger of the two solutions results in positive profit.

- (d) Now consider instead a production process characterized by the equation  $x = A\ell^\alpha$ . Suppose  $\alpha < 1$ . Determine the profit maximizing production plan.

Answer: The last part of the chapter sets up this optimization problem and solves for the production plan

$$\ell = \left( \frac{w}{\alpha A p} \right)^{\frac{1}{\alpha-1}} \text{ and } x = A \left( \frac{w}{\alpha A p} \right)^{\frac{\alpha}{\alpha-1}} \quad (11.5.vii)$$

- (e) What if  $\alpha > 1$ ?

Answer: When  $\alpha > 1$ , the marginal product of labor is increasing throughout — and the “solution” is not a true profit maximum because the firm should produce an infinite amount instead. You can tell that the “solution” makes no sense in this case because, when  $\alpha > 1$ , it would suggest that you hire more labor and produce more output as wages go up and output prices go down.

- (f) What if  $\alpha = 1$ ?

Answer: When  $\alpha = 1$ , the marginal product of labor is constant — and the production frontier is a straight line. The “solution” we calculated is not defined because the exponent  $1/(\alpha - 1)$  is not defined when  $\alpha = 1$ . This is because there is no tangency — either all production plans on the frontier yield zero profit and all such plans are therefore profit maximizing, or the true profit maximum involves zero or an infinite level of output.



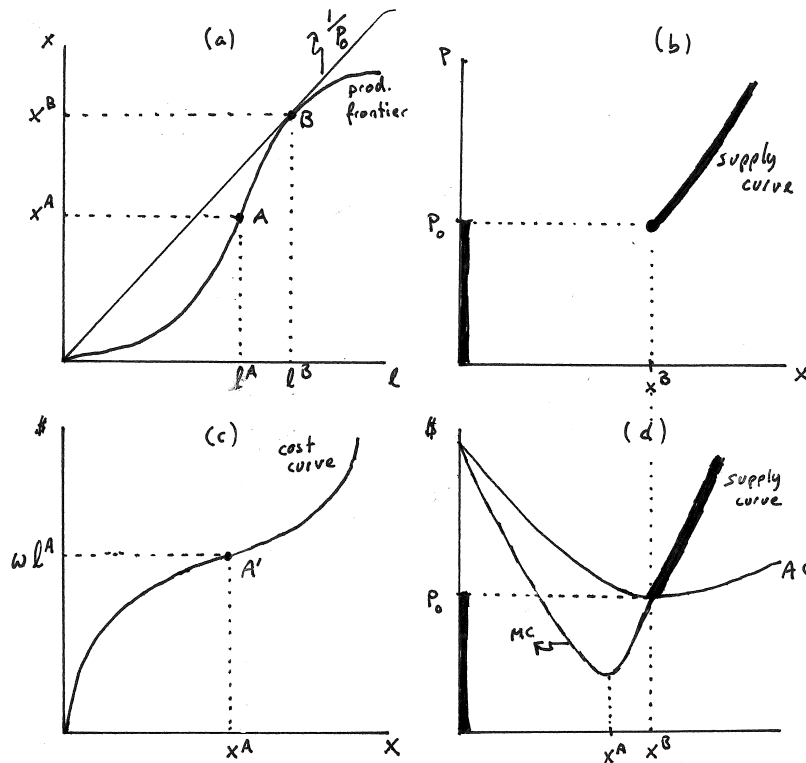
## Exercise 11.7

We have shown that there are two ways in which we can think of the producer as maximizing profits: Either directly, or in a two-step process that begins with cost minimization.

**A:** This exercise reviews this equivalence for the case where the production process initially has increasing marginal product of labor but eventually reaches decreasing marginal product. Assume such a production process throughout.

- (a) Begin by plotting the production frontier with labor on the horizontal and output on the vertical axis. Identify in your graph the production plan  $A = (\ell^A, x^A)$  at which increasing returns turns to decreasing returns.

Answer: This is illustrated in panel (a) of Exercise Graph 11.7 where  $A$  lies at the point at which the production frontier stops increasing at an increasing rate and starts increasing at a decreasing rate. Put differently, at  $A$  the slope stops increasing and starts decreasing.



Exercise Graph 11.7: 2 Ways to Maximize Profit

- (b) Suppose wage is  $w = 1$ . Illustrate in your graph the price  $p_0$  at which the firm obtains zero profit by using a production plan  $B$ . Does this necessarily lie above or below  $A$  on the production frontier?

Answer: This is also illustrated in panel (a) where  $B$  lies on an isoprofit that is tangent to the production frontier at  $B$  and intersects the origin (which implies zero profit). The slope of the isoprofit is  $1/p_0$  since  $w = 1$ . It is apparent from the graph that  $B$  must lie above  $A$  on the production frontier — i.e. it must be the case that the zero-profit price results in production on the decreasing marginal product of labor portion of the production frontier.

- (c) Draw a second graph next to the one you have just drawn. With price on the vertical axis and output on the horizontal, illustrate the amount the firm produces at  $p_0$ .

Answer: This is illustrated in panel (b) of Exercise Graph 11.7 where the point  $(x^B, p_0)$  illustrates the lowest price at which the firm produces positive output.

- (d) Suppose price rises above  $p_0$ . What changes on your graph with the production frontier — and how does that translate to points on the supply curve in your second graph?

Answer: If price rises above  $p_0$ , the isoprofit lines become shallower — which implies the new optimal quantity lies at a tangency higher on production frontier than  $B$ . The isoprofit that is tangent at the new profit maximizing production plan also has positive intercept on the vertical axis — implying profit will be positive. Thus, output increases as  $p$  rises above  $p_0$  — leading to an upward sloping supply curve (as illustrated in panel (b).)

- (e) What if price falls below  $p_0$ ?

Answer: If price falls below  $p_0$ , the isoprofit curves become steeper — implying tangencies to the left of  $B$ . At those tangencies, however, the intercept on the vertical axis will be negative — implying negative profit. Thus, the firm is better off not producing at all — which is why the supply curve in panel (b) is vertical at zero output level up to the price  $p_0$ .

- (f) Illustrate the cost curve on a graph below your production frontier graph. What is similar about the two graphs — and what is different — around the point that corresponds to production plan  $A$ .

Answer: The cost curve, as illustrated in panel (c) of Exercise Graph 11.7, has the inverse shape from the production frontier — because when each additional labor unit increases production more than the last (on the increasing marginal product part of the production frontier), the cost of increasing output rises at a decreasing rate (and vice versa). Around  $A'$  — the point corresponding to  $A$  in panel (a), the cost curve switches from increasing at a decreasing rate to increasing at an increasing rate (just as the switch from increasing at an increasing rate to increasing at a decreasing rate happens on the production frontier.)

- (g) Next to your cost curve graph, illustrate the marginal and average cost curves. Which of these reaches its lowest point at the output quantity  $x^A$ ? Which reaches its lowest point at  $x^B$ ?

Answer: This is illustrated in panel (d) of Exercise Graph 11.7. The marginal cost (MC) curve is the slope of the cost curve — so it reaches its lowest point at the output level  $x^A$  where the slope of the cost curve begins to get steeper. The average cost curve reaches its lowest point where the MC curve crosses it — which is also where the supply curve begins. This occurs at  $x^B$ .

- (h) Illustrate the supply curve on your graph and compare it to the one you derived in parts (c) and (d).

Answer: The supply curve is the part of the MC curve that lies above the AC curve — with output of zero below that. This is highlighted in panel (d) of Exercise Graph 11.7 — with the resulting supply curve being identical to what we derived before in panel (b).

**B:** Suppose that you face a production technology characterized by the function  $x = f(\ell) = \alpha / (1 + e^{-(\ell-\beta)})$ .

- (a) Assuming labor  $\ell$  costs  $w$  and the output  $x$  can be sold at  $p$ , set up the profit maximization problem.

Answer: The profit maximization problem is

$$\max_{\ell, x} px - w\ell \quad \text{subject to} \quad x = \frac{\alpha}{1 + e^{-(\ell-\beta)}} \quad (11.7.i)$$

which can be written as the unconstrained problem

$$\max_{\ell} \frac{p\alpha}{1 + e^{-(\ell-\beta)}} - w\ell. \quad (11.7.ii)$$

- (b) Derive the first order condition for this problem.

Answer: The first order condition is

$$\frac{\alpha p e^{-(\ell-\beta)}}{[1 + e^{-(\ell-\beta)}]^2} = w. \quad (11.7.iii)$$

- (c) Substitute  $y = e^{-(\ell-\beta)}$  into your first order condition and, using the quadratic formula, solve for  $y$ . Then, recognizing that  $y = e^{-(\ell-\beta)}$  implies  $\ln y = -(\ell - \beta)$ , solve for the two implied labor inputs and identify which one is profit maximizing (assuming that an interior production plan is optimal).

Answer: Substituting  $y = e^{-(\ell-\beta)}$ , the first order condition reduces to  $\alpha p y / (1 + y)^2 = w$  which can be written in the form

$$wy^2 + (2w - \alpha p)y + w = 0. \quad (11.7.iv)$$

The quadratic formula then gives two solutions for  $y$ :

$$y_1 = \frac{-(2w - \alpha p) + \sqrt{(2w - \alpha p)^2 - 4w^2}}{2w} = \frac{\alpha p - 2w + \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2w} \quad (11.7.v)$$

and

$$y_2 = \frac{-(2w - \alpha p) - \sqrt{(2w - \alpha p)^2 - 4w^2}}{2w} = \frac{\alpha p - 2w - \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2w}. \quad (11.7.vi)$$

Given that  $y = e^{-(\ell - \beta)}$ , we can take natural logs of both sides to get  $-(\ell - \beta) = \ln y$  or  $\ell = \beta - \ln y$ . Using the two solutions for  $y$ , we therefore get

$$\begin{aligned} \ell_1 &= \beta - \ln \left( \frac{\alpha p - 2w + \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2w} \right) \quad \text{and} \\ \ell_2 &= \beta - \ln \left( \frac{\alpha p - 2w - \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2w} \right). \end{aligned} \quad (11.7.vii)$$

Since  $y_1 > y_2$ , we know that  $\ell_1 < \ell_2$  — and thus the true profit maximizing labor input (assuming an interior production plan is profit maximizing) is given by

$$\ell(w, r, p) = \beta - \ln \left( \frac{\alpha p - 2w - \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2w} \right). \quad (11.7.viii)$$

(d) Use your answer to solve for the supply function (assuming an interior solution is optimal).

Answer: Plugging  $\ell = \beta - \ln y_2$  into the production function, we then get

$$x = \frac{\alpha}{1 + e^{-(\beta - \ln y_2 - \beta)}} = \frac{\alpha}{1 + e^{\ln y_2}} = \frac{\alpha}{1 + y_2}. \quad (11.7.ix)$$

Substituting in  $y_2$  from equation (11.7.vi), this then simplifies to the supply function (assuming an interior optimum)

$$x(w, r, p) = \frac{2\alpha w}{\alpha p - \sqrt{\alpha^2 p^2 - 4\alpha w p}}. \quad (11.7.x)$$

We can also re-write this by multiplying both numerator and denominator by the term  $(\alpha p + \sqrt{\alpha^2 p^2 - 4\alpha w p})$  to get

$$x(w, r, p) = \frac{2\alpha w (\alpha p + \sqrt{\alpha^2 p^2 - 4\alpha w p})}{\alpha^2 p^2 - (\alpha^2 p^2 - 4\alpha w p)} = \frac{\alpha p + \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2p}. \quad (11.7.xi)$$

- (e) Now use the two-step method to verify your answer. Begin by solving the production function for  $\ell$  to determine how much labor is required for each output level assuming none is wasted.

Answer: To do the 2-step optimization, we begin by solving the production function  $x = \alpha / (1 + e^{-(\ell-\beta)})$  for  $\ell$ . We can do this by first multiplying through by  $(1 + e^{-(\ell-\beta)})$ , dividing by  $x$  and subtracting 1 from both sides to get

$$e^{-(\ell-\beta)} = \frac{\alpha}{x} - 1 = \frac{\alpha - x}{x} \quad (11.7.xii)$$

which allows us to write

$$-(\ell - \beta) = \ln\left(\frac{\alpha - x}{x}\right) \quad (11.7.xiii)$$

which then solves to

$$\ell = \beta - \ln\left(\frac{\alpha - x}{x}\right). \quad (11.7.xiv)$$

- (f) Use your answer to derive the cost function and the marginal cost function.

Answer: The minimum cost at which the firm can produce any level of output  $x$  is then simply this inverted production function times the wage; i.e. the cost function is

$$C(w, x) = w\beta - w \ln\left(\frac{\alpha - x}{x}\right). \quad (11.7.xv)$$

From this, we can get the marginal cost function

$$MC(w, x) = \frac{\partial C}{\partial x} = \frac{\alpha w}{(\alpha - x)x}. \quad (11.7.xvi)$$

- (g) Set price equal to marginal cost and solve for the output supply function (assuming an interior solution is optimal). Can you get your answer into the same form as the supply function from your direct profit maximization problem?

Answer: Setting  $MC = p$ , we can then solve for the supply function; i.e. we set  $p = \alpha w / ((\alpha - x)x)$ , multiply through and write it in the form that allows us to once again apply the quadratic formula:

$$x^2 - \alpha x + \frac{\alpha w}{p} = 0 \quad \text{or equivalently} \quad px^2 - \alpha px + \alpha w = 0. \quad (11.7.xvii)$$

Applying the quadratic formula, we get two “solutions”:

$$x_1 = \frac{\alpha p + \sqrt{\alpha^2 p^2 - 4p\alpha w}}{2p} \quad \text{and} \quad x_2 = \frac{\alpha p - \sqrt{\alpha^2 p^2 - 4p\alpha w}}{2p} \quad (11.7.xviii)$$

of which the first one is the true solution (since it is the larger of the two). The supply function (assuming an interior solution) is therefore

$$x(w, r, p) = \frac{\alpha p + \sqrt{\alpha^2 p^2 - 4p\alpha w}}{2p} \quad (11.7.xix)$$

which is equivalent to the previous solution we got in equation (11.7.xi) by solving the profit maximization problem directly.

- (h) Use the supply function and your answer from part (e) to derive the labor input demand function (assuming an interior solution is optimal). Is it the same as what you derived through direct profit maximization in part (c)?

Answer: Plugging the supply function into equation (11.7.xiv), we get

$$\ell = \beta - \ln \left[ \frac{\alpha - \frac{\alpha p + \sqrt{\alpha^2 p^2 - 4p\alpha w}}{2p}}{\frac{\alpha p + \sqrt{\alpha^2 p^2 - 4p\alpha w}}{2p}} \right] = \beta - \ln \left[ \frac{\alpha p - \sqrt{\alpha^2 p^2 - 4p\alpha w}}{\alpha p + \sqrt{\alpha^2 p^2 - 4p\alpha w}} \right]. \quad (11.7.xx)$$

Multiplying both the denominator and numerator within the brackets by the numerator  $(\alpha p - \sqrt{\alpha^2 p^2 - 4p\alpha w})$ , we can then write this as

$$\begin{aligned} \ell &= \beta - \ln \left[ \frac{\alpha^2 p^2 - 2\alpha p \sqrt{\alpha^2 p^2 - 4p\alpha w} + (\alpha^2 p^2 - 4p\alpha w)}{\alpha^2 p^2 - (\alpha^2 p^2 - 4p\alpha w)} \right] \\ &= \beta - \ln \left( \frac{\alpha p - 2w - \sqrt{\alpha^2 p^2 - 4p\alpha w}}{2w} \right) \end{aligned} \quad (11.7.xxi)$$

which is exactly equal to the labor demand function we derived in equation (11.7.viii) through direct profit maximization.

### Exercise 11.9

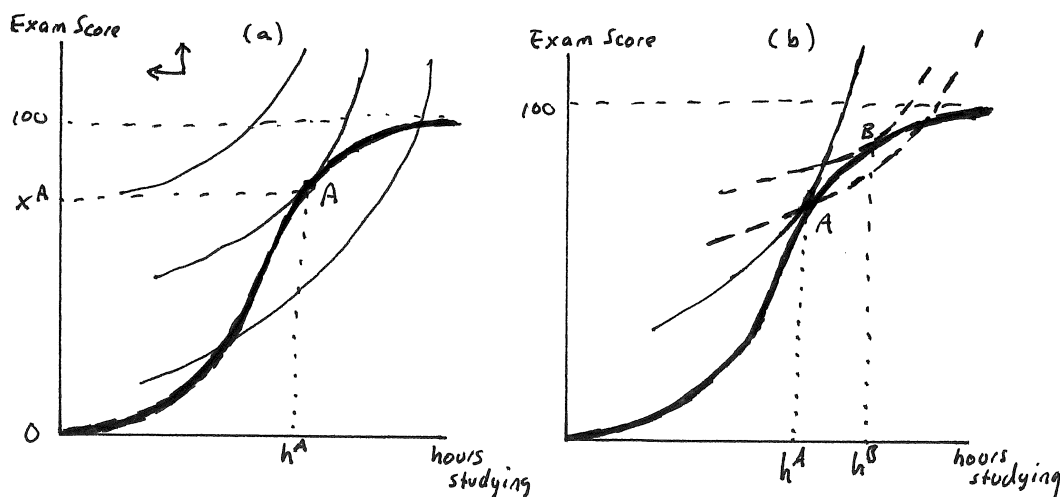
Everyday Application: *Studying for an Exam:* Consider the problem you face as a student as you determine how much to study for an exam by modeling yourself as a “producer of an exam score” between 0 and 100.

**A:** Suppose that the marginal payoff to studying for the initial hours you study increases but that this marginal payoff eventually declines as you study more.

- (a) Illustrate, on a graph with “hours studying for the exam” as an input on the horizontal axis and “exam score” (ranging from 0 to 100) as an output on the vertical axis, what your production frontier will look like.

Answer: The production frontier is graphed as the (dark) frontier plotted in panel (a) of Exercise Graph 11.9.

- (b) Now suppose that your tastes over leisure time (i.e. non-study time) and exam scores satisfies the usual five assumptions about tastes that we outlined in Chapter 4. What will your producer tastes look like. (Be careful



Exercise Graph 11.9 : Studying for an Exam

to recognize that the producer picture has “hours studying” and not leisure hours on the horizontal axis.)

Answer: Three indifference curves are graphed in panel (a) of Exercise Graph 11.9. You become better off as you move to the north-west in the graph — fewer hours of studying, higher exam grades.

- (c) Combining your production frontier with graphs of your indifference curves, illustrate the optimal number of hours you will study.

Answer: Bundle A in panel (a) of the graph is your optimal “production plan” — resulting in  $h^A$  hours of studying and an exam grade of  $x^A$ .

- (d) Suppose that you and your friend differ in that your friend’s marginal rate of substitution at every possible “production plan” is shallower than yours. Who will do better on the exam?

Answer: Your friend will do better, as illustrated in panel (b) of the graph. At your optimal production plan A, your friend’s indifference curve cuts the producer choice set in such a way that all the “better” plans (that lie to the north-west) involve greater effort than  $h^A$ . Your friend’s optimal production plan is B.

- (e) Notice that the same model can be applied to anything we do where the amount of effort is an input and how well we perform a task is the output. As we were growing up, adults often told us: “Anything worth doing is worth doing well.” Is that really true?

Answer: Not really. Both you and your friend could have done very well on the exam — even scoring 100 — by putting more time in. But you also

have other priorities in life — which is why you would prefer to devote less time to studying. It would not be optimal to devote all your time to one dimension — to doing well on this exam.

**B:** Now suppose that you and your friends Larry and Daryl each face the same “production technology”  $x = 3\ell^2 - 0.2\ell^3$  where  $x$  is the exam grade and  $\ell$  is the number of hours of studying. Suppose further that each of you has tastes that can be captured by the utility function  $u(\ell, x) = x - \alpha\ell$ .

(a) Calculate your optimal hours of studying as a function of  $\alpha$ .

Answer: The problem that needs to be solved is

$$\max_{\ell, x} x - \alpha\ell \quad \text{subject to} \quad x = 3\ell^2 - 0.2\ell^3 \quad (11.9.i)$$

which can be written as an unconstrained optimization problem by replacing  $x$  in the objective with the constraint — i.e.

$$\max_{\ell} 3\ell^2 - 0.2\ell^3 - \alpha\ell. \quad (11.9.ii)$$

Taking the first derivative and setting it to zero, we get  $6\ell - 0.6\ell^2 - \alpha = 0$  or, rearranging terms slightly,

$$-0.6\ell^2 + 6\ell - \alpha = 0. \quad (11.9.iii)$$

For an equation of the form  $ax^2 + bx + c = 0$ , the quadratic formula gives the solutions

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \quad (11.9.iv)$$

Using this to solve equation (11.9.iii) (where  $a = -0.6$ ,  $b = 6$  and  $c = -\alpha$ ), we get

$$\ell = \frac{-6 + \sqrt{36 - 2.4\alpha}}{-1.2} \quad \text{and} \quad \ell = \frac{-6 - \sqrt{36 - 2.4\alpha}}{-1.2}. \quad (11.9.v)$$

We know that these two “solutions” are tangencies between the production frontier and two indifference curves — but only the higher is a true optimum. Thus, the actual solution is simply

$$\ell = \frac{-6 - \sqrt{36 - 2.4\alpha}}{-1.2}. \quad (11.9.vi)$$

(b) Suppose the values for  $\alpha$  are 7, 10, and 13 for you, Larry and Daryl respectively. How much time will each of you study?

Answer Plugging these values into the solution of equation (11.9.vi), we get that you, Larry and Daryl will study approximately 8.65, 7.87 and 6.83 hours respectively.



(c) What exam grades will each of you get?

Answer: Plugging these values of hours studied back into the production function, we get that you, Larry and Daryl will receive the following approximate grades respectively: 95, 88.5 and 76.2.

(d) If each of you had 10 hours available that you could have used to study for the exam, could you each have made a 100? If so, why didn't you?

Answer: Yes, each of you could have scored 100 by simply putting 10 hours of effort into preparing for the exam. (The answer is the same for each of you since your "production technology" is the same — just check that when you plug 10 into the production function, you actually get 100.) But you would have had to spend all your time preparing for the exam — which, given your tastes that place value on doing other things, was not worth it.

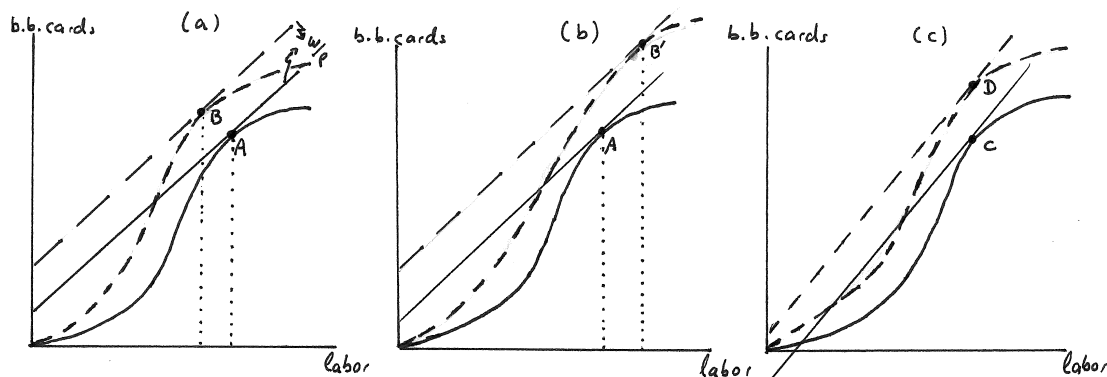
### Exercise 11.11

**Business Application: Technological Change in Production:** Suppose you and your friend Bob are in the business of producing baseball cards.

**A:** Both of you face the same production technology which has the property that the marginal product of labor initially increases for the first workers you hire but eventually decreases. You both sell your cards in a competitive market where the price of cards is  $p$ , and you hire in a competitive labor market where the wage is  $w$ .

(a) Illustrate your profit maximizing production plan assuming that  $p$  and  $w$  are such that you and Bob can make a positive profit.

Answer: This is shown in the solid lines of panel (a) of Exercise Graph 11.11 where A is the profit maximizing production plan given the tangency of the (solid) isoprofit line at A.



Exercise Graph 11.11 : Technological Innovation

- (b) *Now suppose you find a costless way to improve the technology of your firm in a way that unambiguously expands your producer choice set. As a result, you end up producing more than Bob (who has not found this technology). Illustrate how the new technology might have changed your production frontier.*

Answer: Two possibilities are illustrated with the dashed portions of panels (a) and (b) of Exercise Graph 11.11. In panel (a), your new profit maximizing production plan is  $B$  — which results in more baseball cards. In panel (b), your new profit maximizing production plan is  $B'$  which also results in more baseball cards.

- (c) *Can you necessarily tell whether you will hire more or less labor with the new technology?*

Answer: No, it is not clear. In panel (a), you end up hiring fewer workers — while in panel (b) you end up hiring more workers.

- (d) *Can you say for sure that adopting the new technology will result in more profit?*

Answer: Yes, profit increases unambiguously in both panels (as seen in the higher intercept of the dashed isoprofit curve relative to the solid one.)

- (e) *Finally, suppose  $p$  falls. Illustrate how it might now be the case that Bob stops producing but you continue to stay in the business.*

Answer: A drop in  $p$  causes the slope of the isoprofits (i.e.  $w/p$ ) to increase. Panel (c) then illustrates a case where Bob's "optimal" production plan  $C$  is not actually optimal because the isoprofit that is tangent at  $C$  has negative intercept and thus involves negative profit. You, on the other hand, maximize profit at  $D$  along an isoprofit with positive intercept (and thus positive profit). So Bob shuts down and you remain open.

**B:** *You and Bob initially face the production technology  $x = 3A\ell^2 - 0.1\ell^3$ , and you can sell your output for  $p$  and hire workers at a wage  $w$ .*

- (a) *Derive the marginal product of labor and describe its properties.*

Answer: The marginal product of labor is calculated from the production function  $x = f(\ell)$  as

$$MP_\ell = \frac{df}{d\ell} = 6A\ell - 0.3\ell^2. \quad (11.11.i)$$

This function is initially increasing but eventually decreases. To find the quantity of  $\ell$  at which the marginal product of labor begins to decline, all we have to do is determine where the derivative of  $MP_\ell$  is equal to zero; i.e. we need to solve

$$\frac{dMP_\ell}{d\ell} = 6A - 0.6\ell = 0 \quad (11.11.ii)$$

which solves to  $\ell = 10A$ . Thus, marginal product of labor increases until  $\ell = 10A$  and falls thereafter. You can also check that marginal product of labor becomes negative at  $\ell = 20A$ .

- (b) Calculate the optimal number of baseball cards as a function of  $A$  assuming output price is given by  $p$  and the wage is  $w = 20$ . (Use the quadratic formula.)

Answer: We need to solve

$$\max_{\ell, x} px - 20\ell \text{ subject to } x = 3A\ell^2 - 0.1\ell^3, \quad (11.11.iii)$$

which can be written as the unconstrained optimization problem

$$\max_{\ell} p(3A\ell^2 - 0.1\ell^3) - 20\ell. \quad (11.11.iv)$$

The first order condition for this problem is

$$-0.3p\ell^2 + 6Ap\ell - 20 = 0. \quad (11.11.v)$$

For an equation of the form  $ax^2 + bx + c = 0$ , the quadratic formula gives the solutions

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \quad (11.11.vi)$$

We can use this formula to solve equation (11.11.v) and get the two “solutions”

$$\ell = \frac{-6Ap + \sqrt{36A^2p^2 - 24p}}{-0.6p} \text{ and } \ell = \frac{-6Ap - \sqrt{36A^2p^2 - 24p}}{-0.6p}, \quad (11.11.vii)$$

of which the latter is the true optimum (since we know from our intuitive treatment of the material that the larger of the two input amounts represents the profit maximum.)

- (c) How much will you each produce if  $A = 1$  and  $p = 1$ , and how much profit do each of you earn?

Answer: When  $p = 1$  and  $A = 1$  are plugged into the true optimum we just calculated, we get  $\ell \approx 15.77$ . (The “false” optimum in the first equation would come out to be  $\ell \approx 4.23$ .)

We can calculate the profit by first calculating the output you would produce with this much labor. Plugging  $\ell = 15.77$  into the production function, we get  $x \approx 353.96$ . We can then calculate the profit as  $\pi = px - w\ell = (1)(353.96) - 20(15.77) \approx 38.5$ . (You can also verify that the second “solution” for labor input is not an optimum by calculating the profit you would get under that input level — which turns out to be  $-38.5$ .)

- (d) *Now suppose you find a better technology — one that changes your production function from one where  $A = 1$  to one where  $A = 1.1$ . How do your answers change?*

Answer: You will now hire  $\ell = 18.20$ , produce  $x = 493.72$  and earn profit  $\pi = 126.30$ . (You can derive these exactly as you did for the previous case.)

- (e) *Now suppose that competition in the industry intensifies and the price of baseball cards falls to  $p = 0.88$ . How will you and Bob change your production decisions?*

Answer: Bob still operates under  $A = 1$ . Plugging in  $A = 1$  and  $p = 0.88$  into our solution for the optimal labor input, we get  $\ell \approx 14.92$ . Plugging this into the production function, this would give us  $x \approx 335.77$ , and using these to calculate profit, we get a profit of  $\pi \approx -2.99$ . Since the highest profit Bob can get by producing is negative, he will no longer produce.

You, on the other hand, face a production process with  $A = 1.1$ . Following the same procedure, you can then calculate that your optimal production plan is  $(\ell, x) \approx (17.73, 479.93)$  which gives profit of  $\pi \approx 67.8$ . Since profit is positive, you will indeed produce this level of output.

### Exercise 11.13

Policy Application: *Determining Optimal Class Size: Public policy makers are often pressured to reduce class size in public schools in order to raise student achievement.*

**A:** *One way to model the production process for student achievement is to view the “teacher/student” ratio as the input. For purposes of this problem, let  $t$  be defined as the number of teachers per 1000 students; i.e.  $t = 20$  means there are 20 teachers per 1,000 students. Class size in a school of 1000 students is then equal to  $1000/t$ .*

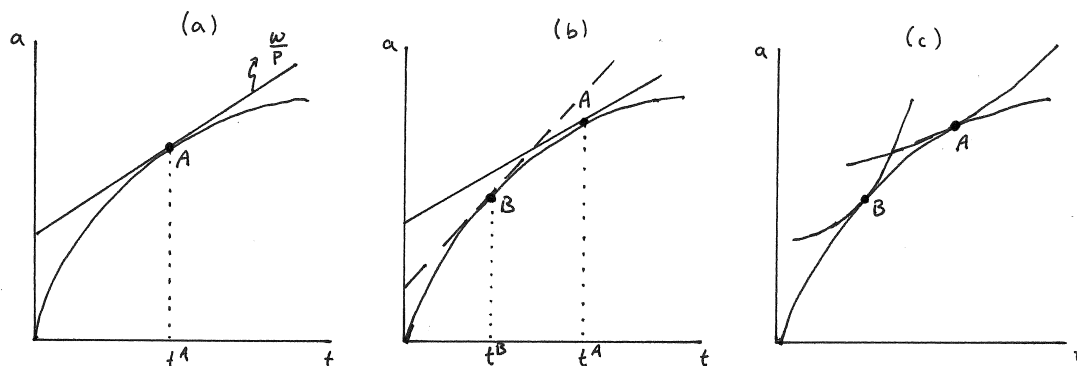
- (a) *Most education scholars believe that the increase in student achievement from reducing class size is high when class size is high but diminishes as class size falls. Illustrate how this translates into a production frontier with  $t$  on the horizontal axis and average student achievement  $a$  on the vertical.*

Answer: This production frontier is pictured in panel (a) of Exercise Graph 11.13 — with large slope (or marginal product) initially that falls as  $t$  increases. Note that  $t$  is not class size — when  $t$  is small, there are few teachers per 1000 students, which implies a large class size. In our graph, class size therefore falls as  $t$  increases.

- (b) *Consider a school with 1,000 students. If the annual salary of a teacher is given by  $w$ , what is the cost of raising the input  $t$  by 1 — i.e. what is the cost per unit of the input  $t$ ?*

Answer: Since  $t$  is the number of teachers per 1000 students, we need to hire one more teacher to increase  $t$  by 1. Thus, the per unit cost of  $t$  is  $w$ .

- (c) *Suppose  $a$  is the average score on a standardized test by students in the school, and suppose that the voting public is willing to pay  $p$  for each unit*



Exercise Graph 11.13 : Setting Class Size

increase in  $a$ . Illustrate the “production plan” that the local school board will choose if it behaves analogously to a profit maximizing firm.

Answer: This is also illustrated in panel (a) of Exercise Graph 11.13 where  $A$  is the optimal “production plan” given the willingness  $p$  of voters to get a one unit increase in achievement. The input  $t^A$  then results in class size of  $1000/t^A$ .

(d) What happens to class size if teacher salaries increase?

Answer: If teacher salaries increase,  $w$  goes up — which implies  $w/p$  increases and the isoprofits become steeper. As a result, the optimal  $t$  falls — which implies class size increases. This is illustrated in panel (b) of Exercise Graph 11.13.

(e) How would your graph change if the voting public’s willingness to pay per unit of  $a$  decreases as  $a$  increases?

Answer: This would in effect imply that  $p$  is high when  $a$  is low but decreases as  $a$  increases. A high  $p$  implies a relatively flat slope — thus, we would get isoprofit curves that, rather than being straight lines as in panels (a) and (b) of the graph, would be shaped as depicted in panel (c). (Note: The isoprofit tangent at  $B$  represents the curve when teacher salaries are higher.)

(f) Now suppose that you are analyzing two separate communities that fund their equally sized schools from tax contributions by voters in each school district. They face the same production technology, but the willingness to pay for marginal improvements in  $a$  is lower in community 1 than in community 2 at every production plan. How do the isoprofit maps differ for the two communities?

Answer: At every “production plan” the isoprofits of the community with lower willingness to pay would be steeper — indicating that a greater increase in achievement would be required for every increase in  $t$  in order for the community to remain equally well off.

- (g) *Illustrate how this will result in different choices of class size in the two communities.*

Answer: This can be illustrated in a graph identical to the one pictured in panel (c) of Exercise Graph 11.13 where the community with the lower willingness to pay optimizes at  $B$  while the other community optimizes at  $A$ .

- (h) *Suppose that the citizens in each of the two communities described above were identical in every way except that those in community 1 have a different average income level than those in community 2. Can you hypothesize which of the two communities has greater average income?*

Answer: Community 1 is poorer. Even though they care as much about education, their willingness to pay is lower simply because they have fewer resources. (This should be familiar from consumer theory — people with identical tastes can have very different demands for goods if they have different incomes.)

- (i) *Higher level governments often subsidize local government contributions to public education, particularly for poorer communities. What changes in your picture of a community's optimal class size setting when such subsidies are introduced?*

Answer: You can think of these subsidies as either reducing the teacher salaries  $w$  (since the higher level government now shares in the expense) or as raising the willingness to pay  $p$  since the voters know that they get more resources if they spend more on education. Whether viewed as a decrease in  $w$  or an increase in  $p$ , the impact on  $w/p$  is the same:  $w/p$  falls — which implies that isoprofit curves become shallower. This in turn has the impact of increasing the optimal choice of  $t$  (as illustrated in panels (b) and (c) of Exercise Graph 11.13) — which is the same as saying that class size will decrease.

**B:** *Suppose the production technology for average student achievement is given by  $a = 100t^{0.75}$ , and suppose again that we are dealing with a school that has 1000 students.*

- (a) *Let  $w$  denote the annual teacher salary in thousands of dollars and let  $p$  denote the community's marginal willingness to pay for an increase in student achievement. Calculate the "profit maximizing" class size.*

Answer: We need to solve the maximization problem

$$\max_{a,t} pa - wt \quad \text{subject to} \quad a = 100t^{0.75} \quad (11.13.i)$$

which can be written as the unconstrained optimization problem

$$\max_t a = 100t^{0.75} p - wt. \quad (11.13.ii)$$

The first order condition for this problem is  $75t^{-0.25}p - w = 0$  and can be solved to get

$$t = \left( \frac{75p}{w} \right)^4. \quad (11.13.iii)$$

Since class size is  $1000/t$ , the optimal class size is

$$\text{Class size} = \frac{1000w^4}{(75p)^4}. \quad (11.13.iv)$$

- (b) *What is the optimal class size when  $w = 60$  and  $p = 2$ ?*

Answer: Plugging these into the expression we just derived, we get an optimal class size of 25.6 students per class.

- (c) *What happens to class size as teacher salaries change?*

Answer: We can simply take the derivative of our optimal class size expression with respect to  $w$  to see if it is positive or negative. Since it is clearly positive, we know that class size increases as teacher salaries increase (and decreases as teacher salaries decrease).

- (d) *What happens to class size as the community's marginal willingness to pay for student achievement changes?*

Answer: This time we would take the derivative of our optimal class size expression with respect to  $p$ . Since this derivative is clearly negative, we know that class size will increase as  $p$  decreases (and decrease as  $p$  increases).

- (e) *What would change if the state government subsidizes the local contribution to school spending?*

Answer: This would effectively lower  $w$  — and thus decrease class size. Alternatively you could think of it as raising the local willingness to spend money on schools — which would also have the effect of decreasing class size.

- (f) *Now suppose that the community's marginal willingness to pay for additional student achievement is a function of the achievement level. In particular, suppose that  $p(a) = Ba^{\beta-1}$  where  $\beta \leq 1$ . For what values of  $\beta$  and  $B$  is the problem identical to the one you just solved?*

Answer: If  $B = p$  and  $\beta = 1$ , the problem is identical to what we solved before.

- (g) *Solve for the optimal  $t$  given the marginal willingness to pay of  $p(a)$ . What is the optimal class size when  $B = 3$  and  $\beta = 0.95$  (assuming again that  $w = 60$ ).*

Answer: We would now solve the problem

$$\max_{a,t} p(a)a - wt = Ba^{\beta-1}a - wt = Ba^{\beta} - wt \quad \text{subject to} \quad a = 100t^{0.75} \quad (11.13.v)$$

which can be written as the unconstrained maximization problem

$$\max_t B(100t^{0.75})^\beta - wt. \quad (11.13.vi)$$

The first order condition for this problem is

$$0.75\beta B(100^\beta)t^{0.75\beta-1} - w = 0, \quad (11.13.vii)$$

which solves to

$$t = \left( \frac{w}{0.75\beta B(100^\beta)} \right)^{1/(0.75\beta-1)} \quad (11.13.viii)$$

Substituting in  $w = 60$ ,  $\beta = 0.95$  and  $B = 3$ , we then get  $t \approx 37.27$ . Given that class size is  $1000/t$ , we get that the optimal class size under these conditions is approximately 26.83 students per class.

- (h) *Under the parameter values just specified, does class size respond to changes in teacher salaries as it did before?*

Answer: Under the parameter values specified, the optimal input level  $t$  becomes

$$t = \left( \frac{w}{0.75(0.95)(3)(100^{0.95})} \right)^{1/(0.75(0.95)-1)} \approx 169.79w^{-3.48}. \quad (11.13.ix)$$

This implies an optimal class size of

$$\text{Class Size} = \frac{1000}{t} = \frac{1000}{169.79w^{-3.48}} \approx 5.89w^{3.48}. \quad (11.13.x)$$

The derivative of class size with respect to  $w$  is then clearly positive — which implies that class size increases as teacher salaries increase.

## Conclusion: Potentially Helpful Reminders

1. Remember that the one-step profit maximization method is completely equivalent to the two-step method that minimizes costs first. The former gives us the profit maximizing condition that  $MRP = w$ ; the latter gives us the condition that  $p = MC$ . Since the two methods are equivalent, the two conditions imply one another in the single-input model; i.e. whenever  $MRP = w$ , it must be that  $p = MC$  and vice versa if we are at a true optimum for the firm. (This insight will generalize to multiple inputs in the next chapter.)
2. Make sure you really understand how the 2-step method for maximizing profit is equivalent to simply maximizing profit in a single step. As the model becomes more complex when we allow for two (rather than one) input in the next chapter, we will increasingly rely on the 2-step method.



3. In the 2-step method, we begin by deriving the cost curve (or function) — and this involves no consideration of the output price  $p$ . Thus, any producer who hires inputs in competitive input markets will get the same cost curves for a given technology. In particular, it does not matter for the cost curves whether the producer is in fact a price taker in the output market — he could be a monopolist who sets price in the output market, but his cost curves would still arise in exactly the same way. This will become important in Chapters 23 and beyond where we will use the same types of cost curves to discuss markets that are not perfectly competitive (where firms are not price takers).
4. It's a good time to get used to the way we use the term *cost*. Whenever we say “cost” in this chapter and throughout this text, we mean “economic” or “opportunity” cost. If something has to be paid regardless of what action I choose, it is not a cost of choosing an action. Ever. (In Chapter 13 we will adopt the term “expense” for cases where we have to spend money regardless of what we do.)