CHAPTER

12

Production with Multiple Inputs

This chapter continues the treatment of producer theory when firms are price takers. Chapter 11 focused on the short run model in which capital is held fixed and labor is therefore the only variable input. This allowed us to introduce the ideas of profit maximization and cost minimization within the simplest possible setting. Chapter 12 now focuses on the long run model in which both capital and labor are variable. The introduction of a second input then introduces the possibility that firms will substitute between capital and labor as input prices change. It also introduces the idea of returns to scale. And we will see that the 2-step profit maximization approach that was introduced at the end of Chapter 11 — i.e. the approach that begins with costs and then adds revenues to the analysis — is much more suited to a graphical treatment than the 1-step profit maximization approach (which would require graphing in 3 dimensions.)

Chapter Highlights

The main points of the chapter are:

1. Profit maximization in the 2-input (long run) model is conceptually the same as it is for the one-input (short run) model — the profit maximizing production plans (that involve positive levels of output) again satisfying the condition that the **marginal revenue products of inputs are equal to the input prices**. The **marginal product** of each input is measured along the vertical slice of the production frontier that holds the *other* input fixed (as already developed for the marginal product of labor in Chapter 11.)

2. **Isoquants** are horizontal slices of the production frontier and are, in a technical sense, similar to indifference curves from consumer theory. Their shape indicates the degree of substitutability between capital and labor, and their slope is the (marginal) **technical rate of substitution** which is equal to the (negative) ratio of the marginal products of the inputs.

3. Unlike in consumer theory where the labeling of indifference curves had no cardinal meaning, the labeling on isoquants has a clear cardinal interpretation since production units are objectively measurable. The rate at which this labeling increases tells us whether the production frontier's slope is increasing at an increasing or decreasing rate — and thus whether the production technology is exhibiting increasing or decreasing **returns to scale**.

4. Cost minimization in the two-input model is considerably more complex than it was in the single-input model of Chapter 11 because there are now many different ways of producing any given output level without wasting inputs (i.e. in a technologically efficient way) as indicated by all input bundles on each isoquant. The **least cost way of producing** any output level then depends on input prices — and is graphically seen as the tangency between **isocosts** and isoquants.

5. For **homothetic production processes**, all cost minimizing input bundles will lie on the same ray from the origin within the isoquant graph. The vertical slice of the 3-D production frontier along that ray is then the relevant slice on which the profit maximizing production plan lies.

6. The **cost curve** is derived from the cost-minimizing input bundles on that same ray from the origin — and, analogous to what we did in Chapter 11, its shape is the inverse of the shape of the production frontier along that slice. (This shape also indicates whether the production process has increasing or decreasing returns to scale). Once we have derived the cost curve, the 2-step profit maximization proceeds exactly as it did in Chapter 11 — with output occurring where $p = MC$.

12A Solutions to Within-Chapter-Exercises for Part A

Exercise 12A.1

Suppose we are modeling all non-labor investments as capital. Is the rental rate any different depending on whether the firm uses money it already has or chooses to borrow money to make its investments?

Answer: No — for the same reason that the rental rate of photocopiers for Kinkos is the same regardless of whether Kinkos owns or rents the copiers. If the firm borrows money from another firm, it is doing so at the interest rate r which then becomes the rental rate for the financial capital it is investing. If the firm uses its own money, it is foregoing the option of lending that money to another firm at the interest rate r — and thus it again costs the firm r per dollar to invest in its own capital.

Exercise 12A.2

Explain why the vertical intercept on a three dimensional isoprofit plane is π/p (where π represents the profit associated with that isoprofit plane).

Answer: A production plan on the vertical intercept has positive x but zero ℓ and k. Profit for a production plan (ℓ, k, x) is given by $\pi = px - w\ell - rk$ — but since $\ell = k = 0$ on the vertical axis, this reduces to $\pi = px$. Put differently, when there are no input costs, profit is the same as revenue for the firm — and revenue is just price times output. Dividing both sides of $\pi = px$ by p, we get π/p — the value of the intercept of the isoprofit plane associated with profit π.

Exercise 12A.3

We have just concluded that $MP_k = r/p$ at the profit maximizing bundle. Another way to write this is that the marginal revenue product of capital $MRP_k = pMP_k$ is equal to the rental rate. Can you explain intuitively why this makes sense?

Answer: The intuition is exactly identical to the intuition developed in Chapter 11 for the condition that marginal revenue product of labor must be equal to wage at the optimum. The marginal product of capital is the additional output we get from one more unit of capital (holding fixed all other inputs). Price times the marginal product of capital is the additional *revenue* we get from one more unit of capital. Suppose we stop hiring capital when the cost of a unit of capital r is exactly equal to this marginal revenue product of capital. Since marginal product is diminishing, this means that the marginal revenue from the previous unit of capital was greater than r — and so I made money on hiring the previous unit of capital. But if I hire past the point where $MRP_k = r$, I am hiring additional units of capital for which the marginal revenue is less than what it costs me to hire those units. Thus, had I stopped hiring before $MRP_k = r$, I would have forgone the opportunity of making additional profit from hiring more capital; if, on the other hand, I hire beyond $MRP_k = r$, I am incurring losses on the additional units of capital.

Exercise 12A.4

Suppose capital is fixed in the short run but not in the long run. *True or False*: If the firm has its long run optimal level of capital k^D (in panel (f) of Graph 12.1), then it will choose ℓ^D labor in the short run. And if ℓ^A in panel (c) is not equal to ℓ^D in panel (f), it must mean that the firm does not have the long run optimal level of capital as it is making its short run labor input decision.

Answer: This is true. If the firm has capital k^D, then it is operating on the short-run slice that holds k^D fixed in panel (f). The short run isoprofit is then just a slice of the long run isoprofit plane — and is tangent at labor input level ℓ^D. If the firm chooses $\ell^A \neq \ell^D$ in the short run, then it is not operating on this slice — and thus does not have the long run profit maximizing capital level of k^D.

Exercise 12A.5

Apply the definition of an isoquant to the one-input producer model. What does the isoquant look like there? (*Hint*: Each isoquant is typically a single point.)

Answer: An isoquant for a given level of output x is the set of all input bundles that result in that level of output without wasting any input. In the one-input model, the only production plans that don't waste inputs are those that lie on the production frontier. For each level of x, we therefore have a single level of (labor) input that can produce that level of x without any input being wasted. This single labor input level is then the isoquant for producing a particular output level x.

Exercise 12A.6

Why do you think we have emphasized the concept of marginal product of an input in producer theory but not the analogous concept of marginal utility of a consumption good in consumer theory?

Answer: The marginal product of an input is the number of additional units of output that can be produced if one more unit of the input is hired. This is an objectively measurable quantity. The marginal utility of a consumption good is the additional utility that will result from consumption of one more unit of the consumption good. Since it is measured in utility terms, it is not objectively measurable (since we have no way to measure "utils" objectively).

Exercise 12A.7

Repeat this reasoning for the case where $MP_\ell = 2$ and $MP_k = 3$.

Answer: Suppose we currently produce some quantity x using ℓ units of labor and k units of capital. If $MP_\ell = 2$ and $MP_k = 3$, this implies that, at my current production plan, capital is 1.5 times as productive as labor. Suppose I want to use one less unit of capital but continue to produce the same amount as before. Then, since capital is 1.5 times as productive as labor, this would imply I would have to hire 1.5 units of labor. In other words, substituting 1 unit of capital for 1.5 units of labor leads to no change in output on the margin — which is another way of saying that my technical rate of substitution is currently $TRS = -1/1.5 = -2/3$ — which is just $-MP_\ell/MP_k$.

Exercise 12A.8

Is there a relationship analogous to equation (12.3) that exists in consumer theory and, if so why do you think we did not highlight it in our development of consumer theory?

Answer: Yes. In exactly the same way, we could derive the relationship

$$MRS = -\frac{MU_1}{MU_2} \qquad (12A.8)$$

where MU_1 and MU_2 are the marginal utility of consuming good 1 and good 2. Since marginal utility is not objectively measurable, we did not emphasize the concept. However, note that "utils" cancel on the right hand side of our equation — implying that MRS is not expressed in util terms. Thus, MRS is a meaningful and measurable concept even if MU is not.

Exercise 12A.9

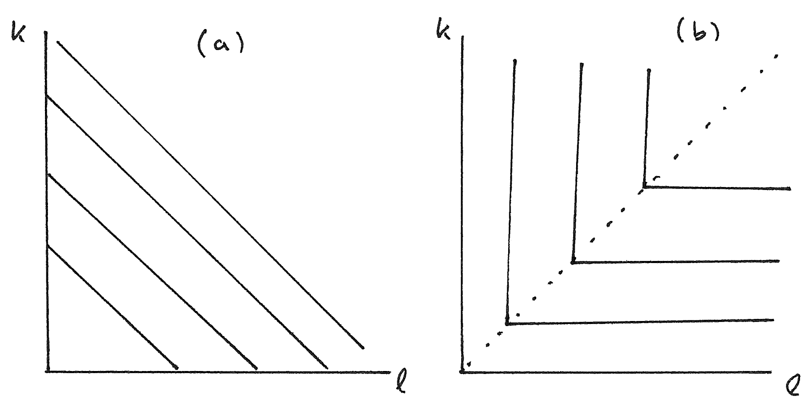
In the "old days", professors used to hand-write their academic papers and then have secretaries type them up. Once the handwritten scribbles were handed to the secretaries, there were two inputs into the production process: secretaries (labor) and typewriters (capital). If one of the production processes in Graph 12.4 represents the production for academic papers, which would it be?

Answer: There is little substitutability between secretaries and typewriters since each secretary has to be matched with one typewriter if papers are to be typed. Thus, panel (c) would come closest to representing the production for academic papers.

Exercise 12A.10

What would isoquant maps with no substitutability and perfect substitutability between inputs look like? Why are they homothetic?

Answer: These are graphed in Exercise Graph 12A.10, with panel (a) representing a production process with perfect substitutability of capital and labor and panel (b) representing perfect complementarity. These are both homothetic because the slope of the isoquants is unchanged along any ray from the origin.



Exercise Graph 12A.10 : Perfect Substitutes and Complements in Production

Exercise 12A.11

Consider a three-dimensional frontier similar to the one graphed in Graph 12.1 but with two goods on the horizontal axes and utility on the vertical. Why would we not think that vertical slices like the ones in 12.1d are meaningful in this case?

Answer: In order for those slices to be meaningful, we would have to think that we can objectively measure utility on the vertical axis. Since we have no such objective measure, the vertical slices don't have particular meaning in consumer theory. In producer theory, we measure output on the vertical axis – and since output is objectively measurable, the vertical slices are meaningful in producer theory.

Exercise 12A.12

Consider the same utility frontier described in the previous exercise. What would the horizontal slices analogous to those in Graph 12.3 be in consumer theory? Why are they meaningful when the vertical slices in Graph 12.1 are not?

Answer: These slices would be indifference curves in consumer theory. They are meaningful because they illustrate the tradeoffs that individuals are willing to make between goods – which can be stated in objective terms. Indifference curves do not require objective measures of utility as vertical slices in Graph 12.1 would.

Exercise 12A.13

Consider a real-world mountain and suppose that the shape of any horizontal slice of this mountain is a perfect (filled in) circle. I have climbed the mountain from every direction — and I have found that the climb typically starts off easy but gets harder and harder as I approach the top because the mountain gets increasingly steep. Does this mountain satisfy any of the two notions of convexity we have discussed?

Answer: A perfect (filled in) circle is a convex set. Thus, the horizontal slices of our mountain are convex sets — which means the mountain satisfies our original notion of convexity. If, however, the mountain gets steeper as we move up, vertical slices of the mountain will not be convex. Thus, our second notion of convexity does not hold.

Exercise 12A.14

Is the vertical slice described in the previous section (including all the points inside the mountain that lie on the slice) a convex set?

Answer: It is a convex set in panel (a) of Graph 12.5 and a non-convex set in panel (b) of Graph 12.5. The non-convexity in panel (b) can be seen in the fact that the line connecting A' and B' lies outside the shaded slice. (In panel (a), all lines connecting any two points in the shaded set lie fully within the set.)

Exercise 12A.15

Consider a single input production process with increasing marginal product. Is this production process increasing returns to scale? What about the production process in Graph 11.10?

Answer: Increasing marginal product in the single input model implies that we can increase the input by a factor t and thereby will raise output by a factor greater than t. Thus, the production process is increasing returns to scale. In Graph 11.10, the production process has this feature initially — but eventually becomes decreasing returns to scale.

Exercise 12A.16

True or False: Homothetic production frontiers can have increasing, decreasing or constant returns to scale.

Answer: This is true. You can take the same isoquant map and attach labels to it that would make the underlying production technology increasing, decreasing or constant returns to scale. (For increasing returns to scale, the labels would be increasing at an increasing rate; for decreasing returns to scale they would be increasing at a decreasing rate; and for constant returns to scale, they would be increasing at a constant rate.)

Exercise 12A.17

If the three panels of Graph 12.6 represented indifference curves for consumers, would there be any meaningful distinction between them? Can you see why the concept of "returns to scale" is not meaningful in consumer theory?

Answer: The distinction would not be meaningful — because the shape of the indifference curves and the ordering of the labels is the same in all three panels. Returns to scale is not meaningful in consumer theory because the statement "as I double the consumption bundle, my utility doubles" is not meaningful when we don't think we can measure utility objectively.

Exercise 12A.18

True or False: If you have decreasing marginal product of all inputs, you might have decreasing, constant, or increasing returns to scale.

Answer: This is true. The most counterintuitive case is the one where you have decreasing marginal product of all inputs but increasing returns to scale. But the fact that output increases at a diminishing rate as we add a single input at a time does not imply that output would not increase at an increasing rate if we increase all inputs simultaneously (which is increasing returns to scale.)

Exercise 12A.19

True or False: In the two-input model, decreasing returns to scale implies decreasing marginal product of all inputs.

Answer: This is true. If output increases at a diminishing rate as we add all inputs simultaneously, it must be that output increases at a diminishing rate as we add one input at a time.

Exercise 12A.20

True or False: In a two-input model, increasing returns to scale implies increasing marginal product of at least one input.

Answer: This is false. It is possible to have decreasing marginal product of all inputs and still have increasing returns to scale. If we did have increasing marginal product of one input, however, we are guaranteed to also have increasing returns to scale.

Exercise 12A.21

True or False: In the single-input model, each isoquant is composed of a single point which implies that all technologically efficient production plans are also economically efficient.

Answer: This is true. An isoquant is the set of input bundles that can produce a given output level without inputs being wasted. Since there is only one input, there is only one way to produce each output level without wasting inputs — thus the isoquant is a single point. It is technologically efficient because no input is wasted — and economically efficient because it is (by default) the least expensive of all the technologically efficient input bundles.

Exercise 12A.22

True or False: In the two input model, every economically efficient production plan must be technologically efficient but not every technologically efficient production plan is necessarily economically efficient.

Answer: This is true. In order for an input bundle to be the economically most efficient — or cheapest — way of producing an output level, it must be the case that no inputs are wastes — i.e. the input bundle must be technologically efficient for this output level. But, when there are many technologically efficient ways of producing a given level of output, some will be more expensive and some less — so they cannot all be economically efficient (i.e. cheapest).

Exercise 12A.23

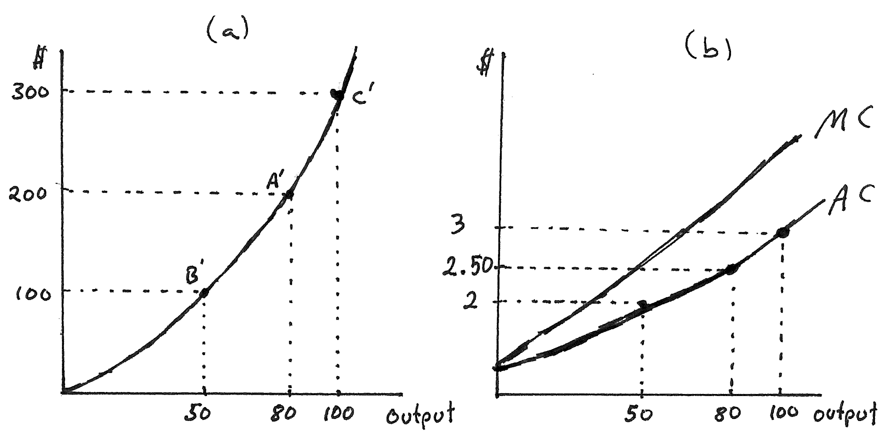
True or False: We have to know nothing about prices, wages or rental rates to determine the technologically efficient ways of producing different output levels, but we cannot generally find the economically efficient ways of producing any output level without knowing these.

Answer: This is true. Technologically efficient production just means production without wasting inputs — and we do not have to know anything about prices in the economy to know whether we are wasting inputs. Put differently, we do not have to know anything about prices to derive isoquants — they just come from the production frontier which is determined by the technology that is available to the producer. Economically efficient production means the “cheapest” way to produce — and that of course has much to do with input prices. (It does not, of course, have anything to do with the output price.)

Exercise 12A.24

Suppose the numbers associated with the isoquants in Graphs 17.7(a) and (b) had been 50, 80 and 100 instead of 50, 100 and 150. What would the total cost, *MC* and *AC* curves look like? Would this be an increasing or decreasing returns to scale production process, and how does this relate to the shape of the cost curves?

Answer: This is illustrated in Exercise Graph 12A.24. This would imply it is getting increasingly hard to produce additional units of output — i.e. the underlying technology represented by the isoquants has decreasing returns to scale. As a result, the cost of producing is increasing at an increasing rate — which causes the *MC* and *AC* curves to slope up.

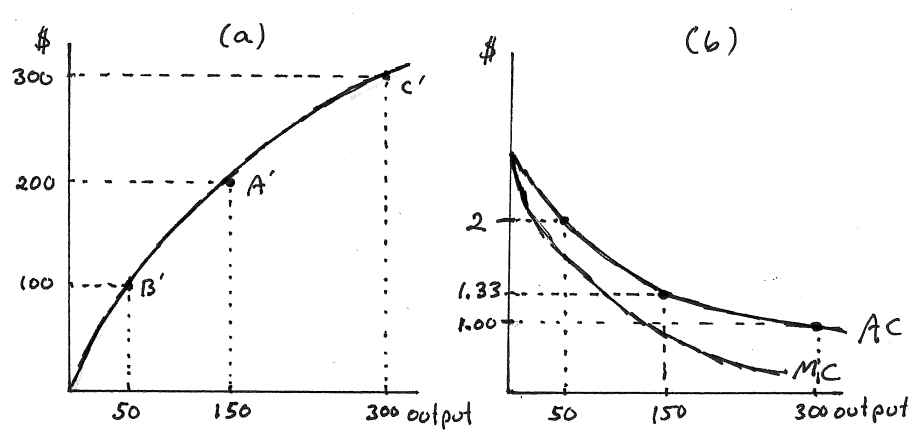


Exercise Graph 12A.24 : Decreasing Returns to Scale Cost Curves

Exercise 12A.25

How would your answer to the previous question change if the numbers associated with the isoquants were 50, 150, and 300 instead?

Answer: This is illustrated in Exercise Graph 12A.25. Production of additional goods is getting increasingly easy — which means the underlying production technology has increasing returns to scale. As a result, increased production causes costs to increase at a decreasing rate — which implies *MC* and *AC* are downward sloping.



Exercise Graph 12A.25 : Increasing Returns to Scale Cost Curves

Exercise 12A.26

If *w* increases, will the economically efficient production plans lie on a steeper or shallower ray from the origin? What if *r* increases?

Answer: If *w* increases, then w/r increases — which means the slope of the isocosts becomes steeper. Thus, the tangencies with isoquants will occur to the left (where the isoquants are steeper) — implying that they will occur on a ray that is steeper. If *r* increases, w/r falls — meaning that the isocosts get shallower. Thus, the tangencies with isoquants will occur to the right (where the isoquants are shallower) — implying that they will occur on a ray that is shallower. This should make sense — as *w* increases, economic efficiency will require a substitution away from labor and toward more capital, and the reverse will happen if *r* increases.

Exercise 12A.27

What is the shape of such a production process in the single input case? How does this compare to the shape of the vertical slice of the 3-dimensional production frontier along the ray from the origin in our graph?

Answer: The shape of such a one-input production process is the usual shape we employed in Chapter 11: On a graph with labor on the horizontal and output on the vertical, the production frontier initially increases at an increasing rate (as it becomes easier and easier to produce additional output) but eventually increases at a decreasing rate (as it becomes increasingly hard to produce additional output.) This is exactly the same shape as the slice along a ray from the origin of the 2-input production process that has initially increasing and eventual decreasing returns to scale.

Exercise 12A.28

True or False: If a producer minimizes costs, she does not necessarily maximize profits, but if she maximizes profits, she also minimizes costs. (*Hint*: Every point on the cost curve is derived from a producer minimizing the cost of producing a certain output level.)

Answer: True. Any production plan that is represented along the cost curve is cost minimizing, but only the plan where $p = MC$ is profit maximizing. But since the profit maximizing point is derived from the cost curve, it implicitly is also cost minimizing.

Exercise 12A.29

Suppose a production process begins initially with increasing returns to scale, eventually assumes constant returns to scale but never has decreasing returns. Would the MC curve ever cross the AC curve?

Answer: No, it would never cross AC. The MC and AC curves would start at the same place, with MC falling faster than AC along the increasing returns to scale portion of production. When we reach the constant returns to scale portion, MC would become flat, and AC would continue to fall at a decreasing rate as it converges (but never quite reaches) the flat MC curve.

Exercise 12A.30

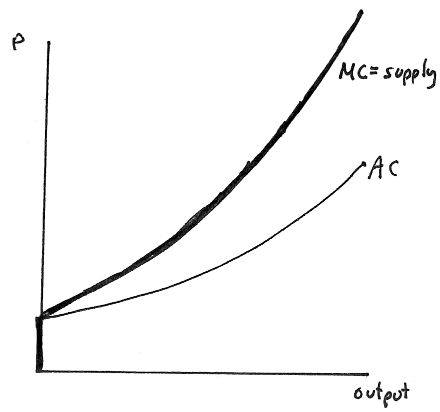
Another special case is the one graphed in Graph 12.7. What are the optimal supply choices for such a producer as the output price changes?

Answer: When $p^* = MC$, any output quantity would be optimal; when $p^* < MC$, it is optimal to produce zero (since profit would be negative); when $p^* > MC$, it would be optimal to produce an infinite amount (since you can keep making profit on each additional unit produced. Thus, the supply curve would lie on the vertical axis between $p = 0$ and $p = p^*$, horizontal at p^* and "vertical at infinity" for $p > p^*$.

Exercise 12A.31

Illustrate the output supply curve for a producer whose production frontier has decreasing returns to scale throughout (such as the case illustrated in Graph 12.1).

Answer: This is illustrated in Exercise Graph 12A.31. Decreasing returns to scale lead to a *MC* curve that is increasing throughout. Since it begins where *AC* begins, the entire *MC* curve lies above *AC* — and thus the entire *MC* curve is the supply curve.



Exercise Graph 12A.31 : Supply Curve with Decreasing Returns to Scale

12B Solutions to Within-Chapter-Exercises for Part B

Exercise 12B.1

Just as we can take the partial derivative of a production function with respect to one of the inputs (and call it the "marginal product of the input"), we could take the partial derivative of a utility function with respect to one of the consumption goods (and call it the "marginal utility from that good"). Why is the first of these concepts economically meaningful but the second is not?

Answer: This is because utility is not objectively measurable whereas output is. It is therefore meaningful to ask "how much additional output will one more unit of labor produce", but it is not meaningful to ask "how much additional utility will one more unit of good x yield."

Exercise 12B.2

Using the same method employed to derive the formula for MRS from a utility function, derive the formula for TRS from a production function $f(\ell, k)$.

Answer: The technical rate of substitution (TRS) is simply the change in k divided by the change in ℓ such that output remains unchanged, or

$$\frac{\Delta k}{\Delta \ell} \text{ such that } \Delta x = 0. \qquad (12B.2.i)$$

Actually, what we mean by a technical rate of substitution is somewhat more precise — we are not looking for just *any* combination of changes in k and ℓ (such that Δx=0). Rather, we are looking for small changes that define the slope around a particular point. Such small changes are denoted in calculus by using "d" instead of "Δ". Thus, we can re-write (12B.2.i) as

$$\frac{dk}{d\ell} \text{ such that } dx = 0. \qquad (12B.2.ii)$$

Changes in output arise from the combined change in k and ℓ, and this is expressed as the total differential (dx)

$$dx = \frac{\partial f}{\partial \ell} d\ell + \frac{\partial f}{\partial k} dk. \qquad (12B.2.iii)$$

Since we are interested in changes in input bundles that result in no change in output (thus leaving us on the same isoquant), we can set expression (12B.2.iii) to zero

$$\frac{\partial f}{\partial \ell} d\ell + \frac{\partial f}{\partial k} dk = 0 \qquad (12B.2.iv)$$

and then solve out for $dk/d\ell$ to get

$$\frac{dk}{d\ell} = -\frac{(\partial f/\partial \ell)}{(\partial f/\partial k)}. \qquad (12B.2.v)$$

Since this expression for $dk/d\ell$ was derived from the expression $dx = 0$, it gives us the equation for small changes in k divided by small changes in ℓ such that production remains unchanged — which is precisely our definition of a technical rate substitution.

Exercise 12B.3

True or False: Producer choice sets whose frontiers are characterized by quasiconcave functions have the following property: All horizontal slices of the choice sets are convex sets.

Answer: This is true — the horizontal slices of the quasiconcave functions are isoquants that satisfy the "averages are better than extremes" property — which means the set of production plans that lie above the isoquant (and thus inside the producer choice set) is convex.

Exercise 12B.4

True or False: All quasiconcave production functions — but not all concave production functions — give rise to convex producer choice sets.

Answer: This is false. Since all concave production functions are also quasiconcave, whatever holds for quasiconcave production functions must hold for concave productions. The statement would be true of the terms "quasiconcave" and "concave" switched places.

Exercise 12B.5

True or False: Both quasiconcave and concave production functions represent production processes for which the "averages are better than extremes" property holds.

Answer: This is true. We have shown that quasiconcave production functions give rise to producer choice sets whose horizontal slices are convex sets — which in turn implies that the isoquants have the usual shape that satisfies "averages are better than extremes." And since all concave functions are also quasiconcave, the same must hold for concave production functions.

Exercise 12B.6

Verify the last statement regarding Cobb-Douglas production functions.

Answer: The Cobb-Douglas production function takes the form $f(\ell, k) = \ell^{\alpha} k^{\beta}$. When we multiply a given input bundle (ℓ, k) by some factor t, we get

$$f(t\ell, tk) = (t\ell)^{\alpha}(tk)^{\beta} = t^{(\alpha+\beta)}\ell^{\alpha}k^{\beta} = t^{(\alpha+\beta)}f(\ell, k). \tag{12B.6}$$

When $\alpha+\beta = 1$, this equation tells us that increasing the inputs be a factor of t results in an increase of output by a factor of t — which is the definition of constant returns to scale. When $\alpha+\beta < 1$, the equation tells us that such an increase in inputs will result in less than a t-fold increase in output — which is the definition of decreasing returns to scale. And when $\alpha+\beta > 1$, output increases by more that t-fold — giving us increasing returns to scale.

Exercise 12B.7

Can you give an example of a Cobb-Douglas production function that has increasing marginal product of capital and decreasing marginal product of labor? Does this production function have increasing, constant or decreasing returns to scale?

Answer: In order for the example to work, the function $f(\ell, k) = \ell^{\alpha}k^{\beta}$ would have to be such that $\beta > 1$ (to get increasing marginal product of capital) and $\alpha < 1$ (to get decreasing marginal product of labor). Since we would still have $\alpha > 0$, this implies that $\alpha+\beta > 1$ — i.e. the production function has increasing returns to scale. This should make intuitive sense: If I can increase just *one* input t-fold and get a greater than t-fold increase in output (as I can if the marginal product of capital is increasing), then I can certainly increase *both* inputs t-fold and get more than a t-fold increase in output. So — as long as we have increasing marginal product in one input, we have increasing returns to scale.

Exercise 12B.8

True or False: It is not possible for a Cobb-Douglas production process to have decreasing returns to scale and increasing marginal product of one of its inputs.

Answer: This follows immediately from our answer to the previous exercise: Increasing marginal product in the Cobb-Douglas production function implies an exponent greater than 1 — but that implies a sum of exponents greater than 1 which is in turn equivalent to increasing returns to scale. Therefore the statement is true — you cannot have decreasing returns to scale and increasing marginal product at the same time.

Exercise 12B.9

In a 3-dimensional graph with x on the vertical axis, can you use the equation (12.18) to determine the vertical intercept of an isoprofit curve $P(\pi, p, w, r)$? What about the slope when k is held fixed?

Answer: At the vertical intercept, $k = \ell = 0$ — which implies the equation simply becomes $\pi = px$ or $x = \pi/p$ which is the intercept on the vertical (x) axis. When k is held fixed at, say, $\bar{k}$, The equation becomes $\pi = px - w\ell - r\bar{k}$. Rearranging terms, we can write this as

$$x = \left(\frac{\pi + r\overline{k}}{p}\right) + \frac{w}{p}\ell. \qquad (12B.9)$$

This is then an equation of the part of the production frontier that falls on the vertical slice that holds k fixed at $\overline{k}$. It has an intercept equal to the term in parenthesis, and it's slope is w/p.

Exercise 12B.10

Define profit and isoprofit curves for the case where land L is a third input and can be rented at a price r_L.

Answer: Profit is then simply

$$\pi = px - w\ell - rk - r_L L, \qquad (12B.10.i)$$

and the isoprofit plane P is

$$P(\pi, p, w, r, r_L) = \left\{(x, \ell, k, L) \in \mathbb{R}^4 \mid \pi = px - w\ell - rk - r_L L\right\}. \qquad (12B.10.ii)$$

Exercise 12B.11

Demonstrate that the problem as written in (12.20) gives the same answer.

Answer: Setting up the Lagrange function for this problem gives

$$\mathcal{L}(x, \ell, k, \lambda) = px - w\ell - rk + \lambda(x - f(\ell, k)), \qquad (12B.11.i)$$

which results in first order conditions

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= p + \lambda = 0,\\
\frac{\partial \mathcal{L}}{\partial \ell} &= -w - \lambda\frac{\partial f(\ell,k)}{\partial \ell} = 0,\\
\frac{\partial \mathcal{L}}{\partial k} &= -r - \lambda\frac{\partial f(\ell,k)}{\partial k} = 0,\\
\frac{\partial \mathcal{L}}{\partial \lambda} &= x - f(\ell, k) = 0.
\end{aligned} \qquad (12B.11.ii)$$

Solving the first of these equations for $\lambda = -p$, substituting this into the second and third equations and rearranging terms then gives

$$w = p\frac{\partial f(\ell,k)}{\partial \ell} \text{ and } r = p\frac{\partial f(\ell,k)}{\partial k}, \qquad (12B.11.iii)$$

which can further be written as

$$w = pMP_\ell = MRP_\ell \text{ and } r = pMP_k = MRP_k. \qquad (12B.11.iv)$$

Exercise 12B.12

Demonstrate that solving the problem as defined in equation (12.27) results in the same solution.

Answer: The Lagrange function for this problem is

$$\mathscr{L}(x,\ell,k,\lambda) = px - w\ell - rk + \lambda(x - 20\ell^{2/5}k^{2/5}). \tag{12B.12.i}$$

The first order conditions for this problem are

$$\begin{aligned}
\frac{\partial \mathscr{L}}{\partial x} &= p + \lambda = 0,\\
\frac{\partial \mathscr{L}}{\partial \ell} &= -w - 8\lambda\ell^{-3/5}k^{2/5} = 0,\\
\frac{\partial \mathscr{L}}{\partial k} &= -r - 8\lambda\ell^{2/5}k^{-3/5} = 0,\\
\frac{\partial \mathscr{L}}{\partial \lambda} &= x - 20\ell^{2/5}k^{2/5} = 0.
\end{aligned} \tag{12B.12.ii}$$

Plugging the $\lambda = -p$ (derived from the first equation) into the second and third equations then gives the condition that input prices are equal to marginal revenue products:

$$w = 8p\ell^{-3/5}k^{2/5} \text{ and } r = 8p\ell^{2/5}k^{-3/5}. \tag{12B.12.iii}$$

From this point forward, the problem solves out exactly as in the text. Solving the second of the two equations for k and plugging it into the first, we get the labor demand function

$$\ell(p,w,r) = \frac{(8p)^5}{r^2w^3}, \tag{12B.12.iv}$$

and plugging this in for ℓ in the second equation, we get the capital demand function

$$k(p,w,r) = \frac{(8p)^5}{w^2r^3}. \tag{12B.12.v}$$

Finally, we can derive the output supply function by plugging equations (12B.12.iv) and (12B.12.v) into the production function $f(\ell,k) = 20\ell^{2/5}k^{2/5}$ to get

$$x(p,w,r) = 20\frac{(8p)^4}{(wr)^2} = 81920\frac{p^4}{(wr)^2}. \tag{12B.12.vi}$$

Exercise 12B.13

Each panel of Graph 12.12 illustrates one of three “slices” of the respective function through the production plan $(x = 1280, \ell = 128, k = 256)$. What are the other two slices for each of the three functions? Do they slope up or down?

Answer: For the supply function, the other two slices are

$$x(5,w,10) = 81920\frac{5^4}{(10w)^2} = \frac{512000}{w^2} \text{ and}$$
$$x(5,20,r) = 81920\frac{5^4}{(20r)^2} = \frac{128000}{r^2}, \quad (12B.13.i)$$

both of which slope down. This makes sense: As input prices increase, less output is produced. For the labor demand function, the two other slices are

$$\ell(p,20,10) = \frac{(8p)^5}{(10^2)(20^3)} \approx 0.0401p^5 \text{ and } \ell(5,20,r) = \frac{(8(5))^5}{20^3 r^2} = \frac{12800}{r^2}. \quad (12B.13.ii)$$

The slope is positive for the first and negative for the second. Thus, labor demand increases as output price increases but decreases as the rental rate of capital increases.

Finally, for the capital demand function, the other two slices are

$$k(p,20,10) = \frac{(8p)^5}{20^2(10^3)} \approx 0.082p^5 \text{ and } k(5,w,10) = \frac{(8(5))^5}{10^3 w^2} = \frac{102400}{w^2}. \quad (12B.13.iii)$$

Again, the slope is positive for the first and negative for the second of these. Thus, capital demand increases as output price increases but decreases as wage increases.

Exercise 12B.14

Did we calculate a "conditional labor demand" function when we did cost minimization in the one-input model?

Answer: Yes, but we did not have to solve a "cost minimization" problem to do so. The only reason we need to solve a cost minimization problem now is that there are many technologically efficient production plans for each output level to choose from — and the problem allows us to determine which of these is the cheapest for a given set of input prices. In the one-input model, there was only one technologically efficient way of producing each output level — so we already knew that this was the cheapest way to produce. Thus, all we needed to do was invert the production function $x = f(\ell)$ — so that we could get the function $\ell(x)$ that told us how much labor input we needed to produce any output level. This function was then our "conditional labor demand" function — it told us, conditional on how much we want to produce, how much labor we will demand. In this case, input price was not part of the function because we knew that we would need that much labor to produce each output level no matter what the input price.

Exercise 12B.15

Why are the conditional input demand functions not a function of output price p?

Answer: Conditional input demands tell us least cost way of producing some output level x. The output price has no relevance for determining what the least cost way of producing is — it is only relevant for determining how much we should produce in order to maximize the difference between cost and revenue. Thus, only unconditional input demands are a function of output price.

Exercise 12B.16

Suppose you are determined to produce a certain output quantity $\overline{x}$. If the wage rate goes up, how will your production plan change? What if the rental rate goes up?

Answer: We can take the partial derivatives of the input demand functions with respect to wage to get

$$\frac{\partial \ell(w,r,\overline{x})}{\partial w} = \frac{-r^{1/2}}{2w^{3/2}}\left(\frac{x}{20}\right)^{5/4} < 0 \text{ and } \frac{\partial k(w,r,\overline{x})}{\partial w} = \frac{1}{(wr)^{1/2}}\left(\frac{x}{20}\right)^{5/4} > 0. \quad (12B.16)$$

Thus, when w increases, you will substitute away from labor and toward capital. The reverse holds if r increases (for similar reasons.)

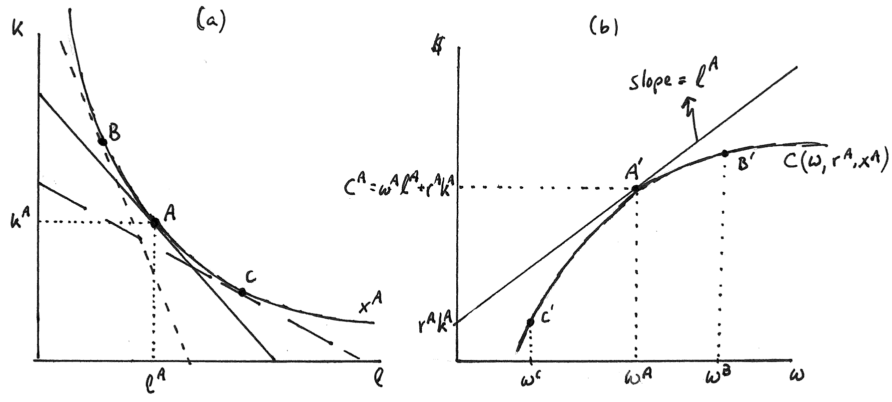
Exercise 12B.17

Can you replicate the graphical proof of the concavity of the expenditure function in the Appendix to Chapter 10 to prove that the cost function is concave in w and r?

Answer: The relevant section in the Appendix to Chapter 10 begins with "Suppose that a consumer initially consumes a bundle A when prices of x_1 and x_2 are p_1^A and p_2^A, and suppose that the consumer attains utility level u^A as a result." Let's re-write this sentence to make it apply to the producer's cost minimization problem: "Suppose that a *producer* initially *employs* a bundle A when prices of ℓ and k are w^A and r^A, and suppose that the *producer produces an output* level x^A as a result." This input bundle A is graphed in panel (a) of Exercise Graph 12B.17 where the slope of the (solid) isocost tangent to the x^A isoquant is $-w^A/r^A$.

The lowest cost at which x^A can be produced when input prices are w^A and r^A is therefore $C(w^A,r^A,x^A) = C^A = w^A\ell^A + r^Ak^A$. This is plotted in panel (b) of the graph where w is graphed on the horizontal and cost is graphed on the vertical axis. Since r^A and x^A are held fixed, we are in essence going to graph the slice of the cost function along which w varies. So far, we have plotted only one such point labeled A'.

Now suppose that w increases. If the producer does not respond by changing her input bundle, her cost will be given by the equation $C = r^Ak^A + w\ell^A$ as w changes — and this is just the equation of a line with intercept r^Ak^A and slope ℓ^A.



Exercise Graph 12B.17 : Concavity of $C(w,r,x)$ in w

This line is plotted in panel (b) of the graph and represents the costs as w changes assuming the producer naively stuck with the same input bundle (ℓ^A, k^A). But of course the producer does not do this — because she can reduce her costs by substituting away from labor and toward more capital as she slides to the new cost-minimizing input bundle B that has the new (steeper) isocost tangent to the x^A isoquant. Thus, as w increases to w^B, her costs will go up by *less* than the naive linear cost line in panel (b) suggests. The same logic implies that the producer's costs will fall by more than what is indicated by the line if w falls to w^C. This results in the cost function slice $C(w, r^A, x^A)$ taking on the concave shape in the graph. Put differently, even if the producer never substituted toward inputs that have become relatively cheaper and away from inputs that have become relatively more expensive, this slice of the cost function would be a straight line (and thus "weakly" concave). Any ability to substitute between inputs then causes the strict concavity we have derived. The same logic applies to changes in r.

Exercise 12B.18

What is the elasticity of substitution between capital and labor if the relationships in equation (12.51) hold with equality?

Answer: If these relationships hold with equality, then this implies that a cost-minimizing producer will not change her input bundle to produce a given output level as input prices change. In other words, as some inputs become relatively cheaper and others relatively more expensive, the producer does not substitute away from the more expensive to the cheaper. This can only be cost-minimizing if in fact the technology is such that substituting between inputs is not possible — which is the same as saying that the elasticity of substitution is zero.

Exercise 12B.19

Demonstrate how these indeed result from an application of the Envelope Theorem.

Answer: Substituting the constraint into the objective, we can write the profit maximization problem in an unconstrained form; i.e.

$$\max_{\ell,k} \pi = pf(\ell,k) - w\ell - rk. \tag{12B.19.i}$$

The "Lagrangian" is then simply equal to $\mathcal{L} = pf(\ell,k) - w\ell - rk$ (since there is no constraint to be multiplied by λ. The solution to the optimization problem is $\ell(w,r,p$ and $k(w,r,p)$. Substituting this solution into the objective function, we get the profit function $\pi(w,r,p)$ that tells us profit or any combination of prices (assuming the producer is profit maximizing). The envelope theorem then tells us that the derivative of this profit function with respect to a parameter (such as input and output prices) is equal to the derivative of the Lagrangian (which is just equal to the π expression in our optimization problem) with respect to that parameter *evaluated at the optimum* — i.e. evaluated at $\ell(w,r,p)$ and $k(w,r,p$. Thus,

$$\frac{\pi(w,r,p)}{\partial w} = \frac{\partial \mathcal{L}}{\partial w}\Big|_{\ell(w,r,p),k(w,r,p)} = -\ell\Big|_{\ell(w,r,p),k(w,r,p)} = -\ell(w,r,p), \tag{12B.19.ii}$$

and

$$\frac{\pi(w,r,p)}{\partial r} = \frac{\partial \mathcal{L}}{\partial r}\Big|_{\ell(w,r,p),k(w,r,p)} = -k\Big|_{\ell(w,r,p),k(w,r,p)} = -k(w,r,p). \tag{12B.19.iii}$$

Finally,

$$\begin{aligned}\frac{\pi(w,r,p)}{\partial r} = \frac{\partial \mathcal{L}}{\partial p}\Big|_{\ell(w,r,p),k(w,r,p)} &= f(\ell,k)\Big|_{\ell(w,r,p),k(w,r,p)} \\ &= f\big(\ell(w,r,p),k(w,r,p)\big) = x(w,r,p).\end{aligned} \tag{12B.19.iv}$$

Exercise 12B.20

How can you tell from panel (a) of the graph that $\pi(x^B,\ell^B) > \pi' > \pi(x^A,\ell^A)$?

Answer: The intercept of the new (magenta) isoprofit is higher than the intercept of the original (blue) isoprofit. Let the new intercept be denoted π^B/p^B and the original intercept as π^A/p^A. We know that

$$\frac{\pi^B}{p^B} > \frac{\pi^A}{p^A} \text{ and } p^B > p^A, \tag{12B.20.i}$$

which can be true only if $\pi^B > \pi^A$. Similarly,

$$\frac{\pi'}{p^B} > \frac{\pi^A}{p^A} \text{ and } p^B > p^A \text{ implies } \pi' > \pi^A. \qquad (12B.20.ii)$$

Finally,

$$\frac{\pi'}{p^B} > \frac{\pi'}{p^B} \text{ implies } \pi^B > \pi'. \qquad (12B.20.iii)$$

These three conclusions together imply $\pi^B > \pi' > \pi^A$.

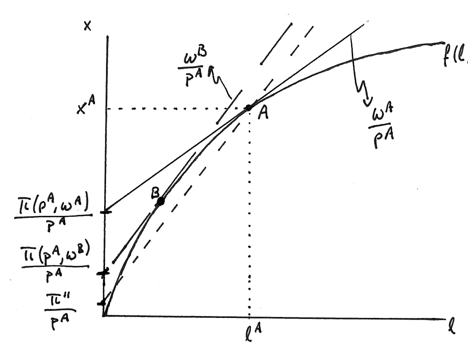
Exercise 12B.21

Use a graph similar to that in panel (a) of Graph 12.14 to motivate Graph 12.15.

Answer: This is done in Exercise Graph 12B.21 where the short run production function $f(\ell)$ is plotted with the originally optimal production plan (ℓ^A, x^A) at the original prices (w^A, p^A). An increase in the wage to w^B causes isoprofits to become steeper — with B becoming the new profit maximizing production plan. Had the producer not responded by changing her production plan, she would have operated on the steeper isoprofit that does through A rather than the one that goes through B — and would have made profit π'' instead of $\pi(p^A, w^B)$. Since the intercepts of the three isoprofits all have p^A in the denominator, it is immediate from the picture that

$$\pi(p^A, w^A) > \pi(p^A, w^B) > \pi'', \qquad (12B.21)$$

exactly as in the graph of the text.



Exercise Graph 12B.21 : Deriving the convexity of the profit function in w

12C Solutions to Odd Numbered End-of-Chapter Exercises

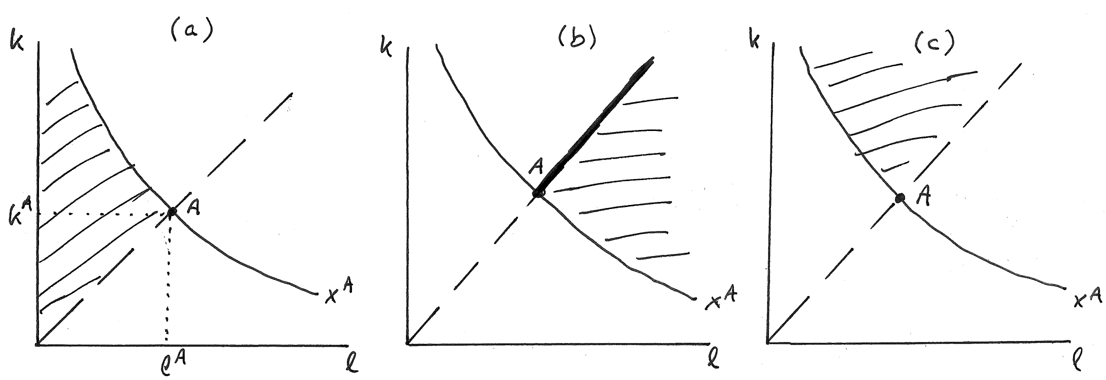
Exercise 12.1

In our development of producer theory, we have found it convenient to assume that the production technology is homothetic.

A: *In each of the following, assume that the production technology you face is indeed homothetic. Suppose further that you currently face input prices (w^A, r^A) and output price p^A — and that, at these prices, your profit maximizing production plan is $A = (\ell^A, k^A, x^A)$.*

(a) *On a graph with ℓ on the horizontal and k on the vertical, illustrate an isoquant through the input bundle (ℓ^A, k^A). Indicate where all cost minimizing input bundles lie given the input prices (w^A, r^A).*

Answer: This is depicted in panel (a) of Exercise Graph 12.1. Since the isocosts must be tangent at the profit maximizing input bundle A, homotheticity implies that all tangencies of isocosts with isoquants lie on the ray from the origin that passes through A.



Exercise Graph 12.1 : Changing Prices and Profit Maximization

(b) *Can you tell from what you know whether the shape of the production frontier exhibits increasing or decreasing returns to scale along the ray you indicated in (a)?*

Answer: You cannot tell whether the production frontier has increasing or decreasing returns to scale along the entire ray from the origin.

(c) *Can you tell whether the production frontier has increasing or decreasing returns to scale around the production plan $A = (\ell^A, k^A, x^A)$?*

Answer: Yes, you can tell that it must have decreasing returns to scale at A — because the isoprofit must be tangent at that point in order for A to be the profit maximizing production plan.

(d) *Now suppose that wage increases to w'. Where will your new profit maximizing production plan lie relative to the ray you identified in (a)?*

Answer: When w increases, the isocosts become steeper — which implies that they are tangent to the isoquants to the left of the ray that goes through A. Thus, the new ray on which all cost minimizing production plans lie is steeper than the ray drawn in panel (a) of Exercise Graph 12.1. Since the new profit maximizing production plan must lie on that ray (because profit maximization implies cost minimization), the new profit maximizing production plan must lie to the left of the ray that passes through A.

(e) *In light of the fact that supply curves shift to the left as input prices increase, where will your new profit maximizing input bundle lie relative to the isoquant for x^A?*

Answer: The leftward shift of supply curves as w increases implies that the profit maximizing output level falls. Thus, the new profit maximizing input bundle must lie *below* the x^A isoquant.

(f) *Combining your insights from (d) and (e), can you identify the region in which your new profit maximizing bundle will lie when wage increases to w'?*

Answer: This is illustrated as the shaded area in panel (a) of Exercise Graph 12.1. The shaded area emerges from the insight in (d) that the new profit maximizing bundle lies to the *left* of the ray through A and from the insight in (e) that it must lie *below* the isoquant for x^A.

(g) *How would your answer to (f) change if wage fell instead?*

Answer: If wage falls instead, then the isocosts become shallower — which implies that all cost minimizing bundles will now lie to the *right* of the ray through A. A drop in w will furthermore shift the output supply curve to the right — which implies that the profit maximizing production plan will involve an increase in the production of x. Thus, the new profit maximizing plan must lie to the *right* of the ray through A (because profit maximization implies cost minimization) and it must lie *above* the isoquant for x^A (because output increases). This is indicated as the shaded area in panel (b) of Exercise Graph 12.1.

(h) *Next, suppose that, instead of wage changing, the output price increases to p'. Where in your graph might your new profit maximizing production plan lie? What if p decreases?*

Answer: When output price p changes, the slopes of the isocosts (which are equal to $-w/r$) remain unchanged. Thus, all cost minimizing production plans remain on the ray through A. Since supply curves slope up, an increase in p will cause an increase in output — implying that the new profit maximizing production plan lies *above* the isoquant for x^A.

Thus, when p increases, the new profit maximizing production plan lies on the bold portion of the ray through A as indicated in panel (b) of Exercise Graph 12.1. When p decreases, on the other hand, output falls — which implies that the new profit maximizing production plan lies on the dashed portion of the ray through A in panel (b) of the graph.

(i) *Can you identify the region in your graph where the new profit maximizing plan would lie if instead the rental rate r fell?*

Answer: If r falls, the isocosts become steeper — implying the ray containing all cost minimizing production plans will be steeper than the ray through A. Thus, cost minimization implies that the new profit maximizing input bundle will lie to the *left* of the ray through A. A decrease in r further implies a shift in the supply curve to the right — which implies that output will increase. Thus, the profit maximizing input bundle must lie *above* the isoquant for x^A. This gives us the region to the *left* of the ray through A and *above* the isoquant x^A — which is equal to the shaded region in panel (c) of Exercise Graph 12.1.

B: *Consider the Cobb-Douglas production function $f(\ell,k) = A\ell^\alpha k^\beta$ with $\alpha,\beta > 0$ and $\alpha+\beta<1$.*

(a) *Derive the demand functions $\ell(w,r,p)$ and $k(w,r,p)$ as well as the output supply function $x(w,r,p)$.*

Answer: These result from the profit maximization problem

$$\max_{\ell,k,x} px - w\ell - rk \text{ subject to } x = A\ell^\alpha k^\beta \tag{12.1.i}$$

which can also be written as

$$\max_{\ell,k} pA\ell^\alpha k^\beta - w\ell - rk. \tag{12.1.ii}$$

Taking first order conditions and solving these, we then get input demand functions

$$\ell(w,r,p) = \left(\frac{pA\alpha^{(1-\beta)}\beta^\beta}{w^{(1-\beta)}r^\beta}\right)^{1/(1-\alpha-\beta)} \text{ and } k(w,r,p) = \left(\frac{pA\alpha^\alpha\beta^{(1-\alpha)}}{w^\alpha r^{(1-\alpha)}}\right)^{1/(1-\alpha-\beta)}. \tag{12.1.iii}$$

Plugging these into the production function and simplifying, we also get the output supply function

$$x(w,r,p) = \left(\frac{Ap^{(\alpha+\beta)}\alpha^\alpha\beta^\beta}{w^\alpha r^\beta}\right)^{1/(1-\alpha-\beta)} \tag{12.1.iv}$$

(b) *Derive the conditional demand functions $\ell(w,r,x)$ and $k(w,r,x)$.*

Answer: We need to solve the cost minimization problem

$$\min_{\ell,k} \; w\ell + rk \;\text{ subject to }\; x = A\ell^{\alpha}k^{\beta}. \tag{12.1.v}$$

Setting up the Lagrangian and solving the first order conditions, we then get the conditional input demand functions

$$\ell(w,r,x) = \left(\frac{\alpha r}{\beta w}\right)^{\beta/(\alpha+\beta)}\left(\frac{x}{A}\right)^{1/(\alpha+\beta)} \text{ and } k(w,r,x) = \left(\frac{\beta w}{\alpha r}\right)^{\alpha/(\alpha+\beta)}\left(\frac{x}{A}\right)^{1/(\alpha+\beta)} \tag{12.1.vi}$$

(c) *Given some initial prices* (w^A, r^A, p^A), *verify that all cost minimizing bundles lie on the same ray from the origin in the isoquant graph.*

Answer: Dividing the conditional input demands by one another, we get

$$\frac{k(w^A,r^A,x)}{\ell(w^A,r^A,x)} = \frac{\beta w^A}{\alpha r^A}. \tag{12.1.vii}$$

Thus, regardless of what isoquant x we try to reach, the ratio of capital to labor that minimizes the cost of reaching that isoquant is independent of x — implying that all cost minimizing input bundles lie on a ray from the origin.

(d) *If w increases, what happens to the ray on which all cost minimizing bundles lie?*

Answer: If w increases to w', the ratio of capital to labor becomes

$$\frac{\beta w'}{\alpha r^A} > \frac{\beta w^A}{\alpha r^A}; \tag{12.1.viii}$$

i.e. the ray becomes steeper as firms substitute away from labor and toward capital.

(e) *What happens to the profit maximizing input bundles?*

Answer: We see from the input demand equations in (12.1.iii) that both labor and capital demand fall as w increases. (Similarly, we see from equation (12.1.iv) that output supply falls.)

(f) *How do your answers change if w instead decreases?*

Answer: When wage falls to w'', we get that the ray on which cost minimizing bundles occur is

$$\frac{\beta w''}{\alpha r^A} < \frac{\beta w^A}{\alpha r^A}; \tag{12.1.ix}$$

i.e. the ray becomes shallower. From the input demand functions, we also see that demand for labor and capital increase — as does output (as seen in the output supply function).

(g) *If instead p increases, does the ray along which all cost minimizing bundles lie change?*

Answer: The ray along which cost minimizing bundles lie is defined by the ratio of conditional capital to conditional labor demand — which is

$$\frac{k(w,r,x)}{\ell(w,r,x)} = \frac{\beta w}{\alpha r}. \tag{12.1.x}$$

Since this does not depend on p, we can see that the ray does not depend on output price. This should make sense: Cost minimization does not take output price into account since all it asks is: "what is the least cost way of producing x?"

(h) *Where on that ray will the profit maximizing production plan lie?*

Answer: Since the ray of cost minimizing input bundles remains unchanged, we know that the new profit maximizing plan lies somewhere on that ray. From the output supply equation (12.1.iv), we see that output increases with p. Thus, the new profit maximizing production plan lies above the initial isoquant and on the same ray as the initial profit maximizing production plan.

(i) *What happens to the ray on which all cost minimizing input bundles lie if r falls? What happens to the profit maximizing input bundle?*

Answer: If r falls to r', we get

$$\frac{\beta w^A}{\alpha r'} > \frac{\beta w^A}{\alpha r^A}; \tag{12.1.xi}$$

i.e. the ray on which cost minimizing input bundles lie will be steeper as firms substitute toward capital and away from labor. From the output supply equation (12.1.iv), we can also see that a decrease in r results in an increase in output — thus, the new profit maximizing input bundle lies above the initial isoquant and to the left of the initial ray along which cost minimizing input bundles occurred.

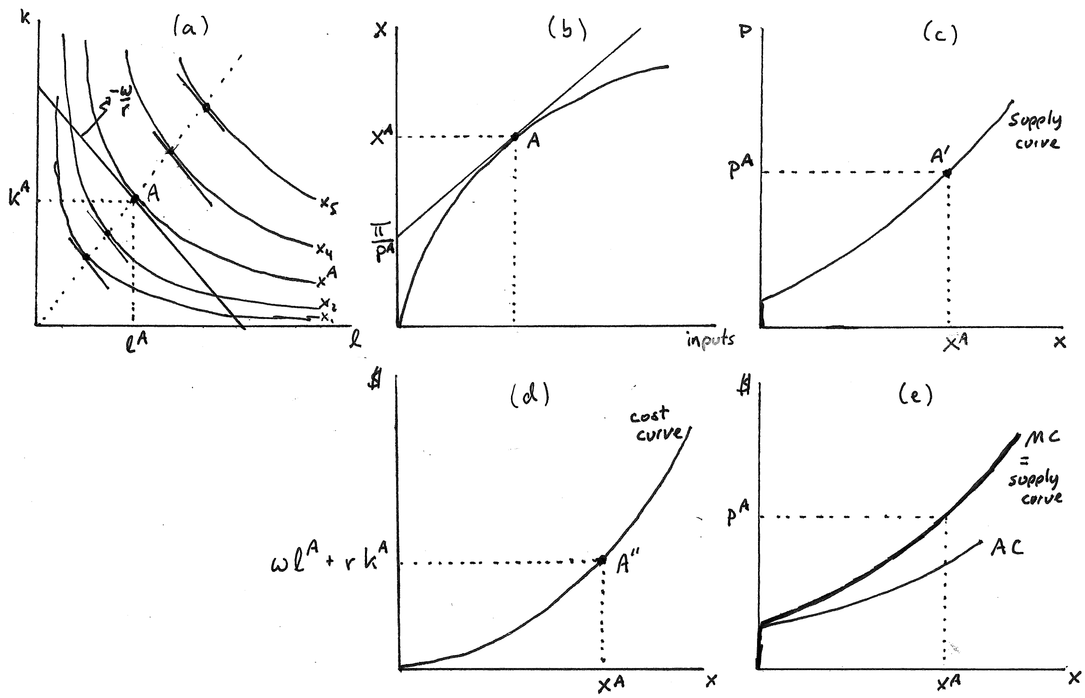
Exercise 12.3

Consider again the two ways in which we can view the producer's profit maximization problem.

A: *Suppose a homoethetic production technology involves two inputs, labor and capital, and that its producer choice set is fully convex.*

(a) *Illustrate the production frontier in an isoquant graph with labor on the horizontal axis and capital on the vertical.*

Answer: This is done in panel (a) of Exercise Graph 12.3. Since the producer choice set is convex, the horizontal slices represented by the isoquants must have the usual convex shape. In addition, the homotheticity property implies that the slopes (or TRS) of the isoquants are the same along any ray from the origin.



Exercise Graph 12.3 : 2 Ways to Derive Output Supply

(b) *Does this production process have increasing or decreasing returns to scale? How would you be able to see this on an isoquant graph like the one you have drawn?*

Answer: It has decreasing returns to scale — because the entire producer choice set is convex. You would only see this in an isoquant map if the isoquants are accompanied by output numbers that increase at a decreasing rate along any ray from the origin.

(c) *For a given wage w and rental rate r, show in your graph where the cost minimizing input bundles lie. What is true at each such input bundle?*

Answer: The input prices give us the slope of the isocost lines — which is $(-w/r)$. The isocost drawn in panel (a) is tangent at A — implying that (ℓ^A, k^A) is the cheapest input bundle that can produce the output level x^A. Since the production process is homothetic, it implies that all isoquants have the same slope along the ray from the origin through A. This further implies that all cost minimizing input bundles for the various output levels (represented by the isoquants) lie on this ray. Put differently, along this ray it is always true that $TRS = -w/r$ — the condition for cost minimization.

(d) *On a separate graph, illustrate the vertical slice (of the production frontier) that contains all these cost minimizing input bundles.*

Answer: Panel (b) of Exercise Graph 12.3 illustrates this vertical slice whose shape emerges from the decreasing returns to scale of the production process.

(e) *Assuming output can be sold at p^A, use a slice of the isoprofit plane to show the profit maximizing production plan. What, in addition to what is true at all the cost-minimizing input bundles, is true at this profit maximizing plan?*

Answer: This is also illustrated in panel (b) where the slice of the isoprofit plane is tangent at A. Since this is the profit maximizing plan, it must also be true that $p^A MP_\ell^A = w$ and $p^A MP_k^A = r$ — i.e. the marginal revenue product of each input is equal to that input's price. (Of course this automatically implies that $TRS^A = -w/r$ — which can be shown by simply dividing the two previous profit maximizing conditions by each other.)

(f) *If output price changes, would you still profit maximize on this vertical slice of the production frontier? What does the supply curve (which plots output on the horizontal and price on the vertical) look like?*

Answer: Yes, you would still produce on the same slice. This can be seen in panel (a) — a change in p changes nothing in panel (a). Thus, the cost minimizing input bundles remain unchanged, and — since profit maximization implies cost minimization — the profit maximizing plan must therefore lie on this slice. As p changes, the slope of the isoprofit line in (b) changes, becoming steeper when p falls and shallower when it rises. Thus, as p increases, output increases — and as p decreases, output decreases. This results in a shape for the supply curve as drawn in panel (c) of Exercise Graph 12.3.

(g) *Now illustrate the (total) cost curve (with output on the horizontal and dollars on the vertical axis). How is this derived from the vertical slice of the production frontier that you have drawn before?*

Answer: The vertical slice of the production frontier in panel (b) illustrates that it gets increasingly difficult to produce additional units of output as the inputs are increased in proportion to one another. This implies that the cost of increasing output will rise faster and faster as output increases — giving us the shape for the cost curve in panel (d) of Exercise Graph 12.3. This shape is essentially the inverse of the shape of the production frontier slice in (a). For the output quantity x^A, for instance, this cost is simply calculated by going back to panel (a) and checking how much of each input is required to produce x^A. We then multiply each input quantity by how much that input costs per unit to determine the total cost of producing x^A.

(h) *Derive the marginal and average cost curves and indicate where in your picture the supply curve lies.*

Answer: This is done in panel (e) of Exercise Graph 12.3 where the MC is simply the slope of the (total) cost curve from (d) — a slope that starts small (i.e. shallow) and becomes increasingly large (i.e. steeper). As always, the AC begins where the MC. Since MC is increasing throughout, this implies that AC always lies below MC. The supply curve is then, as always, the part of the MC curve that lies above the AC.

(i) *Does the supply curve you drew in part (f) look similar to the one you drew in part (h)?*

Answer: Yes — because the two methods of deriving the supply curve are equivalent.

B: *Suppose that the production technology is fully characterized by the Cobb-Douglas production function $x = f(\ell,k) = A\ell^{\alpha}k^{\beta}$ with $\alpha+\beta<1$ and A, α, and β all greater than zero.*

(a) *Set up the profit maximization problem (assuming input prices w and r and output price p). Then solve for the input demand and output supply functions. (Note: This is identical to parts B(b) and (c) of exercise 12.2 — so if you have solved it there, you can simply skip to part (b) here.)*

Answer: Derived in the usual way, the input demand functions we calculated there are

$$\ell(w,r,p) = \left(\frac{pA\alpha^{(1-\beta)}\beta^{\beta}}{w^{(1-\beta)}r^{\beta}}\right)^{1/(1-\alpha-\beta)} \text{ and } k(w,r,p) = \left(\frac{pA\alpha^{\alpha}\beta^{(1-\alpha)}}{w^{\alpha}r^{(1-\alpha)}}\right)^{1/(1-\alpha-\beta)} \tag{12.3.i}$$

and the output supply function was

$$x(w,r,p) = \left(\frac{Ap^{(\alpha+\beta)}\alpha^{\alpha}\beta^{\beta}}{w^{\alpha}r^{\beta}}\right)^{1/(1-\alpha-\beta)}. \tag{12.3.ii}$$

(b) *Now set up the cost minimization problem and solve for the first order conditions.*

Answer: This problem is

$$\min_{\ell,k} w\ell + rk \text{ subject to } x = A\ell^{\alpha}k^{\beta}. \tag{12.3.iii}$$

The Lagrange function for this problem is

$$\mathcal{L}(\ell,k,\lambda) = w\ell + rk + \lambda\left(x - A\ell^{\alpha}k^{\beta}\right) \tag{12.3.iv}$$

giving rise to first order conditions

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\ell} &= w - \lambda A\alpha\ell^{\alpha-1}k^{\beta} = 0\\
\frac{\partial\mathcal{L}}{\partial k} &= r - \lambda A\beta\ell^{\alpha}k^{\beta-1} = 0\\
\frac{\partial\mathcal{L}}{\partial\lambda} &= x - A\ell^{\alpha}k^{\beta} = 0
\end{aligned} \tag{12.3.v}$$

(c) *Solve for the conditional labor and capital demands.*

Answer: Moving the negative terms in each of the first two first order conditions to the other side, and dividing the two conditions by each other, we get

$$\frac{w}{r} = \frac{\alpha k}{\beta \ell} \quad \text{or} \quad k = \frac{\beta w \ell}{\alpha r}. \tag{12.3.vi}$$

Substituting the latter expression for k into the third first order condition and solving for ℓ, we then get the conditional labor demand function

$$\ell(w,r,x) = \left(\frac{\alpha r}{\beta w}\right)^{\beta/(\alpha+\beta)} \left(\frac{x}{A}\right)^{1/(\alpha+\beta)} \tag{12.3.vii}$$

and substituting this back into the expression for k from equation (12.3.vi), we get the conditional capital demand function

$$k(w,r,x) = \left(\frac{\beta w}{\alpha r}\right)^{\alpha/(\alpha+\beta)} \left(\frac{x}{A}\right)^{1/(\alpha+\beta)}. \tag{12.3.viii}$$

(d) *Derive the cost function and simplify the function as much as you can. (*Hint*: You can check your answer with the cost function given for the same production process in exercise 12.4) Then derive from this the marginal and average cost functions.*

Answer: The cost function is simply the sum of the conditional input demands multiplied by the respective input prices; i.e.

$$\begin{aligned} C(w,r,x) &= w\ell(w,r,x) + rk(w,r,x) \\ &= w\left(\frac{\alpha r}{\beta w}\right)^{\beta/(\alpha+\beta)} \left(\frac{x}{A}\right)^{1/(\alpha+\beta)} + r\left(\frac{\beta w}{\alpha r}\right)^{\alpha/(\alpha+\beta)} \left(\frac{x}{A}\right)^{1/(\alpha+\beta)}. \end{aligned} \tag{12.3.ix}$$

This can be written as

$$C(w,r,x) = \left[w\left(\frac{\alpha r}{\beta w}\right)^{\beta/(\alpha+\beta)} + r\left(\frac{\beta w}{\alpha r}\right)^{\alpha/(\alpha+\beta)} \right] \left(\frac{x}{A}\right)^{1/(\alpha+\beta)} \tag{12.3.x}$$

which, with a little algebra, simplifies to

$$C(w,r,x) = (\alpha+\beta)\left(\frac{x w^{\alpha} r^{\beta}}{A \alpha^{\alpha} \beta^{\beta}}\right)^{1/(\alpha+\beta)}. \tag{12.3.xi}$$

The marginal cost function is then simply the derivative of the cost function with respect to x; i.e.

$$MC = \frac{\partial C(w,r,x)}{\partial x} = \left(\frac{w^\alpha r^\beta}{A\alpha^\alpha \beta^\beta}\right)^{1/(\alpha+\beta)} x^{(1-\alpha-\beta)/(\alpha+\beta)}. \quad (12.3.xii)$$

Finally, the average cost function is simply

$$AC(w,r,x) = \frac{C(w,r,x)}{x} = (\alpha+\beta)\left(\frac{w^\alpha r^\beta}{A\alpha^\alpha \beta^\beta}\right)^{1/(\alpha+\beta)} x^{(1-\alpha-\beta)/(\alpha+\beta)}. \quad (12.3.xiii)$$

(e) *Use your answers to derive the supply function. Compare your answer to what you derived in (a).*

Answer: To derive the supply function, we set price equal to marginal cost and solve for x; i.e. we start with

$$MC = \left(\frac{w^\alpha r^\beta}{A\alpha^\alpha \beta^\beta}\right)^{1/(\alpha+\beta)} x^{(1-\alpha-\beta)/(\alpha+\beta)} = p. \quad (12.3.xiv)$$

Dividing through by the term in parentheses and taking both sides to the power $(\alpha+\beta)/(1-\alpha-\beta)$, we get

$$x(w,r,p) = \left[p\left(\frac{A\alpha^\alpha \beta^\beta}{w^\alpha r^\beta}\right)^{1/(\alpha+\beta)}\right]^{(\alpha+\beta)/(1-\alpha-\beta)} = \left(\frac{Ap^{(\alpha+\beta)}\alpha^\alpha \beta^\beta}{w^\alpha r^\beta}\right)^{1/(1-\alpha-\beta)}, \quad (12.3.xv)$$

exactly the same as what we derived in (a) through direct profit maximization. (This entire function is the supply function since MC lies above AC everywhere $x > 0$. You can see this by noticing from the AC and MC functions that $AC = (\alpha+\beta)MC$. Since $(\alpha+\beta) < 1$, this implies $AC < MC$ everywhere.)

(f) *Finally, derive the (unconditional) labor and capital demands. Compare your answers to those in (a).*

Answer: We now simply need to substitute $x(w,r,p)$ from above in for x in the conditional input demand equations (12.3.vii) and (12.3.viii) — and once we do that, we get back the unconditional labor and capital demands that are identical to those in the equations (12.3.i) in part (a).

Exercise 12.5

In the absence of recurring fixed costs (such as those in exercise 12.4), the U-shaped cost curves we will often graph in upcoming chapters presume some particular features of the underlying production technology when we have more than 1 input.

A: *Consider the production technology depicted in Graph 12.6 where output is on the vertical axis (that ranges from 0 to 100) and the inputs capital and labor are on the two horizontal axes. (The origin on the graph is the left-most corner).*

(a) *Suppose that output and input prices result in some optimal production plan A (that is not a corner solution). Describe in words what would be true at A relative to what we described as an isoprofit plane at the beginning of this chapter.*

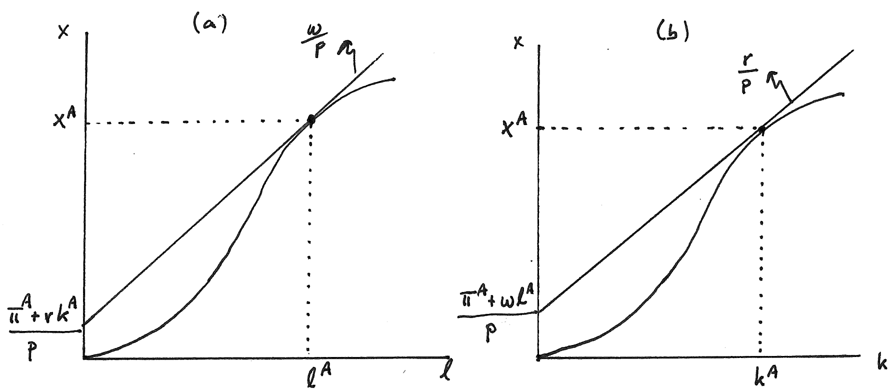
Answer: The isoprofit plane $\pi = px - w\ell - rk$ would have to be tangent to the production frontier — with no other portion of the isoprofit plane intersecting the frontier. It is like a sheet of paper tangent to a "mountain" that is initially getting steeper but eventually becomes shallower. This implies that the isoprofit plane that is tangent at A has a positive vertical intercept.

(b) *Can you tell whether this production frontier has increasing, constant or decreasing returns to scale?*

Answer: The production frontier has initially increasing but eventually decreasing returns to scale — i.e. along every horizontal ray from the origin, the slice of the production frontier has the "sigmoid" shape that we used throughout Chapter 11.

(c) *Illustrate what the slice of this graphical profit maximization problem would look like if you held capital fixed at its optimal level k^A.*

Answer: This is illustrated in panel (a) of Exercise Graph 12.5(1).



Exercise Graph 12.5(1) : Holding k^A and ℓ^A fixed

The tangency of the isoprofit plane shows up as a tangency of the line $x = [(\pi^A + rk^A)/p] + (w/p)\ell$, where the bracketed term is the vertical intercept and the (w/p) term is the slope. (This is just derived from solving the expression $\pi^A = px - w\ell - rk^A$ for x.)

(d) *How would the slice holding labor fixed at its optimal level ℓ^A differ?*

Answer: It would look similar except for re-labeling as in panel (b) of the graph.

(e) *What two conditions that have to hold at the profit maximizing production plan emerge from these pictures?*

Answer: In panels (a) and (b) of Exercise Graph 12.5(1), the slopes of the isoprofit lines are tangent to the slopes of the production frontier with one of the inputs held fixed. The slope of the production frontier at (ℓ^A, x^A) in panel (a) is the marginal product of labor at that production plan; i.e. MP_ℓ^A. And the slope of the production frontier at (k^A, x^A) in panel (b) is the marginal product of capital at that production plan; i.e. MP_k^A. Thus, the conditions that emerge are

$$MP_\ell^A = \frac{w}{p} \text{ and } MP_k^A = \frac{r}{p}. \tag{12.5.i}$$

(f) *Do you think there is another production plan on this frontier at which these conditions hold?*

Answer: Yes — this would occur on the increasing returns to scale portion of the production frontier where an isoprofit "sheet" is tangent to the lower side of the frontier. This "sheet" will, however, have a negative intercept — implying negative profit.

(g) *If output price falls, the profit maximizing production plan changes to once again meet the conditions you derived before. Might the price fall so far that no production plan satisfying these conditions is truly profit maximizing?*

Answer: A decrease in p will cause the isoprofit planes to become steeper — causing the profit maximizing production plan to slide down the production frontier as the tangent isoprofit now happens at a steeper slope. This implies that the vertical intercept also slides down — with profit falling. If the price falls too much, this intercept will become negative — implying that the true profit maximizing production plan becomes (0,0,0). Put differently, if the price falls too much, the firm is better off not producing at all rather than producing at the tangency of an isoprofit with the production frontier.

(h) *Can you tell in which direction the optimal production plan changes as output price increases?*

Answer: As output price increases, the isoprofit plane becomes shallower — which implies that the tangency with the production frontier slides up in the direction of the shallower portion of the frontier. Thus, the production plan will involve more of each input and more output.

B: *Suppose your production technology is characterized by the production function*

$$x = f(\ell, k) = \frac{\alpha}{1 + e^{-(\ell-\beta)} + e^{-(k-\gamma)}} \tag{12.5}$$

where e is the base of the natural logarithm. Given what you might have learned in one of the end-of-chapter exercises in Chapter 11 about the function $x = f(\ell) = \alpha/(1 + e^{-(\ell-\beta)})$, can you see how the shape in Graph 12.16 emerges from this extension of this function?

Answer: Even though this question was not meant to be answered directly, the graph given in part A of the question depicts this function for the case where $\alpha = 100$ and $\beta = \gamma = 5$. The graph was generated using the software package Mathematica (as are the other machine generated graphs in some of the answers in this Chapter). As you can see, the function takes on the shape that has initially increasing and eventually diminishing slope along slices holding each input fixed (as well as along rays from the origin.) Note that ℓ and k enter symmetrically given that $\beta = \gamma$ — and the two inputs appear on the axes in the plane from which the surface emanates. The vertical axis in the graph is output x.

(a) *Set up the profit maximization problem.*

Answer: The problem is

$$\max_{x,\ell,k} px - w\ell - rk \text{ subject to } x = \frac{\alpha}{1+e^{-(\ell-\beta)}+e^{-(k-\gamma)}} \tag{12.5.ii}$$

which can also be written as the unconstrained maximization problem

$$\max_{\ell,k} \frac{\alpha p}{1+e^{-(\ell-\beta)}+e^{-(k-\gamma)}} - w\ell - rk. \tag{12.5.iii}$$

(b) *Derive the first order conditions for this optimization problem.*

Answer: We simply take derivatives with respect to w and r and set them to zero. Thus, we get

$$\frac{\alpha pe^{-(\ell-\beta)}}{\left(1+e^{-(\ell-\beta)}+e^{-(k-\gamma)}\right)^2} = w \text{ and } \frac{\alpha pe^{-(k-\gamma)}}{\left(1+e^{-(\ell-\beta)}+e^{-(k-\gamma)}\right)^2} = r. \tag{12.5.iv}$$

(c) *Substitute $y = e^{-(\ell-\beta)}$ and $z = e^{-(k-\gamma)}$ into the first order conditions. Then, with the first order conditions written with w and r on the right hand sides, divide them by each other and derive from this an expression $y(z,w,r)$ and the inverse expression $z(y,w,r)$.*

Answer: These substitutions lead to the first order conditions becoming

$$\frac{\alpha py}{(1+y+z)^2} = w \text{ and } \frac{\alpha pz}{(1+y+z)^2} = r. \tag{12.5.v}$$

Dividing the two equations by each other, we can then derive

$$y(z,w,r) = \frac{wz}{r} \text{ and } z(y,w,r) = \frac{ry}{w}. \tag{12.5.vi}$$

(d) *Substitute $y(z,w,r)$ into the first order condition that contains r. Then manipulate the resulting equation until you have it in the form az^2+bz+c (where the terms a, b and c may be functions of w, r, α and p). (*Hint*: It is helpful to multiply both sides of the equation by r.) The quadratic formula then allows you to derive two "solutions" for z. Choose the one that uses* *the negative rather than the positive sign in the quadratic formula as your "true" solution $z^*(\alpha, p, w, r)$.*

Answer: Substituting $y(z, w, r)$ into the second expression in equation (12.5.v) and multiplying both sides by the denominator, we get

$$\alpha pz = r\left(1+\frac{wz}{r}+z\right)^2. \qquad (12.5.vii)$$

Multiplying the right hand side by r lets us reduce it to

$$r^2\left(1+\frac{wz}{r}+z\right)^2 = (r+wz+rz)^2 = (r+(w+r)z)^2. \qquad (12.5.viii)$$

Thus, when we multiply both sides of equation (12.5.vii) by r, we get

$$\alpha rpz = (r+(w+r)z)^2. \qquad (12.5.ix)$$

Expanding the left hand side and grouping terms, we then get

$$(w+r)^2z^2 + [2r(w+r)-\alpha rp]z + r^2 = 0. \qquad (12.5.x)$$

This is now in the form we need to apply the quadratic formula to solve for z. The problem tells us to use the version of the formula that has a negative rather than positive sign in front of the square root — thus

$$z^*(\alpha, p, w, r) = \frac{-[2r(w+r)-\alpha rp] - \sqrt{[2r(w+r)-\alpha rp]^2 - 4(w+r)^2r^2}}{2(w+r)^2}. \qquad (12.5.xi)$$

(e) *Substitute $z(y, w, r)$ into the first order condition that contains w and solve for $y^*(\alpha, p, w, r)$ in the same way you solved for $z^*(\alpha, p, w, r)$ in the previous part.*

Answer: Substituting $z(y, w, r)$ into the first expression in equation (12.5.v) and multiplying both sides by the denominator, we get

$$\alpha py = w\left(1+y+\frac{ry}{w}\right)^2. \qquad (12.5.xii)$$

Multiplying the right hand side by w lets us reduce it to

$$w^2\left(1+y+\frac{ry}{w}\right)^2 = (w+wy+ry)^2 = (w+(w+r)y)^2. \qquad (12.5.xiii)$$

Thus, when we multiply both sides of equation (12.5.xii) by w, we get

$$\alpha wpy = (w+(w+r)y)^2. \qquad (12.5.xiv)$$

Expanding the left hand side and grouping terms, we then get

$$(w+r)^2y^2 + [2w(w+r)-\alpha wp]z + w^2 = 0. \qquad (12.5.xv)$$

This is now in the form we need to apply the quadratic formula to solve for y. The problem tells us to use the version of the formula that has a negative rather than positive sign in front of the square root — thus

$$y^*(\alpha, p, w, r) = \frac{-[2w(w+r) - \alpha wp] - \sqrt{[2w(w+r) - \alpha wp]^2 - 4(w+r)^2 w^2}}{2(w+r)^2}. \tag{12.5.xvi}$$

(f) *Given the substitutions you did in part (c), you can now write $e^{-(\ell-\beta)} = y^*(\alpha, p, w, r)$ and $e^{-(k-\gamma)} = z^*(\alpha, p, w, r)$. Take natural logs of both sides to solve for labor demand $\ell(w, r, p)$ and capital demand $k(w, r, p)$ (which will be functions of the parameters α, β and γ.)*

Answer: Taking natural logs of $e^{-(\ell-\beta)} = y^*(\alpha, p, w, r)$ and $e^{-(k-\gamma)} = z^*(\alpha, p, w, r)$ gives us

$$-(\ell - \beta) = \ln y^*(\alpha, p, w, r) \text{ and } -(k - \gamma) = \ln z^*(\alpha, p, w, r) \tag{12.5.xvii}$$

which can be solved for ℓ and k to get the input demand functions:

$$\ell(w, r, p) = \beta - \ln y^*(\alpha, p, w, r) \text{ and } k(w, r, p) = \gamma - \ln z^*(\alpha, p, w, r). \tag{12.5.xviii}$$

(g) *How much labor and capital will this firm demand if $\alpha = 100$, $\beta = \gamma = 5 = p$, $w = 20 = r$? (It might be easiest to type the solutions you have derived into an Excel spreadsheet in which you can set the parameters of the problem.) How much output will the firm produce? How does your answer change if r falls to $r = 10$? How much profit does the firm make in the two cases.*

Answer: The firm would initially hire approximately 8.035 units of labor and capital to produce 91.23 units of output. When $r = 10$, the optimal production plan would change to $(\ell, k, y) = (8.086, 8.780, 93.59)$ — i.e. the firm would increase production primarily by hiring more capital but also by hiring slightly more labor. Profit is 134.74 in the first case and 218.42 in the second.

(h) *Suppose you had used the other "solutions" in parts (d) and (e) — the ones that emerge from using the quadratic formula in which the square root term is added rather than subtracted. How would your answers to (g) be different — and why did we choose to ignore this "solution"?*

Answer: The solution for the initial values given in part (g) would then have been $(\ell, k, y) \approx (3.35, 3.35, 8.77)$ and this would change to $(\ell, k, y) \approx (2.72, 3.42, 6.41)$ when r falls to 10. This would be an odd outcome — with a drop in the input price r, the problem suggests that output will fall. It is wrong because profit in both cases is negative — meaning these are not profit maximizing production plans. (Profit in the first case is -90.19 and in the second -56.61.)

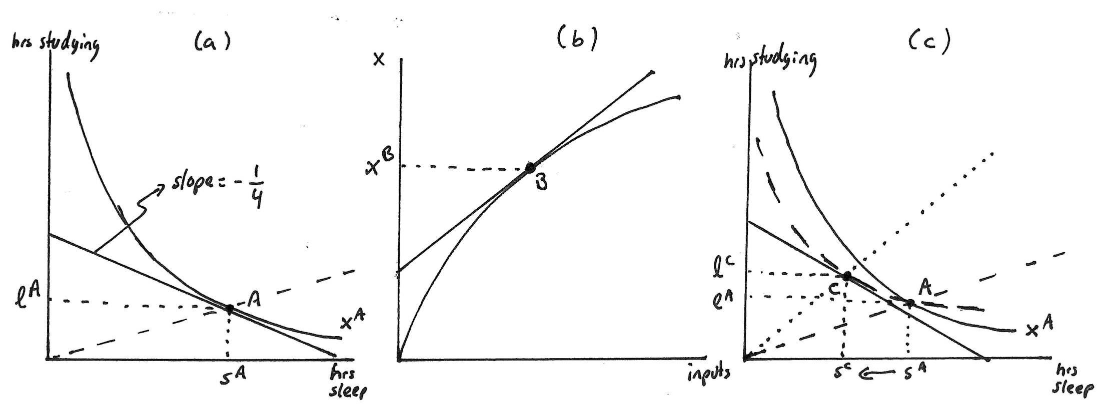
Exercise 12.7

Everyday Application: *To Study or to Sleep?: Research suggests that successful performance on exams requires preparation (i.e. studying) and rest (i.e. sleep). Neither by itself produces good exam grades — but in the right combination they maximize exam performance.*

A: *We can then model exam grades as emerging from a production process that takes hours of studying and hours of sleep as inputs. Suppose this production process is homothetic and has decreasing returns to scale.*

(a) *On a graph with hours of sleep on the horizontal axis and hours of studying on the vertical, illustrate an isoquant that represents a particular exam performance level x^A.*

Answer: This is illustrated in panel (a) of Exercise Graph 12.7 where x^A represents the isoquant with all the input bundles that can produce exam grade x^A.



Exercise Graph 12.7 : Studying and Sleeping

(b) *Suppose you are always willing to pay \$5 to get back an hour of sleep and \$20 to get back an hour of studying. Illustrate on your graph the least cost way to get to the exam grade x^A.*

Answer: This is illustrated in panel (a) of the graph with the addition of the isocost line that is tangent at A — which implies the least cost way to get exam grade x^A is to sleep s^A hours and study ℓ^A hours.

(c) *Since the production process is homothetic, where in your graph are the cost minimizing ways to get to the other exam grade isoquants?*

Answer: The cost minimizing input bundles will all lie on a ray from the origin through A — because the slopes of the isoquants are the same along any such ray, and at A the slope of the isocost is equal to the slope of the isoquant.

(d) *Using your answer to (c), can you graph a vertical slice of the production frontier that contains all the cost minimizing sleep/study input bundles?*

Answer: This is illustrated in panel (b) of Exercise Graph 12.7 where the vertical slice along the ray from the origin in panel (a) is graphed. It has a concave shape because the production process is assumed to have decreasing returns to scale.

(e) *Suppose you are willing to pay \$p for every additional point on your exam. Can you illustrate on your graph from (d) the slice of the "isoprofit" that gives you your optimal exam grade? Is this necessarily the same as the exam grade x^A from your previous graph?*

Answer: This is simply a slice of an isoprofit plane described by $\pi = px - 20\ell - 5s$, where π stands for the highest possible "profit", x is the exam grade, ℓ is the hours spent studying and s is the hours spent sleeping. It is tangent at B — with x^B being the optimal exam grade. This is not necessarily the same as x^A. We had chosen x^A arbitrarily and used it to show on what ray all cost minimizing input bundles lie. x^B lies on that ray — but does not necessarily overlap with x^A.

(f) *What would change if you placed a higher value on each exam point?*

Answer: If you place a higher value on exam grades, nothing in panel (a) will change since none of the items in that graph were derived from the knowledge of p. Neither will the production frontier slice in (b) change since the technology for producing exam grades has not changed — just the value you place on them. The only thing that changes is that the slice of the isoprofit that is tangent to the production frontier in panel (b) becomes shallower — implying that the optimal exam grade is higher.

(g) *Suppose a new caffeine/Gingseng drink comes on the market — and you find it makes you twice as productive when you study. What in your graphs will change?*

Answer: This drink has changed the production technology — so any object in your graphs that comes from the production technology will change. In particular, panel (c) illustrates the original x^A isoquant with the original cost minimizing input bundle A. If the drink makes studying twice as productive, the slope of the new isoquant must be shallower at A than it was before — resulting in the new dashed isoquant. The new cost minimizing input bundle for exam grade x^A is then C — with less sleep and more studying. Since the production technology is homothetic, this implies that all the new cost minimizing ways of getting to different exam grades will lie on the (dotted) ray from the origin through C. The vertical slice of the new production technology will then also differ.

B: *Suppose that the production technology described in part A can be captured by the production function $x = 40\ell^{0.25}s^{0.25}$ — where x is your exam grade, ℓ is the number of hours spent studying and s is the number of hours spent sleeping.*

(a) *Assume again that you'd be willing to pay $5 to get back an hour of sleep and $20 to get back an hour of studying. If you value each exam point at p, what is your optimal "production plan"?*

Answer: We need to solve something quite analogous to a profit maximization problem

$$\max_{\ell,s,x} px - 20\ell - 5s \text{ subject to } x = 40\ell^{0.25}s^{0.25} \tag{12.7.i}$$

which can also be written as the unconstrained optimization problem

$$\max_{\ell,x} 40p\ell^{0.25}s^{0.25} - 20\ell - 5s. \tag{12.7.ii}$$

The two first order conditions are

$$10p\ell^{-0.75}s^{0.25} = 20 \text{ and } 10p\ell^{0.25}s^{-0.75} = 5. \tag{12.7.iii}$$

Solving these, we get the "input demand" equations

$$\ell(p) = 0.50p^2 \text{ and } s(p) = 2p^2. \tag{12.7.iv}$$

And plugging these into the production function, we get the exam grade "supply" function

$$x(p) = 40(0.50p^2)^{0.25}(2p^2)^{0.25} = 40p. \tag{12.7.v}$$

The optimal "production plan" therefore entails getting a grade of $40p$ by studying for $0.5p^2$ hours and sleeping $2p^2$ hours.

(b) *Can you arrive at the same answer using the Cobb-Douglas cost function (given in problem 12.4?*

Answer: Using this cost function and substituting $A = 40$, $\alpha = \beta = 0.25$, $w = 20$ and $r = 5$, we get

$$C(x) = 0.5\left(\frac{20^{0.25}(5^{0.25})x}{40(0.25^{0.25})(0.25^{0.25})}\right)^2 = 0.0125x^2. \tag{12.7.vi}$$

The marginal cost is then

$$MC(x) = \frac{\partial C(x)}{\partial x} = 0.025x. \tag{12.7.vii}$$

Setting this equal to p and solving for x, we get $x(p) = 40p$, exactly as we did before.

(c) *What is your optimal production plan when you value each exam point at $2?*

Answer: You would study for 2 hours, sleep for 8 hours and earn an 80 on the exam.

(d) *How much would you have to value each exam point in order for you to put in the effort and sleep to get a 100 on the exam.*

Answer: You would have to value each exam point at $2.50. You would then study for 3.125 hours, sleep for 12.5 hours and earn a 100.

(e) *What happens to your optimal production plan as the value you place on each exam point increases?*

Answer: It is easy to see from the equations (12.7.iv) and (12.7.v) that p always enters positively. As the value you place on your exam increases, you will therefore study and sleep more — and earn a higher grade.

(f) *What changes if the caffeine/Gingseng drink described in A(g) is factored into the problem?*

Answer: The underlying technology changes — which means the production function would have to change in a way that reflects this. For every 1 hour of studying, you would now get the benefit that you previously received from 2 hours of studying. Thus, the new production function would be

$$x = 40(2\ell)^{0.25}s^{0.25} \approx 47.57\ell^{0.25}s^{0.25}. \tag{12.7.viii}$$

For the previous values of sleep and study time, you can check that you would have to value an exam point by only about $1.77 in order to make a 100 on the exam — and you would put in 2.22 hours of study time with 8.86 hours of sleep.

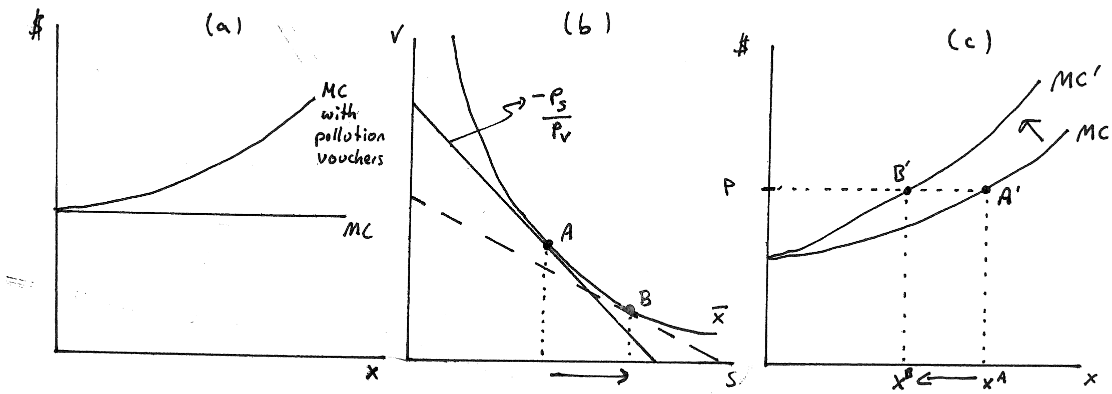
Exercise 12.9

Business and Policy Application: *Investing in Smokestack Filters under Cap-and-Trade: On their own, firms have little incentive to invest in pollution abating technologies such as smokestack filters. As a result, governments have increasingly turned to "cap-and-trade" programs. Under these programs, discussed in more detail in Chapter 21, the government puts an overall "cap" on the amount of permissible pollution and firms are permitted to pollute only to the extent to which they own sufficient numbers of pollution permits or "vouchers". If a firm does not need all of its vouchers, it can sell them at a market price p_v to firms that require more.*

A: *Suppose a firm produces x using a technology that emits pollution through smokestacks. The firm must ensure that it has sufficient pollution vouchers v to emit the level of pollution that escapes the smokestacks, but it can reduce the pollution by installing increasingly sophisticated smokestack filters s.*

(a) *Suppose that the technology for producing x requires capital and labor and, without considering pollution, has constant returns to scale. For a given set of input prices (w, r), what does the marginal cost curve look like?*

Answer: The MC curve is flat when the production technology has constant returns to scale. This is depicted in panel (a) of Exercise Graph 12.9.



Exercise Graph 12.9 : Cap-and-Trade and Smokestack Filters

(b) *Now suppose that relatively little pollution is emitted initially in the production process, but as the factory is used more intensively, pollution per unit of output increases — and thus more pollution vouchers have to be purchased per unit absent any pollution abating smokestack filters. What does this do to the marginal cost curve assuming some price p_v per pollution voucher and assuming the firm does not install smokestack filters?*

Answer: It causes the *MC* curve to be upward sloping as depicted in panel (a) of Exercise Graph 12.9.

(c) *Considering carefully the meaning of "economic cost", does your answer to (b) depend on whether the government gives the firm a certain amount of vouchers or whether the firm starts out with no vouchers and has to purchase whatever quantity is necessary for its production plan?*

Answer: It does not depend on whether the vouchers are owned by the firm or the firm has to purchase them. In both cases, the opportunity cost of using a pollution voucher to emit pollution in production is p_v. If the firm owns the voucher, it foregoes the opportunity to sell it at p_v to another firm that wishes to buy more vouchers. If the firm does not own vouchers, it must directly pay p_v per voucher.

(d) *Suppose that smokestack filters are such that initial investments in filters yield high reductions in pollution, but as additional filters are added, the marginal reduction in pollution declines. You can now think of the firm as using two additional inputs — pollution vouchers and smokestack filters — to produce output x legally. Does the overall production technology now have increasing, constant or decreasing returns to scale?*

Answer: The overall technology now has decreasing returns to scale. This is because, whether the firm uses pollution vouchers or smokestack filters or some combination of the two, it has to expend increasing resources to deal with its pollution output for any marginal increase in production.

(e) *Next, consider a graph with "smokestack filters" s on the horizontal and "pollution vouchers" v on the vertical axis. Illustrate an isoquant that shows different ways of reaching a particular output level $\overline{x}$ legally — i.e. without polluting illegally. Then illustrate the least cost way of reaching this output level (not counting the cost of labor and capital) given p_v and p_s.*

Answer: This is illustrated in panel (b) of Exercise Graph 12.9 where A is the cost minimizing bundle of smokestack filters and pollution vouchers to produce $\overline{x}$ when the prices of filters and vouchers are p_s and p_v.

(f) *If the government imposes additional limits on pollution by removing some of the pollution vouchers from the market, p_v will increase. How much will this affect the number of smokestack filters used in any given firm assuming output does not change? What does your answer depend on?*

Answer: The increase in p_v will cause isocosts to become shallower. If output does not change from $\overline{x}$, this will lead to a change in the cost minimizing bundle to B — causing the firm to use fewer vouchers and more smokestack filters. The size of the adjustment depends on the degree of substitutability between vouchers and smokestack filters in production. In other words, if it is relatively easy for the firm to install additional smokestack filters, the effect will be bigger than if it is not.

(g) *What happens to the overall marginal cost curve for the firm (including all costs of production) as p_v increases? Will output increase or decrease?*

Answer: This is illustrated in panel (c) of Exercise Graph 12.9. The marginal cost of production increases as p_v increases, rotating the MC curve from MC to MC'. For a given output price p, this implies that the profit maximizing output falls from x^A to x^B.

(h) *Can you tell whether the firm will buy more or fewer smokestack filters as p_v increases? Do you think it will produce more or less pollution?*

Answer: It is not clear whether the firm will buy more or fewer smokestack filters — because it is not clear by how much the firm will reduce its output. We know from panel (c) that the firm will produce less, and we know from panel (b) that, for the same level of output, it will buy more filters. But if the firm decreases production sufficiently much, it may end up buying fewer filters. No matter what, however, it will produce less pollution — because it produces less output with more filters for that level of output than it would have used before.

(i) True or False*: The Cap-and-Trade system reduces overall pollution by getting firms to use smokestack filters more intensively and by causing firms to reduce how much output they produce.*

Answer: This is true. As we have shown, the firm uses more smokestack filters for any given output level (panel (b) of the graph) but also produces less output (panel (c)).

B: *Suppose the cost function (not considering pollution) is given by $C(w,r,x) = 0.5w^{0.5}r^{0.5}x$, and suppose that the tradeoff between using smokestack filters* *s and pollution vouchers v to achieve legal production is given by the Cobb-Douglas production technology* $x = f(s, v) = 50s^{0.25}v^{0.25}$.

(a) *In the absence of cap-and-trade policies, does the production process have increasing, decreasing or constant returns to scale?*

Answer: The marginal cost function derived from $C(w, r, x)$ is

$$MC(w,r,x) = \frac{\partial C(w,r,x)}{\partial x} = 0.5w^{0.5}r^{0.5}. \qquad (12.9.i)$$

This function is independent of x — i.e. the marginal cost is constant, which implies constant returns to scale.

(b) *Ignoring for now the cost of capital and labor, derive the cost function for producing different output levels as a function of* p_s *and* p_v *— the price of a smokestack filter and a pollution voucher.(You can derive this directly or use the fact that we know the general form of cost functions for Cobb-Douglas production functions from what is given in problem 12.4).*

Answer: Plugging in $A = 50$ and $\alpha = \beta = 0.25$ into the cost function given in problem 12.4, we get

$$C(p_s, p_v, x) = 0.5\left(\frac{xp_s^{0.25}p_v^{0.25}}{50(0.25^{0.25})(0.25^{0.25})}\right)^2 = 0.0008p_s^{0.5}p_v^{0.5}x^2. \qquad (12.9.ii)$$

(c) *What is the full cost function* $C(w, r, p_s, p_v)$*? What is the marginal cost function?*

Answer: The cost of producing output level x is then simply the cost of labor and capital plus the cost of complying with the requirement that pollution is produced legally; i.e.

$$C(w,r,p_s,p_v) = 0.5w^{0.5}r^{0.5}x + 0.0008p_s^{0.5}p_v^{0.5}x^2. \qquad (12.9.iii)$$

The marginal cost function is then

$$MC(w,r,p_s,p_v) = 0.5w^{0.5}r^{0.5} + 0.0016p_s^{0.5}p_v^{0.5}x. \qquad (12.9.iv)$$

(d) *For a given output price p, derive the supply function.*

Answer: We set p equal to MC and solve for x to get

$$x(w,r,p_s,p_v,p) = \frac{p - 0.5w^{0.5}r^{0.5}}{0.0016p_s^{0.5}p_v^{0.5}}. \qquad (12.9.v)$$

(e) *Using Shephard's lemma, can you derive the conditional smokestack filter demand function?*

Answer: Shephard's lemma tells us that the partial derivative of the cost function with respect to an input price is equal to the conditional input demand function for that input; i.e.

$$s(w,r,p_s,p_v,x) = \frac{\partial C(w,r,p_s,p_v)}{\partial p_s} = 0.0004\frac{p_v^{0.5}x^2}{p_s^{0.5}}. \quad (12.9.vi)$$

(f) *Using your answers, can you derive the (unconditional) smokestack filter demand function?*

Answer: If we plug the supply function $x(w,r,p_s,p_v,p)$ into the conditional smokestack filter demand function $s(w,r,p_s,p_v,x)$, we will get the unconditional smokestack filter demand function. We then get

$$s(w,r,p_v,p_s,p) = \frac{625(p-0.5w^{0.5}r^{0.5})^2}{4p_v^{0.5}p_s^{1.5}}. \quad (12.9.vii)$$

(g) *Use your answers to illustrate the effect of an increase in p_v on the demand for smokestack filters holding output fixed as well as the effect of an increase in p_v on the profit maximizing demand for smokestack filters.*

Answer: The derivative of the conditional demand function $s(w,r,p_s,p_v,x)$ with respect to p_v is positive — indicating that we will buy more smokestack filters conditional on producing the same quantity of output as before. The derivative of the unconditional filter demand $s(w,r,p_v,p_s,p)$ with respect to p_v, however, is negative — indicating that we will buy fewer pollution filters when we arrive at our new profit maximizing production plan. This is not because we pollute more — but rather because our supply function $x(w,r,p_s,p_v,p)$ tells us that we will produce sufficiently less such that we will need fewer overall filters even through we use more filters for the quantity that we do produce than we would have before.

Conclusion: Potentially Helpful Reminders

1. Profit maximization implies that marginal product equals input price for *ALL* inputs. Short run profit maximization therefore implies just that $MP_\ell = w$, while long run profit maximization implies that both $MP_\ell = w$ and $MP_k = r$.

2. Cost minimization implies that $TRS = -w/r$ which, since $-TRS = MP_\ell/MP_k$, is equivalent to saying $MP_\ell/MP_k = w/r$. You should be able to show that the profit maximization conditions ($MP_\ell = w$ and $MP_k = r$) imply that the cost minimization condition holds, but the reverse does not hold.

3. Profit maximization can be seen graphically as a tangency of the vertical production frontier slices that hold one input fixed with the slice of the isoprofit plane. You should understand how that tangency is equivalent to saying $MP_\ell = w$ and $MP_k = r$.

4. Cost minimization can be seen graphically as tangencies of isocosts and isoquants. You should understand how the condition $-TRS = MP_\ell/MP_k = w/r$ must logically hold at all production plans that minimize cost. You should

also understand that, when the production frontier is homothetic, *ALL* such tangencies will happen along a single ray from the origin for a given w and r. And you should understand why typically only one of those tangencies represents a *profit maximizing* production plan.

5. End-of-Chapter problem 12.1 is a good problem to practice with concepts contained in the above points — and a good problem to use for preparation for Chapter 13.

6. At the end of Chapter 11, we showed that the supply curve is the part of the marginal cost curve that lies above average cost. The same is true in this chapter when there are 2 inputs — and the same will always be true, in the short and long run, so long as we define costs correctly.

7. One of the points emphasized in end-of-chapter exercises (but only partially emphasized in the text chapter) is that U-shaped average cost curves can arise in one of two ways: (1) because of production technologies that initially exhibit increasing returns to scale but eventually turn to decreasing returns to scale; and (2) because of the existence of a recurring *fixed cost*. This idea is further developed in end-of-chapter exercise 12.4 and then in Chapter 13.