CHAPTER

6

Doing the “Best” We Can

While Chapter 4 introduced us to a general way of thinking about tastes, Chapter 5 gets much more specific and introduces particular dimensions along which we might differentiate between tastes. In particular, we differentiate tastes based on

1. The curvature of individual indifference curves — or how quickly the MRS changes *along an indifference curve*;

2. The relationships between indifference curves — or how the MRS changes *across indifference curves within an indifference map*; and

3. Whether or not indifference curves *cross horizontal or vertical axes* or whether they *converge to the axes*.

The first of these in turn determines the degree to which consumers are willing to substitute between goods (and will lead to what we call the "substitution effect" in Chapter 7) while the second of these determines how consumer behavior responds to changes in income (and will lead to what we call the "income effect" in Chapter 7). Finally, the third category of taste differences becomes important in Chapter 6 where we will see how corner versus interior optimal solutions for a consumer emerge.

Chapter Highlights

The main points of the chapter are:

1. The **degree of substitutability** or, in part B language, the **elasticity of substitution** for a consumer at a particular consumption bundle arises from the **curvature** of the indifference curve at that bundle. There may be no substitutability (as in perfect complements) or perfect substitutability (perfect substitutes) or an infinite number of cases in between these extremes.

2. **Quasilinearity** and **Homotheticity** of tastes represent special cases that describe how indifference curves from the same map relate to one another. These properties have no direct relationship to the concept of substitutability. Tastes are quasilinear in a good x if the MRS only depends on the level of x consumption (and not the level of other goods' consumption. Tastes are homothetic when the MRS depends only on the relative levels of the goods in a bundle.

3. Sometimes it is reasonable to assume that indifference curves only converge to the axes without ever crossing them; other times we assume that they cross the axes. When an indifference curve crosses an axis, it means that we can gain utility beyond what we have by not consuming even if we consume none of one of the goods. When indifference curves only converge to the axes, then some consumption of all goods is necessary in order for a consumer to experience utility above what she would experience by not consuming at all.

4. If you are reading part B of the chapter, you should begin to understand the family of **constant elasticity of substitution utility functions** — with perfect complements, perfect substitutes and Cobb-Douglas tastes as special cases. You should also be able to demonstrate whether a utility function is homothetic or quasilinear. (Most utility functions we use in this text tend to be one or the other.)

6A Solutions to Within-Chapter-Exercises for Part A

Exercise 6A.1

In Chapter 2 we discussed a scenario under which my wife gives me a coupon that reduces the effective price of pants to \$10 a pair. Assuming the same tastes, what would be my best bundle?

Answer: In that case, the slope of the budget constraint is $-p_1/p_2 = -1$ — so the optimal bundle would have to have $MRS = -1$ as well. In describing tastes here, we said that the MRS is equal to -1 at bundles where I have an equal number of shirts and pants — that is, along the 45 degree line. Thus, the optimal bundle would occur at the midpoint of the budget line that has intercepts of 20 on each axis — which is at the bundle (10,10) — 10 pants and 10 shirts.

Exercise 6A.2

Suppose both you and I have a bundle of 6 pants and 6 shirts, and suppose that my MRS of shirts for pants is -1 and yours is -2. Suppose further that neither one of us has access to Wal-Mart. Propose a trade that would make both of us better off.

Answer: In this case, you are willing to trade 2 shirts for 1 pair of pants whereas I am willing to trade them one for one. Assuming we can trade fractions of shirts and pants, a trade in which you give me 1.5 shirts for 1 pair of pants would make you better off (because you would have been willing to give up as many as 2 shirts for 1 pair of pants) and would also make me better off (because I would have been willing to accept as little as 1 shirt for 1 pair of pants). If we don't want to assume we can trade in fractions of goods, then the trade of 3 shirts for 2 pants would work similarly.

Exercise 6A.3

We keep using the phrase "at the margin" — as, for example, when we say that tastes for those leaving Wal-Mart will be the "same at the margin." What do economists mean by this "at the margin" phrase?

Answer: "At the margin" means approximately around the bundle that we are discussing. To say that tastes are the same "at the margin" is the same as saying that around the bundles that individuals currently have (as they leave Wal-Mart), their tastes are the same — but that's not necessarily the same as saying that tastes are the same everywhere. "At the margin" restricts our attention to just a small subset of the larger space in which tastes reside.

Exercise 6A.4

In the previous section, we argued that Wal-Mart's policy of charging the same price to all consumers insures that there are no further gains from trade for goods contained in the shopping baskets of individuals that leave Wal-Mart. The argument assumed that all consumers end up at an interior solution, not a corner solution. Can you see why the conclusion still stands when some people optimize at corner solutions where their *MRS* may be quite different from the *MRS*'s of those who optimize at interior solutions?

Answer: When everyone optimizes at an interior solution, everyone's *MRS* must be the same as everyone else's when they leave Wal-Mart — i.e. our tastes are the same at the margin, thus allowing for no further gains from trade. Now imagine that we consider shirts and pants — and someone leaves Wal-Mart with only shirts and no pants. That person, call her person A, is therefore at a corner solution — and for that corner solution to be optimal, it is almost certainly the case that this person's indifference curve is steeper than the budget constraint at the corner optimum. Thus, this person's tastes are not the same at the margin as those of the other consumers who optimized at a point where the slope of their budget constraint was equal to the slope of their indifference curve. Suppose, then, that person A's *MRS* is -4 and person B's *MRS* is -2 — with person B at an interior solution and person A at a corner solution where she buys only pants. Just looking at the *MRS*'s of the two people, we could say that a trade in which person A gives up 3 shirts in exchange for one pair of pants from person B would make both better off. After all, person B is willing to accept as few as 2 shirts for a pair of pants but would now get 3 instead, and person A is willing to give up as many as 4 shirts for a pair of pants but, under this trade, would only have to give up 3. The problem, however, is that person A has only pants — and therefore has not shirts to give up in a trade. Since person A's *MRS* is higher in absolute value than person B's (and since this has to be the case in order for person A to be at a corner solution with only pants when person B is at an interior solution), the only potential trades that benefit both are those that have shirts going from A to B — but none of those trades is possible because A is at a corner solution and therefore without shirts to give up. Thus, when A and B leave Wal-Mart, there are no further gains from trade even if one (or both) of them is at a corner solution and their tastes are not the same at the margin. Either people who leave Wal-Mart are at an interior solution — in which case they have the same tastes on the margin as everyone else who is at an interior solution and thus can't trade with each other anymore; OR people are at a corner solution and don't have the same tastes as others on the margin but can't trade with them because they already have traded away every unit of the thing they value less at the margin than others who are at an interior solution. Either way, all gains from trade are exhausted in Wal-Mart — and the distribution of goods for people leaving Wal-Mart is efficient.

Exercise 6A.5

Suppose the prices of Coke and Pepsi were the same. Illustrate that now there are many optimal bundles for someone with my kind of tastes. What would be my “best" bundle if Pepsi is cheaper than Coke?

Answer: When the prices of Coke and Pepsi are the same, then the budget constraint has the same slope as all the indifference curves. Therefore, one indifference curve lies right on top of the budget constraint and is therefore “tangent” at every point on the budget constraint. In that case, all bundles on the budget constraint are optimal bundles for the consumer. This makes intuitive sense — if Coke and Pepsi are priced the same and if I can't tell the difference between the two, it doesn't matter how I allocate my spending across Coke and Pepsi.

Exercise 6A.6

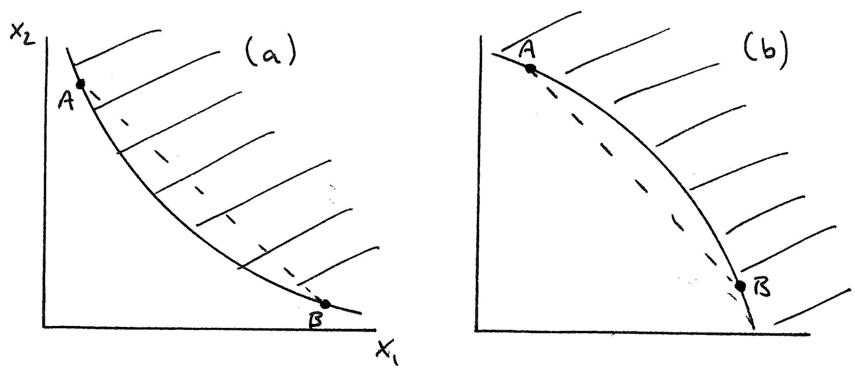
Consider a set of points that compose a solid sphere. Is this set convex? What about the set of points contained in a donut?

Answer: Any line connecting two points in a solid sphere must necessarily be entirely contained within the sphere. Thus, a solid sphere is a convex set. If I pick two points on opposite sides of a donut, on the other hand, the line connecting them will lie (at least partially) outside the donut as it passes through the hole in the middle of the donut. Thus, a donut is not a convex set.

Exercise 6A.7

We have just defined what it means for a set of points to be convex — it must be the case that any line connecting two points in the set is fully contained in the set as well. In Chapter 4, we defined tastes to be convex when "averages are better than (or at least as good as) extremes". The reason such tastes are called "convex" is because the set of bundles that is better than any given bundle is a convex set. Illustrate that this is the case with an indifference curve from an indifference map of convex tastes.

Answer: Panels (a) and (b) of Exercise Graph 6A.7 illustrate two indifference curves, one from a map in which indifference curves satisfy the convexity property, and one from a map of indifference curves that does not satisfy convexity. In both, the set of "better' bundles is shaded. Two bundles, *A* and *B*, on each indifference curve are chosen and the line connecting them is indicated. That line lies fully in the shaded "better than" set in panel (a) but fully outside the shaded "better than" set in panel (b). Thus, convexity of tastes implies convex "better than" sets for each indifference curves, while non-convexities in tastes imply non-convex "better than" sets for some indifference curves.



Exercise Graph 6A.7 : Convexity and Tastes

Exercise 6A.8

True/False: If a choice set is non-convex, there are definitely multiple "best" bundles for a consumer whose tastes satisfy the usual assumptions.

Answer: False. Non-convexities in choice sets imply that there *might* be multiple best bundles, not that there necessarily are for any given tastes of a consumer. In other words, it is easy to construct an indifference curve that only has one tangency on a non-convex budget constraint, but it is also possible to construct an indifference curve (that satisfies the convexity of tastes property) which has more than one tangency on a non-convex budget constraint.

Exercise 6A.9

True/False: If a choice set is convex, then there will be a unique "best" bundle assuming consumer tastes satisfy our usual assumptions and averages are strictly better than extremes.

Answer: This is true. A convex choice set either bends out from the origin or is a straight line with negative slope and positive intercepts. A strictly convex indifference curve, on the other hand, bends toward the origin. Thus, as we move out to higher indifference curves, there will come a point where the budge constraint (that forms the boundary of a convex choice set) contains a single point in common with the indifference curve (that forms a convex "better than" set.)

Exercise 6A.10

Suppose that the choice set is defined by linear budget constraint and tastes satisfy the usual assumptions but contain indifference curves with linear components (or "flat spots"). True/False: Then there might be multiple "best" bundles but we can be sure that the set of "best" bundles is a convex set.

Answer: True. When indifference curves have "flat spots", there is the potential that the line segment of the indifference curve (i.e. the "flat spot") has the same slope as the budget constraint and therefore each bundle on that segment is optimal (much like all bundles are optimal in the case of perfect substitutes when prices were the same for Coke and Pepsi in exercise 6A.5). The set of optimal bundles is then a line segment. Take any two points on the line segment, and it has to be the case that all points that lie on the line (between the points) connecting them also lies in the set of optimal bundles. Thus, the set of optimal bundles is itself a convex set. Of course it might also be the case that, with such indifference curves, the optimal bundle does not occur on the flat spot — and is therefore just a single point. But a set composed of a single point is trivially also a convex set.

Exercise 6A.11

True/False: When there are multiple "best" bundles due to non-convexities in tastes, the set of "best" bundles is also non-convex (assuming convex choice sets).

Answer: True. When there are non-convexities in tastes, that means that the indifference curves at some point bend away from the origin. If multiple optimal bundles arise from that, it means that these bundles will not be connected as in the previous exercise — which means that the line connecting them will contain bundles that are not optimal. Thus, the set of optimal bundles is then non-convex.

6B Solutions to Within-Chapter-Exercises for Part B

Exercise 6B.1

Solve for the optimal quantities of x_1, x_2 and x_3 in the problem defined in equation 6.11. (Hint: The problem will be considerably easier to solve if you take the logarithm the utility function (which you can do since logarithms are order preserving transformations that do not alter the shapes of indifference curves.))

Answer: Taking the hint in the problem, we can write the utility function as $v(x_1, x_2, x_3) = 0.5\ln x_1 + 0.5\ln x_2 + 0.5\ln x_3$ and the corresponding Lagrange function as

$$\mathcal{L}(x_1, x_2, x_3, \lambda) = 0.5\ln x_1 + 0.5\ln x_2 + 0.5\ln x_3 + \lambda(200 - 20x_1 - 10x_2 - 5x_3). \quad \text{(6B.1.i)}$$

Taking first order conditions with respect to each good, we get

$$\begin{aligned} 0.5x_1^{-1} &= 20\lambda \\ 0.5x_2^{-1} &= 10\lambda \\ 0.5x_3^{-1} &= 5\lambda \end{aligned} \quad \text{(6B.1.ii)}$$

Dividing the first equation by the second and solving for x_2, we get $x_2 = 2x_1$. Dividing the first equation by the third and solving for x_3 we get $x_3 = 4x_1$. Substituting these into the budget constraint (which is the fourth first order condition taken with respect to λ), we get

$$20x_1 + 10(2x_1) + 5(4x_1) = 60x_1 = 200, \quad \text{(6B.1.iii)}$$

which implies $x_1 = 3.33$. Then, using the fact that $x_2 = 2x_1$ and $x_3 = 4x_1$, we get $x_2 = 6.67$ and $x_3 = 13.33$.

Exercise 6B.2

Set up the Lagrange function for this problem and solve it to see whether you get the same solution.

Answer: The Lagrange function is

$$\mathcal{L}(x_1, x_2, \lambda) = \alpha \ln x_1 + x_2 + \lambda(200 - 20x_1 - 10x_2). \quad \text{(6B.2.i)}$$

Taking first order conditions with respect to each variable in the Lagrange function, we get

$$\begin{aligned} &\frac{\alpha}{x_1} - 20\lambda = 0 \\ &1 - 10\lambda = 0 \\ &200 - 20x_1 - 10x_2 = 0 \end{aligned} \quad \text{(6B.2.ii)}$$

The second equation implies that $\lambda = 1/10$. Substituting this into the first equation, we get $x_1 = \alpha/2$, and substituting this into the last equation, we get $x_2 = (200 - 10\alpha)/10$.

Exercise 6B.3

Demonstrate how the Lagrange method (or one of the related methods we introduced earlier in this chapter) fails even worse in the case of perfect substitutes. Can you explain what the Lagrange method is doing in this case?

Answer: Consider the utility function $u(x_1, x_2) = x_1 + x_2$. The Lagrange function would then be

$$\mathscr{L}(x_1, x_2, \lambda) = x_1 + x_2 + \lambda(I - p_1 x_1 - p_2 x_2), \tag{6B.3.i}$$

with the first two first order conditions of

$$\begin{aligned} 1 &= \lambda p_1 \\ 1 &= \lambda p_2. \end{aligned} \tag{6B.3.ii}$$

Dividing these, we would get $p_1/p_2 = 1$ or $p_1 = p_2$. But that makes no sense — the prices are taken as given by the consumer. So, suppose $p_1 = 1$ and $p_2 = 2$. The first order conditions would then give us the "result" that $p_1 = 1 = p_2 = 2$. The Lagrange method fails because, as we have seen in the intuitive section of the chapter, there generally are no interior solutions to the optimization problem for a consumer whose tastes treat the goods as perfect substitutes. Instead, the consumer simply consumes only the good that is cheaper. The only time there are interior solutions occurs when $p_1 = p_2$ (our "result" from the Lagrange method) — but in that case any bundle on the budget line is in fact optimal.

Exercise 6B.4

At what value for α will the Lagrange method correctly indicate an optimal consumption of zero shirts? Which of the panels of Graph 6.10 illustrates this?

Answer: It would have to be the case that the MRS is equal to $-p_1/p_2 = -2$ at $x_1 = 10$. The MRS for the utility function $u(x_1, x_2) = \alpha \ln x_1 + x_2$ is

$$MRS = -\frac{\partial u/\partial x_1}{\partial u/\partial x_2} = -\frac{\alpha/x_1}{1} = -\frac{\alpha}{x_1}. \tag{6B.4.i}$$

Thus, when α is such that $-\alpha/10 = -2$, the MRS at $x_1 = 10$ is exactly equal to the slope of the budget constraint. Solving for α we get $\alpha = 20$.

You can check that this is correct by solving the optimization problem with Lagrange function

$$\mathscr{L}(x_1, x_2, \lambda) = 20 \ln x_1 + x_2 + \lambda(200 - 20x_1 - 10x_2). \tag{6B.4.ii}$$

The first two first order conditions of this problem are

$$\frac{20}{x_1} = 20\lambda$$
$$1 = 10\lambda. \tag{6B.4.iii}$$

These solve to give us $x_1 = 10$ and, plugging this back into the budget constraint, $x_2 = 0$. This is exactly what is illustrated in panel (a) of Graph 6.10.

Exercise 6B.5

In the previous section, we concluded that the first order conditions of the Lagrange problem may be misleading when goods are not essential. Are these conditions either necessary or sufficient in that case?

Answer: No. The conditions might not hold at the optimum (as we have seen in the case of corner solutions) — which means they are not necessary conditions for an optimum when goods are not essential. When they do hold, they might hold (as we have seen) at negative consumption levels when corner solutions are optimal — and so they are not sufficient. They are only sufficient for us to conclude we are at an optimum if they lead to positive consumption levels — in that case we would have an interior solution despite the fact that the goods are not essential.

Exercise 6B.6

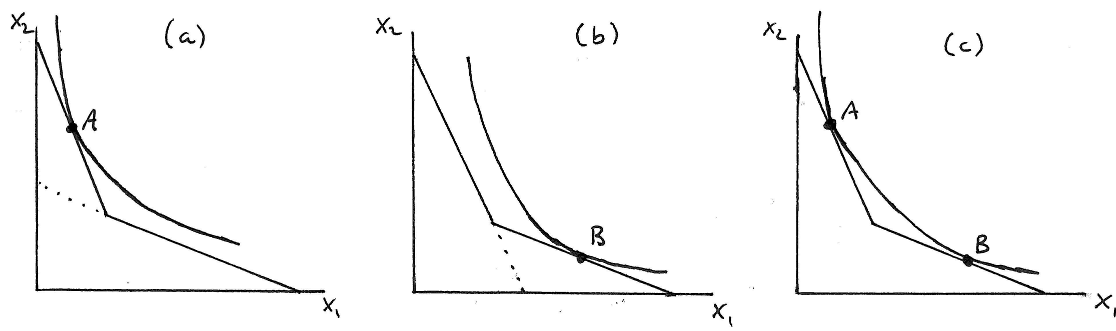
Is it necessary for the indifference curve at the kink of the budget constraint to have a kink in order for both problems in (6.26) to result in x_1=6?

Answer: No, it is not necessary so long as the kink points out rather than in. At the bundle (6,14), the indifference curve can have a slope between −2 and −1 and the kink point will in fact be optimal. (If the kink points in, however, then only an indifference curve that is also kinked at that bundle can result in this bundle being an optimum.)

Exercise 6B.7

Using the intuitions from graphical analysis similar to that in Graph 6.14, illustrate how you might go about solving for the true optimum when a choice set is non-convex due to an "inward" kink.

Answer: Essentially, there are three different possibilities, depicted in panels (a) through (c) of Exercise Graph 6B.7. In panel (a), the optimal bundle is clearly bundle *A* which in fact will be the solution to the Lagrange problem that uses the steeper budget line. The Lagrange problem that uses the shallower budget line might produce an "optimal" bundle that lies on the dashed portion of that shallower budget — in which case we know it can't be optimal given that the steeper budget contains bundles that have strictly more of everything. Alternatively, the Lagrange problem that uses the shallower budget might result in an "optimal" bundle that lies on the solid portion of that shallower budget — but when we determine the utility level at that bundle and compare it to A we would find the utility at A to be higher.



Exercise Graph 6B.7 : Optimization with an Inward Kink

In panel (b), the optimal bundle is B on the shallower portion of the budget constraint. In that case, the Lagrange problem that uses the shallower budget will find this optimal bundle. The Lagrange problem that uses the steeper budget might find an "optimal" bundle on the dashed portion of the steeper budget (in which case we would immediately know that it was not truly optimal since bundles with more of everything are in fact available) or on the solid portion. In the latter case, we we would compare the utility at that bundle to that from B and find that the utility at B is greater.

Finally, panel (c) illustrates the special case where the Lagrange problem with the steeper budget gives us A as the optimal bundle and the Lagrange problem with the shallower budget gives us B — and when we plug both of them back into the utility function, we find that they give the same utility. In that case, we have found two optimal bundles.

6C Solutions to Odd Numbered End-of-Chapter Exercises

Exercise 6.1

I have two 5-year old girls — Ellie and Jenny — at home. Suppose I begin the day by giving each girl 10 toy cars and 10 princess toys. I then ask them to plot their indifference curves that contain these endowment bundles on a graph with cars on the horizontal and princess toys on the vertical axis.

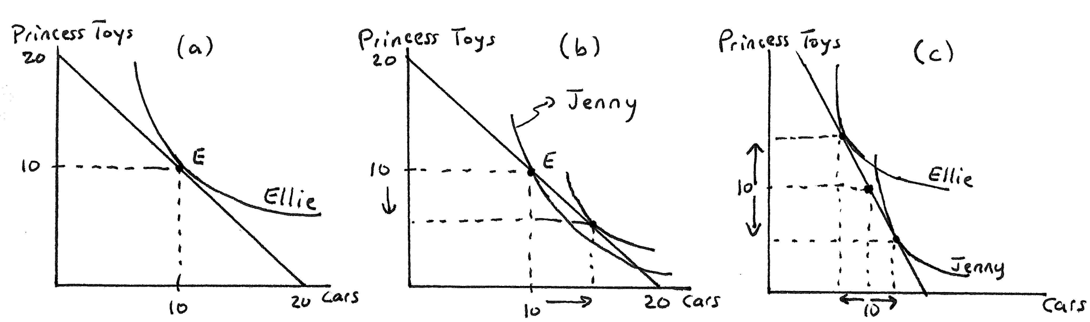
A: *Ellie's indifference curve appears to have a marginal rate of substitution of -1 at her endowment bundle, while Jenny's appears to have a marginal rate of substitution of -2 at the same bundle.*

(a) *Can you propose a trade that would make both girls better off?*

Answer: Any trade under which Jenny would give up x princess toys for 1 car and Ellie would accept x princess toys in exchange for giving up 1 car would work so long as $1 < x < 2$. This is because Jenny would be willing to give up as many as 2 princess toys for 1 car — so the trade will make her better off because she has to give up less; and Ellie would be willing to accept as little as 1 princess toy to give up 1 car — so the trade will make her better off because she gets more without giving up more.

(b) *Suppose the girls cannot figure out a trade on their own. So I open a store where they can buy and sell any toy for \$1. Illustrate the budget constraint for each girl.*

Answer: The budget constraints would be the same for the two girls — because they both have the same endowment point (10,10) and both face the same prices (that result in a slope of -1). These constraints are illustrated in panels (a) and (b) of Exercise Graph 6.1, with the endowment point labeled E.



Exercise Graph 6.1 : Toy Cars and Princess Toys

(c) *Will either of the girls shop at my store? If so, what will they buy?*

Answer: We can then add Ellie's indifference curve through her endowment point in panel (a) and Jenny's indifference curve through her endowment point in panel (b). We know that Ellie's is tangent to her budget constraint because the budget constraint has a slope of -1 and the MRS described in A is also -1 at the endowment bundle E. So Ellie does not want to buy or sell anything at my store at these prices. Jenny's indifference curve at E, on the other hand, has slope -2 — and thus we know her indifference curve cuts her budget constraint at E from above. This implies that Jenny will have better points available in her choice set — with all better points lying to the right of E. Jenny will therefore want to sell princess toys and buy toy cars at my store.

(d) *Suppose I do not actually have any toys in my store and simply want my store to help the girls make trades among themselves. Suppose I fix the price at which princess toys are bought and sold to $1. Without being specific about what the price of toy cars would have to be, illustrate, using final indifference curves for both girls on the same graph, a situation where the prices in my store result in an efficient allocation of toys.*

Answer: It would have to be that the girls have the same tastes at the margin when they leave my store. Thus, they would have to be at indifference curves that are tangent to the same budget line (because their budget goes through the same endowment bundle and has the same slope). Since Jenny likes cars more than Ellie does at their endowment points, this implies that Jenny will end up selling princess toys and buying cars while Ellie will sell car toys and buy princess toys. For the allocation of toys to be efficient, the price of cars will have to be set so that the number of cars Ellie wants to sell is exactly equal to the number of cars that Jenny wants to buy, and the number of princess toys Ellie wants to buy is exactly equal to the number of princess toys Jenny wants to sell. Thus, the arrows on each axis in panel (c) of the graph have to be the same size.

(e) *What values might the price for toy cars take to achieve the efficient trades you described in your answer to (d)?*

Answer: We concluded in (a) that mutually beneficial trades had to have terms of trades under which x princess toys are traded for 1 car, with x falling between 1 and 2. The price of toy cars must therefore be between 1 and 2 times the price of princess toys, allowing consumers to buy between 1 and 2 times as many princess toys as toy cars with any given dollar amount. Since the price of princess toys is fixed at $1, this implies that the price of cars must lie between $1 and $2. You can see from panel (b) that the price of cars can't possibly be lower than $1 because at a price of $1 Jenny wants to buy cars and sell princess toys but Ellie is willing to do neither. Thus, the price of cars has to go up in order to induce Ellie to be willing to sell cars and to induce Jenny to demand fewer cars. At the same time, we could similarly show that the price can't be higher than $2 — because at a price of $2, Jenny would no longer want to trade but Ellie would definitely want to sell cars for princess toys. Depending on exactly what the indifference maps look like, some price between \$1 and \$2 will therefore be just right.

B: *Now suppose that my girls' tastes could be described by the utility function* $u(x_1, x_2) = x_1^{\alpha} x_2^{(1-\alpha)}$*, where* x_1 *represents toy cars,* x_2 *represents princess toys and* $0 < \alpha < 1$.

(a) *What must be the value of* α *for Ellie (given the information in part A)? What must the value be for Jenny?*

Answer: The MRS for this utility function is

$$MRS = -\frac{\partial u/\partial x_1}{\partial u/\partial x_2} = -\frac{\alpha x_1^{\alpha-1} x_2^{1-\alpha}}{(1-\alpha) x_1^{\alpha} x_2^{-\alpha}} = -\frac{\alpha x_2}{(1-\alpha) x_1}. \quad \text{(6.1.i)}$$

At the bundle (10,10), Ellie's MRS is -1 — which implies that $\alpha = 0.5$ for Ellie. Similarly, for Jenny the MRS is -2 at the bundle (10,10) — which implies that $\alpha/(1-\alpha) = 2$ or $\alpha = 2/3$ for Jenny.

(b) *When I set all toy prices to \$1, what exactly will Ellie do? What will Jenny do?*

Answer: We can solve the general optimization problem in terms of α by writing the Lagrange function as

$$\mathscr{L}(x_1, x_2, \lambda) = x_1^{\alpha} x_2^{(1-\alpha)} + \lambda(20 - x_1 - x_2), \quad \text{(6.1.ii)}$$

where the 20 in the parentheses following λ is simply the value of the endowment of 10 car toys and 10 princess toys when the price of each is set to 1. The first two first order conditions of this problem are

$$\begin{aligned} &\alpha x_1^{\alpha-1} x_2^{1-\alpha} = \lambda \\ &(1-\alpha) x_1^{\alpha} x_2^{-\alpha} = \lambda. \end{aligned} \quad \text{(6.1.iii)}$$

Since the right hand side of each of these is equal to λ, we can just set the left hand sides equal to each other and solve for x_2 to get

$$x_2 = \frac{(1-\alpha)}{\alpha} x_1. \quad \text{(6.1.iv)}$$

Plugging this into the budget constraint $20 = x_1 + x_2$, we can solve for x_1 to get $x_1 = 20\alpha$. Plugging this back into equation (6.1.iv), we can also get $x_2 = 20(1-\alpha)$.

Since $\alpha = 0.5$ for Ellie, this implies Ellie's optimal bundle is $(x_1, x_2) = (10, 10)$ — i.e. Ellie will not trade. Since $\alpha = 2/3$ for Jenny, it means Jenny's optimal bundle is $(x_1, x_2) = (13.33, 6.67)$. Jenny will therefore want to trade 3.33 princess toys for 3.33 toy cars.

(c) *Given that I am fixing the price of princess toys at $1, do I have to raise or lower the price of car toys in order for me to operate a store in which I don't keep inventory but simply facilitate trades between the girls?*

Answer: As we already concluded in part A(e), I will have to raise the price of cars to somewhere between $1 and $2.

(d) *Suppose I raise the price of car toys to $1.40, and assume that it is possible to sell fractions of toys. Have I found a set of prices that allow me to keep no inventory?*

Answer: The Lagrange function written in terms of α is then

$$\mathscr{L}(x_1, x_2, \lambda) = x_1^{\alpha} x_2^{(1-\alpha)} + \lambda(24 - 1.4x_1 - x_2), \tag{6.1.v}$$

where 24 is the value of the endowment (10,10). The first two first order conditions are

$$\begin{aligned} \alpha x_1^{\alpha-1} x_2^{1-\alpha} &= 1.4\lambda \\ (1-\alpha) x_1^{\alpha} x_2^{-\alpha} &= \lambda. \end{aligned} \tag{6.1.vi}$$

Dividing the first by the second (and thus canceling λ), we can solve for x_2 in terms of x_1 to get

$$x_2 = \frac{1.4(1-\alpha)}{\alpha} x_1. \tag{6.1.vii}$$

Substituting this into the budget constraint and solving for x_1, we get $x_1 = (24/1.4)\alpha = 17.143\alpha$. Plugging this back into equation (6.1.vii) and solving for x_2, we get $x_2 = 24(1-\alpha)$.

For Ellie, $\alpha = 0.5$ — which implies her optimal bundle will be (8.571,12). Thus, she wants to give up 1.429 of x_1 in exchange for receiving 2 of x_2. For Jenny, $\alpha = 2/3$ — which implies her optimal bundle will be (11.429,8). Jenny therefore wants to get 1.429 of x_1 in exchange for giving up 2 of x_2. The trades exactly offset each other — thus I have to keep no inventory at these prices. I am simply facilitating efficient trade between Ellie and Jenny by setting the price of cars equal to $1.40 (while setting the price of princess toys to $1.00.

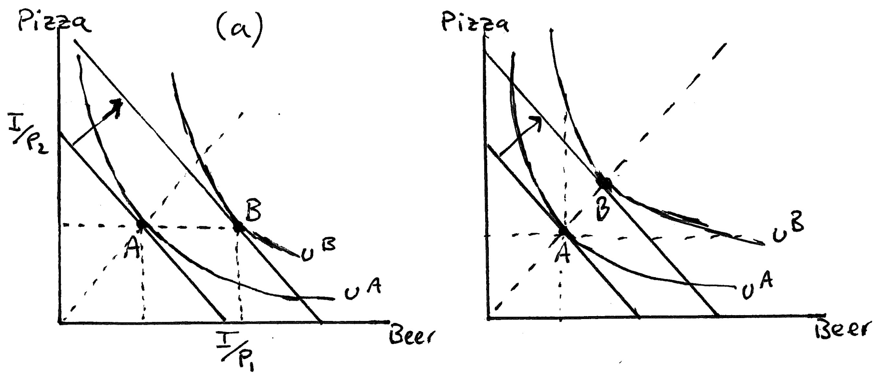
Exercise 6.3

Pizza and Beer: Sometimes we can infer something about tastes from observing only two choices under two different economic circumstances.

A: *Suppose we consume only beer and pizza (sold at prices p_1 and p_2 respectively) with an exogenously set income I.*

(a) *With the number of beers on the horizontal axis and the number of pizzas on the vertical, illustrate a budget constraint (clearly labeling intercepts and the slope) and some initial optimal (interior) bundle A.*

Answer: Panel (a) of Exercise Graph 6.3 illustrates the original budget line containing the optimal bundle A.



Exercise Graph 6.3 : Beer and Pizza

(b) *When your income goes up, I notice that you consume more beer and the same amount of pizza. Can you tell whether my tastes might be homothetic? Can you tell whether they might be quasilinear in either pizza or beer?*

Answer: The shift in income is also indicated in panel (a), with the new optimal bundle B containing more beer but the same amount of pizza. Since the two indifference curves have the same MRS along the horizontal line that holds pizza fixed at its original quantity, the tastes might indeed be quasilinear in pizza. But the tastes could not be homothetic — because, on the ray that passes through A from the origin, the MRS is greater in absolute value along the higher indifference curve than along the lower. The only way this would not be the case is if pizza and beer were perfect substitutes *and* the price of pizza is the same as the price of beer. In that case, all points on both budgets are optimal — including A initially and B after the income change. This would be the one case where tastes are both quasilinear and homothetic.

(c) *How would your answers change if I had observed you decreasing your beer consumption when income goes up?*

Answer: If I simply would have observed a decrease in your beer consumption, I could say for sure that your tastes are not quasilinear in beer (unless beer and pizza are perfect substitutes and prices happen to be such that the slopes of the budget constraints are equal to the MRS everywhere). I could similarly conclude that your tastes are not quasilinear in pizza — because, if you consume less beer with an increase in income, you must be consuming more pizza (if pizza and beer is all you consume). Finally, I could also say for sure that your tastes are not homothetic — because under homothetic tastes, consumption of all goods goes up with increases in income. Again, the one exception is the case where pizza and beer are perfect substitutes with MRS equal to the slopes of the budgets. In that case, we would again have tastes that are both quasilinear and homothetic.

(d) *How would your answers change if both beer and pizza consumption increased by the same proportion as income?*

Answer: This case is graphed in the second panel of Exercise Graph 6.3. The original bundle A and the new optimal bundle B lie on the same ray from the origin — with the indifference curves at both bundles tangent to the same slope. Thus, along this ray, the two indifference curves we know about have the same slope — which is consistent with tastes being homothetic. But the vertical and horizontal lines through A will contain bundles along u^B where the MRS differs from that at A — which implies that the tastes are not quasilinear, at least so long as we rule out the special case that the goods are perfect substitutes and the ratio of prices happens to be such that the budget lines have the same slope as the indifference curves everywhere.

B: *Suppose your tastes over beer (x_1) and pizza (x_2) can be summarize by the utility function $u(x_1,x_2) = x_1^2 x_2$ and that p_1=2, p_2=10 and weekly income I=180.*

(a) *Calculate your optimal bundle A of weekly beer and pizza consumption by simply using the fact that, at any interior solution, $MRS = -p_1/p_2$.*

Answer: Using the fact that we know $MRS = -p_1/p_2 = -2/10 = -1/5$ at the optimum, we can write

$$MRS = -\frac{\partial u/\partial x_1}{\partial u/\partial x_2} = -\frac{2x_1x_2}{x_1^2} = -\frac{2x_2}{x_1} = -\frac{1}{5}, \tag{6.3.i}$$

and the last equality can be written as $x_2 = x_1/10$. Plugging this into the budget constraint $180 = 2x_1 + 10x_2$, we get

$$180 = 2x_1 + 10\frac{x_1}{10} = 3x_1, \tag{6.3.ii}$$

which solves to $x_1 = 60$. Plugging this back into $x_2 = x_1/10$, we also get $x_2 = 6$.

(b) *What numerical label does this utility function assign to the indifference curve that contains your optimal bundle?*

Answer: $u(60,6) = (60^2)(6) = 21,600$.

(c) *Set up the more general optimization problem where, instead of using the prices and income given above, you simply use p_1, p_2 and I. Then, derive your optimal consumption of x_1 and x_2 as a function of p_1, p_2 and I.*

Answer: The more general optimization problem is

$$\max_{x_1,x_2} u(x_1,x_2) = x_1^2 x_2 \text{ subject to } p_1x_1 + p_2x_2 = I, \tag{6.3.iii}$$

with corresponding Lagrange function

$$\mathcal{L}(x_1, x_2, \lambda) = x_1^2 x_2 + \lambda(I - p_1 x_1 - p_2 x_2). \tag{6.3.iv}$$

The first two first order conditions are then

$$\begin{aligned} 2x_1 x_2 &= \lambda p_1 \\ x_1^2 &= \lambda p_2. \end{aligned} \tag{6.3.v}$$

Dividing the first by the second equation, we get $2x_2/x_1 = p_1/p_2$ which can be solved for x_2 to get $x_2 = (p_1 x_1)/(2p_2)$. Substituting this into the budget constraint $I = p_1 x_1 + p_2 x_2$ (which is also the third first order condition), we get

$$I = p_1 x_1 + p_2 \frac{p_1 x_1}{2p_2} = \frac{2p_1 x_1}{2} + \frac{p_1 x_1}{2} = \frac{3p_1 x_1}{2}, \tag{6.3.vi}$$

and this can be solved for x_1 as $x_1 = 2I/(3p_1)$. Plugging this back into the expression $2x_2/x_1 = p_1/p_2$, we can then solve for x_2 as $x_2 = I/(3p_2)$.

(d) *Plug the values p_1=2, p_2=10 and I=180 into your answer to B(c) and verify that you get the same result you originally calculated in B(a).*

Answer: Our solution so far was $x_1 = 2I/(3p_1)$ and $x_2 = I/(3p_2)$. Plugging in the specific values for prices and income, we therefore get $x_1 = 2(180)/(3(2)) = 360/6 = 60$ and $x_2 = 180/(3(10)) = 180/30 = 6$ — 60 beers and 6 pizzas just as we concluded in B(a).

(e) *Using your answer to part B(c), verify that your tastes are homothetic.*

Answer: You can tell how consumption of each good changes with income by taking the derivative of $x_1 = 2I/(3p_1)$ and $x_2 = I/(3p_2)$ with respect to I. This gives

$$\frac{\partial x_1}{\partial I} = \frac{2}{3p_1} \text{ and } \frac{\partial x_2}{\partial I} = \frac{1}{3p_2}. \tag{6.3.vii}$$

Thus, as income increases, consumption of both goods increases linearly. Put differently, as income doubles, consumption of both goods doubles. This is true only for homothetic tastes where the MRS is the same along rays from the origin — which implies that optimal bundles lie on rays from the origin as income changes.

(f) *Which of the scenarios in A(b) through (d) could be generated by the utility function $u(x_1, x_2) = x_1^2 x_2$?*

Answer: Only the last scenario in A(d) could be generated by this utility function since we know it represents homothetic tastes. The scenario in A(b) has tastes that are quasilinear in pizza, while the scenario in A(c) has beer consumption decreasing with an increase in income (which is inconsistent with what we derived before).

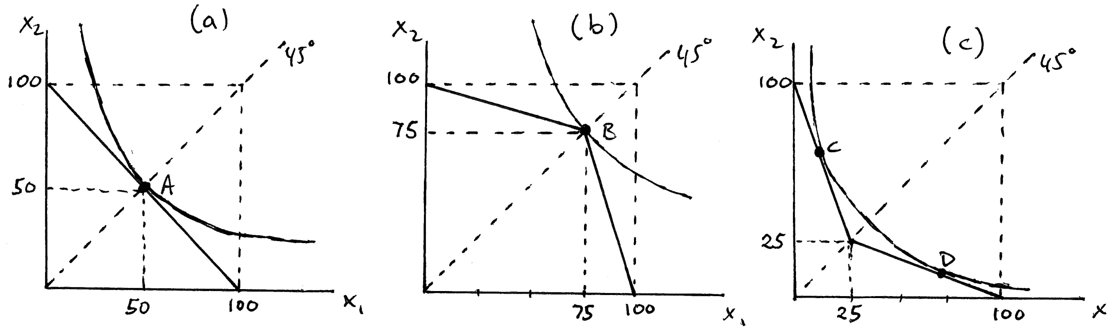
Exercise 6.5

Suppose you have an income of $100 to spend on goods x_1 and x_2.

A: *Suppose that you have homothetic tastes that happen to have the special property that indifference curves on one side of the 45 degree line are mirror images of indifference curves on the other side of the 45 degree line.*

(a) *Illustrate your optimal consumption bundle graphically when $p_1 = 1 = p_2$.*

Answer: Panel (a) of Exercise Graph 6.5(1) illustrates the budget line in this case. Symmetry around the 45-degree line implies that the slope of indifference curves on the 45 degree line must be -1. Since the budget constraint in this case also has slope -1, the optimum must occur on the 45 degree line. This is indicated as point A in the graph.



Exercise Graph 6.5(1) : Homothetic Tastes and Optimization

(b) *Now suppose the price of the first 75 units of x_1 you buy is 1/3 while the price for any additional units beyond that is 3. The price of x_2 remains at 1 throughout. Illustrate your new budget and optimal bundle.*

Answer: This implies that the first 75 units of x_1 cost $25, leaving you with $75 to spend on x_2. The kink point therefore happens at the bundle (75,75). Since the price of x_1 is 3 from then on, you can buy at most 25 more units with the $75 you have left after buying the first 75 units of x_1. The budget constraint therefore looks as it does in panel (b) of the graph. The symmetry of the indifference curves then still implies that the optimum happens on the 45 degree line at the kink point B.

(c) *Suppose instead that the price for the first 25 units of x_1 is 3 but then falls to 1/3 for all units beyond 25 (with the price of x_2 still at 1). Illustrate this budget constraint and indicate what would be optimal.*

Answer: After buying 25 units of x_1 at $3 per unit, you have only $25 left. Thus, the new kink point happens at (25,25). Since the resulting budget line (graphed in panel (c)) is symmetric around the 45 degree line, the symmetry of the indifference curves implies that there will be two optimal bundles (indicated by C and D). These may happen anywhere along the budget line depending on how substitutable the two goods are for one another. If the indifference curves themselves are kinked at the 45-degree line, it may even be the case that $C = D$ so long as the kink is more severe than the kink of the budget constraint (as would be the case for perfect complements).

(d) *If the homothetic tastes did not have the symmetry property, which of your answers might not change?*

Answer: Without the symmetry property, the optimal bundle in (a) would be to the left or right of the 45 degree line, and there would not be two optimal bundles at symmetric distances from the 45 degree line in panel (c). (There might still be two optimal bundles, or there might only be one.) But in panel (b), the optimum might well still occur at the kink point because many different marginal rates of substitution can be "tangent" at that kink.

B: *Suppose that your tastes can be summarized by the Cobb-Douglas utility function* $u(x_1, x_2) = x_1^{1/2} x_2^{1/2}$.

(a) *Does this utility function represent tastes that have the symmetry property described in A?*

Answer: Yes. The MRS for this utility function is $-x_2/x_1$ — which is equal to -1 when $x_1 = x_2$ on the 45 degree line. We can furthermore see that the symmetry holds — if we place x_1 on the vertical instead of the horizontal axis, the MRS simply switches to $-x_1/x_2$ and thus retains the same shape as before.

(b) *Calculate the optimal consumption bundle when* $p_1 = 1 = p_2$.

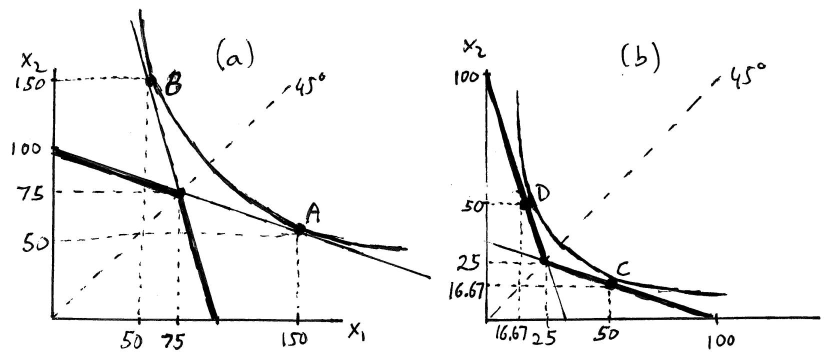
Answer: The optimum occurs where $MRS = -p_1/p_2$ which is $-x_2/x_1 = -1$. Solving for x_2 we get $x_2 = x_1$, and plugging this into the budget constraint, we get $x_1 + x_2 = x_1 + x_1 = 2x_1 = 100$ or $x_1 = 50$ (which then also implies $x_2 = 50$).

(c) *Derive the two equations that make up the budget constraint you drew in part A(b) and use the method described in the appendix to this chapter to calculate the optimal bundle under that budget constraint.*

Answer: The first segment of the budget constraint is $x_2 = 100 - (1/3)x_1$ and the second line segment is $x_2 = 300 - 3x_1$. Optimal tangencies occur where $MRS = -x_2/x_1 = -p_1/p_2$, which implies $x_2 = (p_1 x_1)/p_2$ or $x_2 = p_1 x_1$ since $p_2 = 1$.

Along the first line segment, $p_1 = 1/3$. Substituting $x_2 = p_1 x_1 = (1/3)x_1$ into $x_2 = 100 - (1/3)x_1$, we get $(1/3)x_1 = 100 - (1/3)x_1$ or $(2/3)x_1 = 100$. Solving for x_1, we get $x_1 = 150$ which lies on the portion of the budget line that is not truly part of the kinked budget. This is illustrated as bundle A in panel (a) of Exercise Graph 6.5(2).

Along the second line segment, $p_1 = 3$. Substituting $x_2 = p_1 x_1 = 3x_1$ into $x_2 = 300 - 3x_1$, we get $3x_1 = 300 - 3x_1$ or $6x_1 = 300$. Solving for x_1, we get $x_1 = 50$ which also lies on the portion of the budget line that is not truly part of the kinked budget. This is illustrated as bundle B in panel (a) of Exercise Graph 6.5(2). Note that, due to the symmetry of the indifference curves, bundles A and B lie on the same indifference curve.



Exercise Graph 6.5(2) : Homothetic Tastes and Optimization: Part 2

Since both optimization problems — i.e. the problems using both of the extended line segments as budgets — result in solutions outside the actual kinked budget, the actual optimum lies at the kink point.

(d) *Repeat for the budget constraint you drew in A(c).*

Answer: The first segment of the budget constraint is now $x_2 = 100 - 3x_1$ and the second line segment is $x_2 = 33.33 - (1/3)x_1$. Optimal tangencies occur again where $MRS = -x_2/x_1 = -p_1/p_2$, which implies $x_2 = (p_1x_1)/p_2$ or $x_2 = p_1x_1$ since $p_2 = 1$.

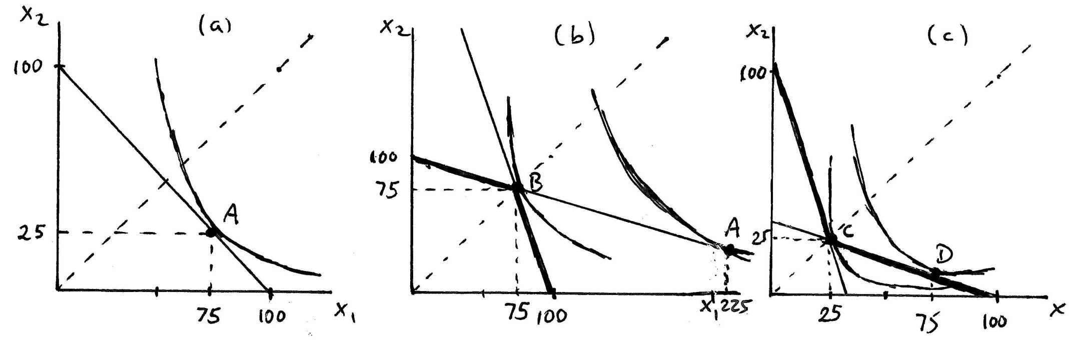
Along the first line segment, $p_1 = 3$. Substituting $x_2 = p_1x_1 = 3x_1$ into $x_2 = 100-3x_1$, we get $3x_1 = 100-3x_1$ or $6x_1 = 100$. Solving for x_1, we get $x_1 = (100/6) = 16.67$ which lies on the portion of the budget line that is in fact part of the kinked budget. This is illustrated as bundle C in panel (b) of Exercise Graph 6.5(2).

Along the second line segment, $p_1 = (1/3)$. Substituting $x_2 = p_1x_1 = (1/3)x_1$ into $x_2 = 33.33 - (1/3)x_1$, we get $(1/3)x_1 = 33.33 - (1/3)x_1$ or $(2/3)x_1 = 33.33$. Solving for x_1, we get $x_1 = 50$ which also lies on the portion of the budget line that is in fact part of the kinked budget. This is illustrated as bundle D in panel (b) of Exercise Graph 6.5(2). Note again that, due to the symmetry of the indifference curves, bundles C and D lie on the same indifference curve. Both of these bundles are therefore optimal.

(e) *Repeat (b) through (d) assuming instead $u(x_1,x_2) = x_1^{3/4}x_2^{1/4}$ and illustrate your answers in graphs.*

Answer: The MRS for this function is $MRS = -3x_2/x_1$. Thus, optimal solutions occur at $MRS = -3x_2/x_1 = -p_1/p_2$ or, equivalently, where $x_2 = p_1x_1/3p_2$ which can furthermore be simplified to $x_2 = p_1x_1/3$ since $p_2 = 1$.

When $p_1 = p_2 = 1$ as in part (b), our optimality condition reduces to $x_2 = x_1/3$. Putting this into the budget constraint, we get $x_1 + x_2 = x_1 + (x_1/3) = 100$ or $(4/3)x_1 = 100$. Solving for x_1 we then get $x_1 = 75$ which implies $x_2 = 25$. This is graphed as A in panel (a) of Exercise Graph 6.5(3).



Exercise Graph 6.5(3) : Homothetic Tastes and Optimization: Part 3

In the scenario of part (c), the first segment of the budget constraint is $x_2 = 100 - (1/3)x_1$ and the second line segment is $x_2 = 300 - 3x_1$. Substituting our optimality condition $x_2 = p_1x_1/3$ into the first equation and letting $p_1 = 1/3$, we get $x_2 = (1/3)(x_1/3) = 100 - (1/3)x_1$ or $(1/9)x_1 = 100 - (1/3)x_1$ which solves to $x_1 = 225$ which is clearly outside the actual kinked budget and is illustrated as A in panel (b) of Exercise Graph 6.5(3). Similarly, substituting our optimality condition $x_2 = p_1x_1/3$ into the second equation and letting $p_1 = 3$, we get $x_2 = 3x_1/3 = 300 - 3x_1$ or $x_1 = 300 - 3x_1$. Solving for x_1, we get $x_1 = 75$. This is exactly the kink point — and is therefore the optimal solution, illustrated as B in panel (b) of Exercise Graph 6.5(3).

In the scenario of part (d), the first segment of the budget constraint is $x_2 = 100 - 3x_1$ and the second line segment is $x_2 = 33.33 - (1/3)x_1$. Substituting our optimality condition $x_2 = p_1x_1/3$ into the first equation and letting $p_1 = 3$, we get $x_2 = 3(x_1/3) = 100 - 3x_1$ or $x_1 = 100 - 3x_1$ which solves to $x_1 = 25$. This happens right at the kink point — which means it could not possibly be an optimum since the indifference curve cuts the other part of the budget constraint. This is illustrated in panel (c) of Exercise Graph 6.5(3) where the kink point is denoted C. Substituting our optimality condition $x_2 = p_1x_1/3$ into the second equation and letting $p_1 = 1/3$, we get $x_2 = (1/3)x_1/3 = 33.33 - (1/3)x_1$ or $(1/9)x_1 = 33.33 - (1/3)x_1$. Solving for x_1, we get $x_1 = 75$. This, illustrated as D in panel (c) of Exercise Graph 6.5(3), is in fact on the actual kinked budget and is therefore the optimal bundle.

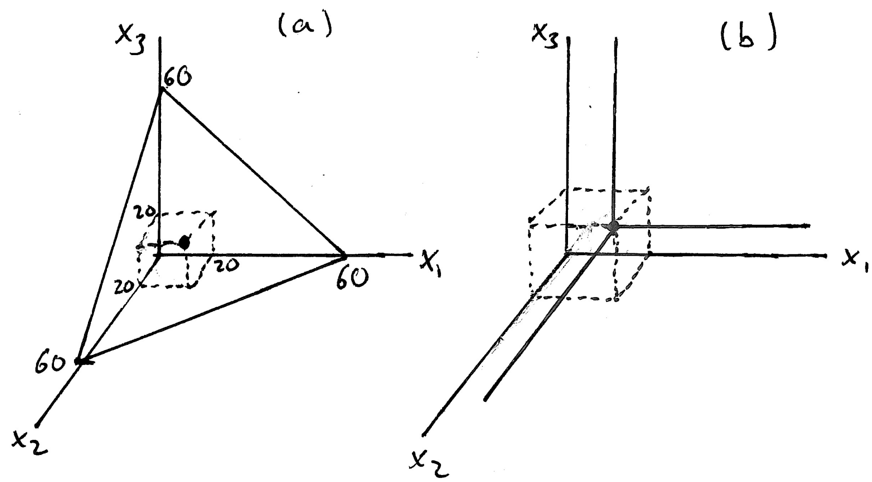
Exercise 6.7

Coffee, Milk and Sugar: Suppose there are three different goods: cups of coffee (x_1), ounces of milk (x_2) and packets of sugar (x_3).

A: *Suppose each of these goods costs 25 cents and you have an exogenous income of $15.*

(a) *Illustrate your budget constraint in three dimensions and carefully label all intercepts.*

Answer: At 25 cents a piece, I can buy as much as 60 of any one of the goods with $15 assuming I don't buy anything else. Thus, panel (a) in Exercise Graph 6.7 has intercept of 60 on each axis (which makes all the slopes equal to −1.)



Exercise Graph 6.7 : Coffee, Milk and Sugar

(b) *Suppose that the only way you get enjoyment from a cup of coffee is to have at least one ounce of milk and one packet of sugar in the coffee, the only way you get enjoyment from an ounce of milk is to have at least one cup of coffee and one packet of sugar, and the only way you get enjoyment from a packet of sugar is to have at least one cup of coffee and one ounce of milk. What is the optimal consumption bundle on your budget constraint.*

Answer: The three goods are therefore perfect complements. This would mean that you would want to consume equal amounts of all three goods — which, given the prices and income, you would do when $x_1 = x_2 = x_3 = 20$. Put differently, you would want to consume 20 perfectly balanced cups of coffee (with an ounce of milk and a packet of sugar in each).

(c) *What does your optimal indifference curve look like?*

Answer: The indifference curve has a corner along the ray from the origin on which all goods are represented in identical quantities. The rest of the indifference "curve" is composed of planes parallel to each of the planes formed by the axes in the graph but ending at the corner. Panel (b) of Exercise Graph 6.7 is an attempt to graph this. Essentially, the indifference "curve" is like three sides of a box with the corner of the box pointing toward the origin and located along the ray that holds all goods equal to one another.

(d) *If your income falls to $10 — what will be your optimal consumption bundle?*

Answer: You would still want to consume the three goods in equal amounts — which means now you could consume 2/3 of what you did before. Before, you were able to consume 20 cups of coffee (with milk and sugar). Now you can only consume 40/3=13.33 cups (with milk and sugar).

(e) *If instead of a drop in income the price of coffee goes to 50 cents, how does your optimal bundle change?*

Answer: Because the goods are perfect complements, it would still need to be the case that you buy the same quantity of each of the goods. Thus, $0.5x_1 + 0.25x_2 + 0.25x_3 = 15$ but $x_1 = x_2 = x_3$ at any optimum. Thus, letting x denote the quantity of each of the goods, $0.5x + 0.25x + 0.25x = 15$ or $x = 15$. Thus, you would drink 15 cups of coffee with milk and sugar.

(f) *Suppose your tastes are less extreme and you are willing to substitute some coffee for milk, some milk for sugar and some sugar for coffee. Suppose that the optimal consumption bundle you identified in (b) is still optimal under these less extreme tastes. Can you picture what the optimal indifference curve might look like in your picture of the budget constraint?*

Answer: The indifference "curve" would still point toward the origin but would now be more "bowl-shaped" rather than "box-shaped" since the corner on the indifference curve would become smooth.

(g) *If tastes are still homothetic (but of the less extreme variety discussed in (f)), would your answers to (d) or (e) change?*

Answer: If 20 cups of coffee with 20 ounces of milk and 20 packets of sugar is optimal under the original income of $15, and if tastes are homothetic, then the ratio of the goods will remains the same if income changes. Thus, the answer to (d) does not change — you would consume 13.33 cups of coffee with as many sugars and ounces of milk when income falls to $10. But when opportunity costs change — as in (e) where the price of a cup of coffee doubles, you will now substitute away from coffee and toward milk and sugar. Thus, you would drink fewer cups of coffee than we concluded in (e), but the coffee would be lighter (because of more milk) and sweeter (because of more sugar).

B: *Continue with the assumption of an income of $15 and prices for coffee, milk and sugar of 25 cents each.*

(a) *Write down the budget constraint.*

Answer: $0.25x_1 + 0.25x_2 + 0.25x_3 = 15$.

(b) *Write down a utility function that represents the tastes described in A(b).*
Answer: $u(x_1,x_2,x_3) = \min\{x_1,x_2,x_3\}$.

(c) *Suppose that instead your tastes are less extreme and can be represented by the utility function $u(x_1,x_2,x_3) = x_1^{\alpha} x_2^{\beta} x_3$. Calculate your optimal consumption of x_1, x_2 and x_3 when your economic circumstances are described by the prices p_1, p_2 and p_3 and income is given by I.*
Answer: It becomes notationally a bit easier to just take the log of the utility function before doing this problem. Thus, we can use the function $v(x_1,x_2,x_3) = \alpha \ln x_1 + \beta \ln x_2 + \ln x_3$. This gives us an optimization problem that can be written as

$$\max_{x_1,x_2,x_3} \alpha \ln x_1 + \beta \ln x_2 + \ln x_3 \text{ subject to } p_1x_1 + p_2x_2 + p_3x_3 = I. \quad (6.7.i)$$

The Lagrange function for this problem is

$$\mathcal{L}(x_1,x_2,x_3,\lambda) = \alpha \ln x_1 + \beta \ln x_2 + \ln x_3 + \lambda(I - p_1x_1 + p_2x_2 + p_3x_3), \quad (6.7.ii)$$

which gives us first order conditions of

$$\begin{aligned} \frac{\alpha}{x_1} &= \lambda p_1 \\ \frac{\beta}{x_2} &= \lambda p_2 \\ \frac{1}{x_3} &= \lambda p_3 \\ p_1x_1 + p_2x_2 &+ p_3x_3 = I. \end{aligned} \quad (6.7.iii)$$

Solving the third equation for λ and substituting this into the first and second equations, we can solve for x_1 and x_2 to get

$$x_1 = \frac{\alpha p_3 x_3}{p_1} \text{ and } x_2 = \frac{\beta p_3 x_3}{p_2}. \quad (6.7.iv)$$

We can then substitute these into the final first order condition (which is equal to the budget constraint) to get

$$p_1x_1 + p_2x_2 + p_3x_3 = p_1\frac{\alpha p_3 x_3}{p_1} + p_2\frac{\beta p_3 x_3}{p_2} + p_3x_3 = (\alpha+\beta+1)p_3x_3 = I. \quad (6.7.v)$$

Solving for x_3, and then using this to plug into equations (6.7.iv), gives

$$x_1 = \frac{\alpha I}{(\alpha+\beta+1)p_1},\ x_2 = \frac{\beta I}{(\alpha+\beta+1)p_2} \text{ and } x_3 = \frac{I}{(\alpha+\beta+1)p_3}. \quad (6.7.vi)$$

(d) *What values must α and β take in order for the optimum you identified in A(b) to remain the optimum under these less extreme tastes?*

Answer: In A(b), $p_1 = p_2 = p_3 = 0.25$ and $I = 15$. Thus, the solutions in (6.7.vi) become

$$x_1 = \frac{60\alpha}{(\alpha+\beta+1)}, \; x_2 = \frac{60\beta}{(\alpha+\beta+1)} \text{ and } x_3 = \frac{60}{(\alpha+\beta+1)}. \quad (6.7.vii)$$

In order for the solution to be $x_1 = x_2 = x_3 = 20$ as in A(b), this implies that $\alpha = \beta = 1$.

(e) *Suppose α and β are as you concluded in part B(d). How does your optimal consumption bundle under these less extreme tastes change if income falls to \$10 or if the price of coffee increases to 50 cents? Compare your answers to your answer for the more extreme tastes in A(d) and (e).*

Answer: Using $\alpha = \beta = 1$ as we have just concluded, the expressions become $x_1 = I/(3p_1)$, $x_2 = I/(3p_2)$ and $x_3 = I/(3p_3)$. Substituting $p_1 = p_2 = p_3 = 0.25$ and $I = 10$, we get $x_1 = x_2 = x_3 = 13.33$ which is identical to what we concluded in A(d) under the more extreme tastes. Substituting $p_1 = 0.50$, $p_2 = p_3 = 0.25$ and $I = 15$, on the other hand, we get $x_1 = 10$, $x_2 = 20$ and $x_3 = 20$. This differs from the answer in A(e) where no substitutability between the goods was permitted — now you end up drinking less coffee but with more milk and sugar in each cup.

(f) True or False*: Just as the usual shapes of indifference curves represent two dimensional "slices" of a 3-dimensional utility function, 3-dimensional "indifference bowls" emerge when there are three goods — and these "bowls" represent slices of a 4-dimensional utility function.*

Answer: This is true. The utility function with three goods can be plotted in 4 dimensions — one for each good and one to indicate the utility level of each bundle — but the indifference "curves" hold utility fixed and can therefore be represented in 3 dimensions. This is analogous to slicing a 3 dimensional utility function with two goods to get two dimensional indifference curves.

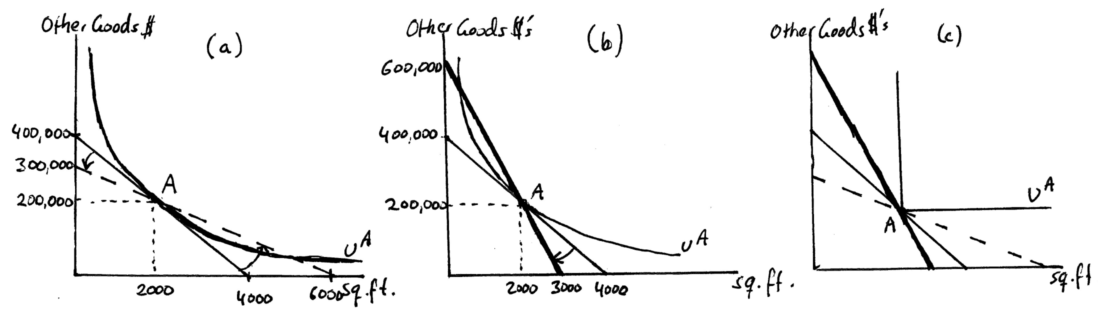
Exercise 6.9

Everyday Application: *Price Fluctuations in the Housing Market: Suppose you have \$400,000 to spend on a house and "other goods" (denominated in dollars).*

A: *The price of 1 square foot of housing is \$100 and you choose to purchase your optimally sized house at 2000 square feet. Assume throughout that you spend money on housing solely for its consumption value (and not as part of your investment strategy).*

(a) *On a graph with "square feet of housing" on the horizontal axis and "other goods" on the vertical, illustrate your budget constraint and your optimal bundle A.*

Answer: The budget constraint would have vertical intercept of \$400,000 (since this is how much other goods you can consume if you buy no housing) and horizontal intercept of 4,000 square feet of housing (since that is how much you can afford at \$100 per square foot if you spend all your money on housing.) The slope of this budget is −100. The budget is depicted as the solid line in panel (a) of Exercise Graph 6.9.



Exercise Graph 6.9 : Housing Price Fluctuations

(b) *After you bought the house, the price of housing falls to \$50 per square foot. Given that you can sell your house from bundle A if you want to, are you better or worse off?*

Answer: The (dashed) new budget line is also drawn in panel (a) of the graph. Note that it has to go through A because A is your endowment point once you have bought the 2000 square foot house. Thus, you can always choose to consume that bundle regardless of what happens to prices. But you can also sell your 2000 square foot house for \$100,000 — which would give you \$300,000 in consumption, your new vertical intercept. Or you can take that \$300,000 and spend it on a new house and thereby buy as much as a 6,000 square foot house since housing now only costs \$50 per square foot. Since your indifference curve at A is tangent to your original budget line, the new (shallower) budget line cuts that indifference curve from below at bundle A. All the new bundles that are now affordable and that lie above the original indifference curve u^A therefore lie to the right of A. You are better off at any of those bundles on the dashed line that lie above the indifference curve u^A.

(c) *Assuming you can easily buy and sell houses, will you now buy a different house? If so, is your new house smaller or larger than your initial house?*

Answer: You will buy a larger house — since all the better bundles on the dashed line in panel (a) are to the right of A and therefore include a house larger than 2000 square feet.

(d) *Does your answer to (c) differ depending on whether you assume tastes are quasilinear in housing or homothetic?*

Answer: No — in both cases you would end up better off consuming a larger house.

(e) *How does your answer to (c) change if the price of housing went up to $200 per square foot rather than down to $50.*

Answer: Panel (b) of Exercise Graph 6.9 illustrates this change in prices. The original budget constraint (from $400,000 on the vertical to 4,000 square feet on the horizontal axis) with bundle A is replicated from panel (a) and illustrates the budget when the price per square foot of housing is $100. The steeper bold line going through A illustrates the new budget line when A is the endowment point and the price of housing goes to $200 per square foot. If you sell your 2000 square foot house at $200 per square foot, you would get $400,000 for it — which, added to the $200,000 you have would give you as much as $600,000 in consumption if you choose not to buy another house. If you do buy another house, the largest possible house at the new prices is now a 3000 square foot house. But you can always choose to stay at A — so A too is on the new budget line. The bundles on the new bold budget that also lie above the indifference curve u^A all lie to the left of A — indicating that the new house that you would purchase would be smaller than your original 2000 square foot house.

(f) *What form would tastes have to take in order for you to not sell your $2000 square foot house when the price per square foot goes up or down?*

Answer: The indifference curve through A would have to have a kink in it, as would be the case if housing and other goods are perfect complements. This is illustrated in panel (c) of Exercise Graph 6.9 where all three budget lines are drawn, as is an indifference curve u^A that treats the two goods as perfect complements. Technically, it could also be the case that the indifference curve through A has a less severe kink at A — one where the slope to the left of A is steeper than the bold budget line and the slope to the right of A is shallower than the slope of the dashed budget line. What is important is that there is a sufficiently severe kink — with no substitutability on the margin between the goods at the kink point. If there is no kink at A — i.e. if there is any substitutability at the margin between housing and other goods at A — then the bold and dashed indifference curves must necessarily cut the indifference curve at A in the ways (though not necessarily with the magnitudes) illustrated in (a) and (b).

(g) True or False*: So long as housing and other consumption is at least somewhat substitutable, any change in the price per square foot of housing makes homeowners better off (assuming it is easy to buy and sell houses.)*

Answer: This is true, as just argued in the previous answer.

(h) True or False*: Renters are always better off when the rental price of housing goes down and worse off when it goes up.*

Answer: This is true. Renters do not have endowment points in this model as homeowners do. So changes in the rental price of housing rotate the budget line through the vertical intercept — which implies that a drop in housing prices unambiguously expands the budget set at every level of housing and an increase in housing prices unambiguously shrinks the choice set at every level of housing.

B: *Suppose your tastes for "square feet of housing" (x_1) and "other goods" (x_2) can be represented by the utility function $u(x_1, x_2) = x_1 x_2$.*

(a) *Calculate your optimal housing consumption as a function of the price of housing (p_1) and your exogenous income I (assuming of course that p_2 is by definition equal to 1.)*

Answer: We want to solve the problem

$$\max_{x_1, x_2} u(x_1, x_2) = x_1 x_2 \text{ subject to } p_1 x_1 + x_2 = I. \tag{6.9.i}$$

The Lagrange function for this problem is

$$\mathscr{L}(x_1, x_2, \lambda) = x_1 x_2 + \lambda(I - p_1 x_1 - x_2), \tag{6.9.ii}$$

which give us first order conditions

$$\begin{aligned} x_2 &= \lambda p_1 \\ x_1 &= \lambda \\ p_1 x_1 + x_2 &= I. \end{aligned} \tag{6.9.iii}$$

Substituting the second equation into the first, we get $x_2 = x_1 p_1$, and substituting this into the last equation, we get $p_1 x_1 + p_1 x_1 = I$ or $x_1 = I/(2p_1)$. Finally, plugging this back into $x_2 = x_1 p_1$, we get $x_2 = I/2$.

(b) *Using your answer, verify that you will purchase a 2000 square foot house when your income is \$400,000 and the price per square foot is \$100.*

Answer: We just concluded that $x_1 = I/(2p_1)$. When $p_1 = 100$ and $I = 400,000$, this implies $x_1 = 400,000/(2(100)) = 2000$.

(c) *Now suppose the price of housing falls to \$50 per square foot and you choose to sell your 2000 square foot house. How big a house would you now buy?*

Answer: By selling your 2000 square foot house at \$50 per square foot, you would make \$100,000. Added to the \$200,000 you had left over after you bought your original 2000 square foot house, this gives you a total income of \$300,000. Plugging I=300,000 and $p_1 = 50$ into our equation for the optimal housing quantity $x_1 = I/(2p_1)$, we get x_1=300,000/(2(50))=3000. Thus, you will buy a 3000 square foot house.

(d) *Calculate your utility (as measured by your utility function) at your initial 2000 square foot house and your new utility after you bought your new house? Did the price decline make you better off?*

Answer: Your initial consumption bundle was (2000, 200000). That gives utility

$$u(2000, 200000) = 2000(200000) = 400,000,000. \tag{6.9.iv}$$

When price fell, you end up at the bundle (3000,150000) which gives utility

$$u(3000, 150000) = 3000(150000) = 450,000,000. \quad (6.9.v)$$

Since your utility after the price decline is higher than before, you are better off.

(e) *How would your answers to B(c) and B(d) change if, instead of falling, the price of housing had increased to $200 per square foot?*

Answer: Again, we have already calculated that $x_1 = I/(2p_1)$ and $x_2 = I/2$. When price increases to $200 and you already own a 2000 square foot house, you can now sell your house for $400,000 which, added to the $200,000 you had left over after buying your original house, gives you up to $600,000 to spend. Treating this as your new I and plugging in the new housing price $p_1 = 200$, we then get that your new optimal bundle has $x_1 = 600000/(2(200)) = 1500$ and $x_2 = 600000/2 = 300,000$. Thus you will buy a 1500 square foot house and consume $300,000 in other goods. This gives you utility

$$u(1500, 300000) = 1500(300000) = 450,000,000, \quad (6.9.vi)$$

which is greater than the utility you had originally and equal to the utility you received from the price decrease before. Thus, a price increase to $200 per square foot makes you better off, exactly as much as a drop in price to $50 per square foot. You are therefore indifferent between the price increase and the price decrease.

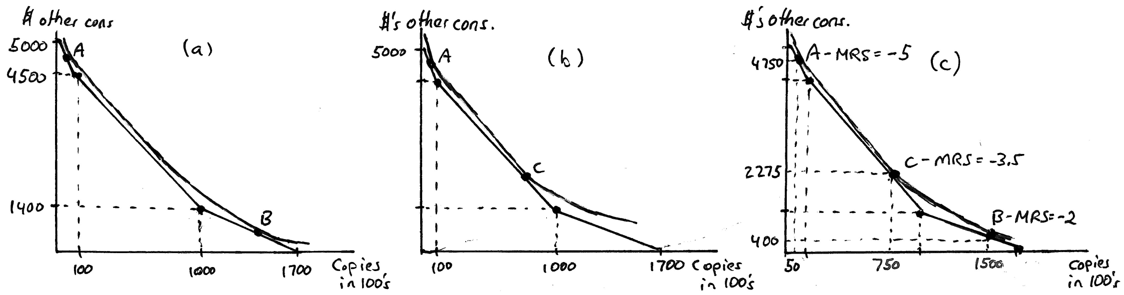
Exercise 6.11

Business Application: *Quantity Discounts and Optimal Choices*: *In end-of-chapter exercise 2.9, you illustrated my department's budget constraint between "pages copied in units of 100" and "dollars spent on other goods" given the quantity discounts our local copy service gives the department. Assume the same budget constraint as the one described in 2.9A.*

A: *In this exercise, assume that my department's tastes do not change with time (or with who happens to be department chair). When we ask below whether someone is "respecting the department's tastes" we mean whether that person is using the department's tastes to make optimal decisions for the department given the circumstances faced by the department. Assume throughout that my department's tastes are convex.*

(a) True or False*: If copies and other expenditures are very substitutable for my department, then you should observe either very little or a great deal of photocopying by our department at the local copy shop.*

Answer: This is true. Panel (a) of Exercise Graph 6.11 illustrates one possibility of this with a single indifference curve tangent at low and high numbers of photocopies (bundles A and B). If the indifference curves have more curvature, then the tangency would lie on the middle portion of the budget constraint with a single optimal quantity that lies in between what one might consider as high and low. It is of course also possible that indifference curves with very little curvature are steeper that the steepest part of the budget — leading to an extreme corner solution on one end of the budget; or that they are very shallow leading to an extreme corner solution on the other end.



Exercise Graph 6.11 : Discounts and Photocopies

(b) *Suppose that I was department chair last year and had approximately 5,000 copies per month made. This year, I am on leave and an interim chair has taken my place. He has chosen to make 150,000 copies per month. Given that our department's tastes are not changing over time, can you say that either I or the current interim chair is not respecting the department's tastes?*

Answer: No, we cannot say that with any certainty. In fact, the indifference curve in panel (a) of Exercise Graph 6.11 illustrates the case where both chairs are respecting the department's tastes despite making very different decision. The reason for this is the non-convexity in the budget set created by the discount policy of the photocopy store.

(c) *Now the interim chair has decided to go on vacation for a month — and an interim interim chair has been named for that month. He has decided to purchase 75,000 copies per month. If I was respecting the department's tastes, is this interim interim chair necessarily violating them?*

Answer: No, not necessarily. Panel (b) of the graph gives and example of an indifference curve that would make both choices, *A* and *C*, optimal from the department's perspective.

(d) *If both I and the initial interim chair were respecting the department's tastes, is the new interim interim chair necessarily violating them?*

Answer: Again, not necessarily. This is illustrated in panel (c) of Exercise Graph 6.11.

B: *Consider the decisions made by the 3 chairs as described above.*

(a) *If I and the second interim chair (i.e. the interim interim chair) both respected the department's tastes, can you approximate the elasticity of substitution of the department's tastes?*

Answer: The elasticity of substitution σ is given by

$$\sigma = \left|\frac{\%\Delta(x_2/x_1)}{\%\Delta MRS}\right| = \left|\left(\frac{(x_2^A/x_1^A)-(x_2^C/x_1^C)}{(x_2^A/x_1^A)}\right)\left(\frac{MRS^A}{MRS^A - MRS^C}\right)\right|, \quad (6.11.i)$$

where bundle A is (50,4750) and C is (750,2275) as depicted in panel (c) of the graph. We furthermore know that $MRS^A = -5$ and $MRS^C = -3.5$. Thus

$$\sigma = \left|\left(\frac{(4750/50)-(2275/750)}{4750/50}\right)\left(\frac{-5}{-5-(-3.5)}\right)\right| = 3.23. \quad (6.11.ii)$$

(b) *If the first and second interim chairs both respected the department's tastes, can you approximate the elasticity of substitution for the department?*

Answer: Now the relevant bundles are C=(750,2275) and B=(1500,400) with $MRS^C = -3.5$ and $MRS^B = -2$, which implies

$$\sigma = \left|\left(\frac{(2275/750)-(400/1500)}{2275/750}\right)\left(\frac{-3.5}{-3.5-(-2)}\right)\right| = 2.13. \quad (6.11.iii)$$

(c) *Could the underlying tastes under which all three chairs respect the department's tastes be represented by a CES utility function?*

Answer: Since we get different elasticity of substitution estimates from the different pairs of choices, tastes that rationalize all three choices given the budget constraint cannot be represented by a *constant* elasticity of substitution utility function that has the same elasticity of substitution everywhere.

Exercise 6.13

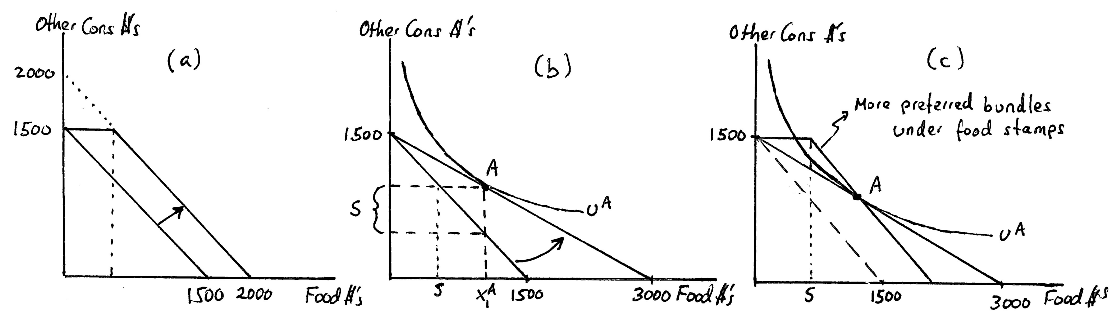
Policy Application: *Food Stamps versus Food Subsidies*: *In exercise 2.13, you considered the food stamp programs in the US. Under this program, poor households receive a certain quantity of "food stamps" — stamps that contain a dollar value which is accepted like cash for food purchases at grocery stores.*

A: *Consider a household with monthly income of $1,500 and suppose that this household qualifies for food stamps in the amount of $500.*

(a) *Illustrate this household's budget, both with and without the food stamp program, with "dollars spent on food" (on the horizontal axis) and "dollars spent on other goods" on the vertical. What has to be true for the household to be just as well off under this food stamp program as it would be if the government simply gave $500 in cash to the household (instead of food stamps)?*

Answer: Panel (a) of Exercise Graph 6.13 illustrates these two budgets. The budget under food stamps has a flat spot at the top because the first $500 in food consumption can be paid for through the food stamps but non-food items cannot be paid for with those stamps. As long as the household would have purchased at least $500 in food under a budget of

$2,000 per month, the food stamp program is exactly like a cash subsidy program for this household. Put differently, so long as the indifference curve tangent to the extended outer budget in panel (a) is tangent at food consumption levels greater than $500, there is no difference between the two types of programs.



Exercise Graph 6.13 : Food Stamps, Cash and Food Subsidies

(b) *Consider the following alternate policy: Instead of food stamps, the government tells this household that it will reimburse 50% of the household's food bills. On a separate graph, illustrate the household's budget (in the absence of food stamps) with and without this alternate program.*

Answer: Panel (b) of the graph illustrates the initial budget (going from $1500 on the vertical axis to $1500 on the horizontal) and the new budget that has shallower slope because $1 of food now only costs 50 cents.

(c) *Choose an optimal bundle A on the alternate program budget line and determine how much the government is paying to this household (as a vertical distance in your graph). Call this amount S.*

Answer: This is also illustrated in panel (b) of the graph. At bundle A, the household is consuming x_1^A in food. We can then read off the vertical axis how much in other consumption the household was able to undertake at A and compare it to how much it would have been able to consume of other goods had it consumed x_1^A in food prior to the subsidy. The difference between these two amounts is S.

(d) *Now suppose the government decided to abolish the program and instead gives the same amount S in food stamps. How does this change the household's budget?*

Answer: This change is illustrated in panel (c) of the graph. In both cases, the bundle A will be available to the consumer because the government is giving S under both programs. However, under the food stamp program, the subsidy amount remains the same regardless of how much food the household consumes, whereas under the price subsidy program the amount of government transfer decreases if the household consumes less food and increases if it consumes more food. Put differently, there is no change in opportunity costs under the cash subsidy, with the price of food going back up to $1 for every $1 of food.

(e) *Will this household be happy about the change from the first alternate program to the food stamp program?*

Answer: The household prefers the food stamps to the price subsidy. You can see this in panel (c) where the indifference curve that makes A optimal under the price subsidy is tangent to the shallower (price subsidy) budget. But this means that the new food stamp budget cuts this indifference curve from above, making a set of new bundles that lie above the indifference curve u^A available to the household.

(f) *If some politicians want to increase food consumption by the poor and others just want to make the poor happier, will they differ on what policy is best?*

Answer: Yes, they will differ. The food price subsidy causes the poor to consume more food whereas the equally costly food stamp program is more preferred by poor households (i.e. makes them happier).

(g) True or False*: The less substitutable food is for other goods, the greater the difference in food consumption between equally funded cash and food subsidy programs.*

Answer: This is false. Imagine making u^A in panel (c) of our graph the shape that presumes food and other goods are perfect complements. In that case, the equally costly food stamp program, which still contains A, will no longer cut the indifference curve u^A — thus eliminating the "better" bundles on the food stamp budget that we identified in panel (c). The household would therefore consume the same amount of food under either program. Then imagine increasing the substitutability between food and other goods at point A — as you do so, more and more "better" bundles become available.

(h) *Consider a third possible alternative — giving cash instead of food stamps.* True or False*: As the food stamp program becomes more generous, the household will at some point prefer a pure cash transfer over an equally costly food stamp program.*

Answer: This is true and relates to our answer to part (a). Since food stamps can only be spent on food, they are equivalent to cash so long as the household would choose to spend at least the value of food stamps on food even if the stamps were replaced by cash. But as the food stamp program becomes more generous, it will at some point be the case that the household would in fact use the food stamps to buy non-food items if it could — and it is at that point that the household would strictly prefer the cash program over the food stamp program.

B: *Suppose this household's tastes for spending on food (x_1) and spending on other goods (x_2) can be characterized by the utility function* $u(x_1, x_2) = \alpha \ln x_1 + \ln x_2$.

(a) *Calculate the level of food and other good purchases as a function of I and the price of food p_1 (leaving the price of dollars on other goods as just 1).*

Answer: We are asked to solve the problem

$$\max_{x_1,x_2} u(x_1,x_2) = \alpha \ln x_1 + \ln x_2 \text{ subject to } p_1x_1 + x_2 = I. \tag{6.13.i}$$

The Lagrange function for this problem is

$$\mathscr{L}(x_1,x_2,\lambda) = \alpha \ln x_1 + \ln x_2 + \lambda(I - p_1x_1 - x_2), \tag{6.13.ii}$$

which gives rise to first order conditions, the first two of which are

$$\begin{aligned} \frac{\alpha}{x_1} &= \lambda p_1 \\ \frac{1}{x_2} &= \lambda \end{aligned} \tag{6.13.iii}$$

Substituting the second equation into the first for λ, we get $x_2 = p_1x_1/\alpha$. And substituting this into the budget constraint (which is the third first order condition), we get $p_1x_1 + x_2 = p_1x_1 + p_1x_1/\alpha = I$ which we can solve for x_1 to get

$$x_1 = \frac{\alpha I}{(\alpha+1)p_1}, \tag{6.13.iv}$$

and substituting this back into $x_2 = p_1x_1/\alpha$,

$$x_2 = \frac{I}{(\alpha+1)}. \tag{6.13.v}$$

(b) *For the household described in part A, what is the range of α that makes the \$500 food stamp program equivalent to a cash gift of \$500?*

Answer: The food stamps are equivalent to a cash gift so long as the household would have spent at least the value of the food stamps on food were it to receive the cash gift instead. Our household has income $I = 1500$ and the price of food is p_1 =\$1 in the absence of a price subsidy. To determine the value of α at which the household would buy exactly \$500 of food with a cash gift of \$500, we need to substitute \$2,000 for I and \$1 for p_1 into our equation for x_1, set it to \$500 and solve for α; i.e.

$$\frac{2000\alpha}{\alpha+1} = 500 \text{ implies } \alpha = \frac{1}{3}. \tag{6.13.vi}$$

Thus, for $\alpha > 1/3$, the cash subsidy is equivalent to the food stamp program of \$500.

(c) *Suppose for the remainder of the problem that $\alpha = 0.5$. How much food will this household buy under the alternate policy described in A(b)?*

Answer: Under this policy, p_1 drops to 1/2 while I remains at 1500. The household will therefore buy

$$x_1 = \frac{\alpha I}{(\alpha+1)p_1} = \frac{0.5(1500)}{1.5(0.5)} = 1000. \qquad (6.13.vii)$$

(d) *How much does this alternate policy cost the government for this household? Call this amount S.*

Answer: If the household buys \$1,000 of food and the government reimburses half, then $S = 500$.

(e) *How much food will the household buy if the government gives S as a cash payment and abolishes the alternate food subsidy program?*

Answer: In that case, $I = 1500 + S = 1500 + 500 = 2000$ and p_1 goes back to 1. Thus,

$$x_1 = \frac{\alpha I}{(\alpha+1)p_1} = \frac{0.5(2000)}{1.5(1)} = 666.67. \qquad (6.13.viii)$$

(f) *Determine which policy — the price subsidy that leads to an amount S being given to the household, or the equally costly cash payment in part (e) — is preferred by the household.*

Answer: Under the price subsidy policy, the household pays \$500 to get \$1000 of food, leaving it with \$1000 in other consumption. Thus, it consumes a bundle (1000,1000). This gives utility

$$u(1000,1000) = 0.5\ln(1000) + \ln(1000) = 10.362. \qquad (6.13.ix)$$

Under the cash subsidy policy, the household gets \$500 in cash to raise its total income to \$2000 of which it spends \$666.67 on food, leaving it with \$1333.33 in other spending; i.e. under cash subsidy, the household consumes bundle (667.67,1333.33). This gives utility

$$u(666.67,1333.33) = 0.5\ln(666.67) + \ln(1333.33) = 10.447. \qquad (6.13.x)$$

The household is happier under the cash subsidy policy.

(g) *Now suppose the government considered subsidizing food more heavily. Calculate the utility that the household will receive from three* equally funded *policies: a 75% food price subsidy (i.e. a subsidy where the government pays 75% of food bills), a food stamp program and a cash gift program.*

Answer: First, consider the price subsidy program that lowers the price p_1 from 1 to 0.25 while keeping I at \$1,500. This will result in food and other good consumption of

$$x_1 = \frac{\alpha I}{(\alpha+1)p_1} = \frac{0.5(1500)}{(1.5)(0.25)} = 2000 \text{ and } x_2 = \frac{I}{(\alpha+1)} = \frac{(1500)}{1.5} = 1000. \tag{6.13.xi}$$

The utility of this bundle is then

$$u(\text{price subsidy}) = u(2000, 1000) = 0.5\ln(2000) + \ln(1000) = 10.708. \tag{6.13.xii}$$

Since food consumption under the price subsidy is 2000, this implies $S = 0.75(2000) = 1500$. If $S = 1500$ is simply given as cash (where income then becomes \$3000 and p_1 goes back up to 1), this will result in food and other consumption of

$$x_1 = \frac{\alpha I}{(\alpha+1)p_1} = \frac{0.5(3000)}{(1.5)(1)} = 1000 \text{ and } x_2 = \frac{I}{(\alpha+1)} = \frac{(3000)}{1.5} = 2000, \tag{6.13.xiii}$$

giving utility of

$$u(\text{cash}) = u(1000, 2000) = 0.5\ln(1000) + \ln(2000) = 11.055. \tag{6.13.xiv}$$

Finally, under the food stamp program of size $S = 1500$, the household would be forced to consume \$1500 of food rather than \$1000 of food that it would have chosen had the money been given in terms of unrestricted cash. Thus, under food stamps, the consumer would buy the bundle (1500,1500) giving utility

$$u(\text{food stamps}) = u(1500, 1500) = 0.5\ln(1500) + \ln(1500) = 10.970. \tag{6.13.xv}$$

Here the food stamp program has gotten so large that it is no longer equivalent to getting cash — and so the consumer prefers the cash to the food stamps but still prefers the food stamps to the food price subsidy.

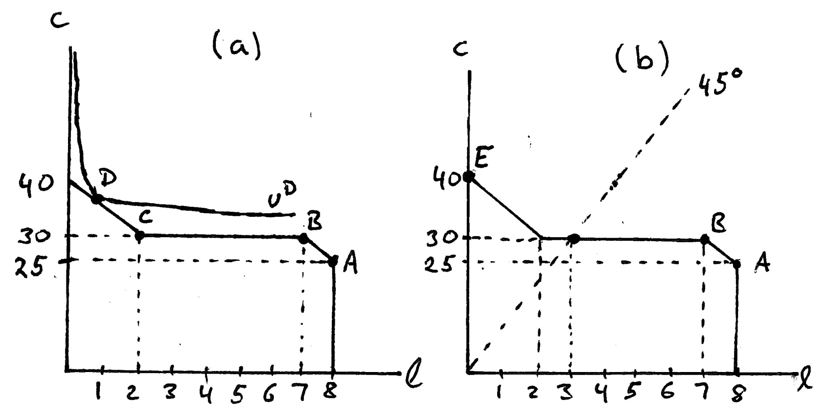
Exercise 6.15

Policy Application: *AFDC and Work Disincentives*: *Consider the AFDC program for an individual as described in end-of-chapter exercise 3.18.*

A: *Consider again an individual who can work up to 8 hours per day at a wage of \$5 per hour.*

(a) *Replicate the budget constraint you were asked to illustrate in 3.18A.*

Answer: This is done in panel (a) of Exercise Graph 6.15(1), with leisure hours on the horizontal and consumption dollars on the vertical axis.



Exercise Graph 6.15(1) : AFDC and Work Disincentives

(b) True or False*: If this person's tastes are homothetic, then he/she will work no more than 1 hour per day.*

Answer: This is false. Suppose, for instance, that leisure and consumption were perfect complements in the sense that this person wants to consume 1 hour of leisure with every \$35 of consumption. Indifference curves would then be L-shaped, with corners happening at bundles like (1,35) and (2,70). This would imply an optimal choice at (1,35) where the worker takes exactly 1 hour of leisure per day and works 7 hours per day. Such tastes are homothetic, as are less extreme tastes that allow for some (but not too much) substitutability between leisure and consumption. An example of an indifference curve u^D from a somewhat less extreme indifference map is illustrated in panel (a) of the graph — with tangency at D.

(c) *For purposes of defining a 45-degree line for this part of the question, assume that you have drawn hours on the horizontal axis 10 times as large as dollars on the vertical. This implies that the 45-degree line contains bundles like (1,10), (2,20), etc. How much would this person work if his tastes are homothetic and symmetric across this 45-degree line? (By "symmetric across the 45-degree line" I mean that the portions of the indifference curves to one side of the 45 degree line are mirror images to the portions of the indifference curves to the other side of the 45 degree line.)*

Answer: Panel (b) of the graph depicts this "45 degree line" where \$10 on the vertical axis is the same distance as 1 hour on the horizontal. In order for indifference curves to be symmetric around this line, it must be that the slope of the indifference curve for bundles on the 45 degree line is -1. But since we are measuring \$10 as geometrically equivalent to 1 hour, a slope of -1 is really a slope, or MRS of -10. If we were to draw a line from the point (0,40) to (3,30), this line would have a slope of $-10/3$. But any indifference curve has a slope of -10 on the 45 degree line — so we know that the indifference curve at (3,30) has a slope of -10 at that point and gets steeper to the left. So all indifference curves going through (3,30) or above on the 45 degree line pass above the budget constraint to the left of the 45 degree line. Thus, such "symmetric" tastes will have an optimum to the right of the 45 degree line — most likely at B but plausibly between B and A.

(d) *Suppose you knew that the individual's indifference curves were linear but you did not know the MRS. Which bundles on the budget constraint could in principle be optimal and for what ranges of the MRS?*

Answer: Bundles on the budget between A and B could be optimal, as could bundle E. In particular for MRS between 0 and $-10/7$, E would be optimal and the individual would work all the time and take no leisure. This is because indifference curves would be straight lines with sufficiently shallow slope to make the corner solution E optimal. For MRS between $-10/7$ and -5, B would be optimal. For $MRS = -5$, any bundle on the budget between B and A is optimal, with all these bundles lying on one indifference curve that is also the highest possible indifference curve for such an individual. Finally, for MRS less than -5, A becomes the optimal bundle.

(e) *Suppose you knew that, for a particular person facing this budget constraint, there are two optimal solutions. How much in AFDC payments does this person collect at each of these optimal bundles (assuming the person's tastes satisfy our usual assumptions)?*

Answer: The only way there can be exactly two optimal solutions is if one of these is B and the other lies anywhere from E to C. The person collects no AFDC between E and C but the full \$25 daily benefit at B.

B: *Suppose this worker's tastes can be summarized by the Cobb-Douglas utility function $u(\ell,c)=\ell^{1-\alpha}c^{\alpha}$ where ℓ stands for leisure and c for consumption.*

(a) *Forget for a moment the AFDC program and suppose that the budget constraint for our worker could simply be written as $c = I - 5\ell$. Calculate the optimal amount of consumption and leisure as a function of α and I.*

Answer: We need to solve the problem

$$\max_{\ell,c}\ u(\ell,c)=\ell^{1-\alpha}c^{\alpha} \text{ subject to } c = I-5\ell. \tag{6.15.i}$$

Setting up the Lagrangian, taking first order conditions and solving for ℓ and c, we get

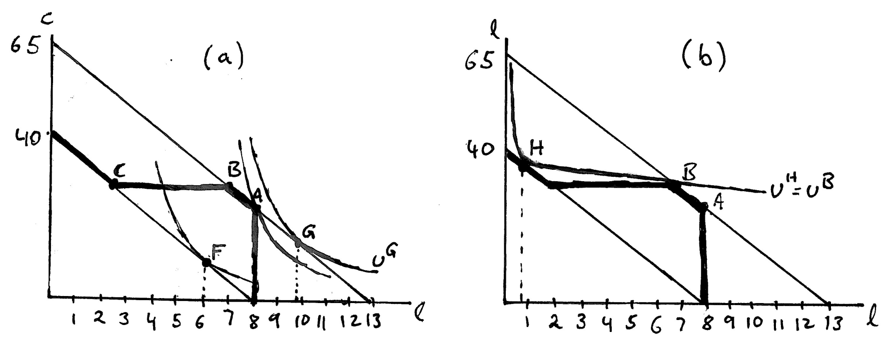
$$\ell=\frac{(1-\alpha)I}{5} \text{ and } c=\alpha I. \tag{6.15.ii}$$

(b) *On your graph of the AFDC budget constraint for this worker, there are two line segments with slope −5 — one for 0-2 hours of leisure and another for 7-8 hours of leisure. Each of these lie on a line defined by $c = I - 5\ell$ except that I is different for the two equations that contain these line segments. What are the relevant I's to identify the right equations on which these budget constraint segments lie?*

Answer: It's easy to see from the graph that I is 40 for the lower line and 65 for the higher.

(c) *Suppose $\alpha = 0.25$. If this worker were to optimize using the two budget constraints you have identified with the two different I's, how much leisure would he choose under each constraint? Can you illustrate what you find in a graph and tell from this where on the real AFDC budget constraint this worker will optimize?*

Answer: When $I = 40$, he would optimize at $\ell = (1-0.25)40/5 = 6$ and when $I = 65$, he would optimize at $\ell = (1-0.25)65/5 = 9.75$. This is illustrated in panel (a) of Exercise Graph 6.15(2) where F with 6 hours of leisure occurs on the lower budget line and G with 9.75 hours of leisure occurs on the higher.



Exercise Graph 6.15(2) : AFDC and Work Disincentives: Part 2

F cannot be optimal inside the (bold) AFDC budget because it lies inside that budget. G, on the other hand, lies outside the (bold) AFDC budget and is therefore not feasible. But we do see that the indifference curve u^G is steeper than −5 on the ray connecting the origin to the kink point A — which implies the highest possible indifference curve on the bold AFDC budget goes through that kink point. Utility at $A = (8,25)$, for instance, would be $u(8,25) = 8^{0.75}25^{0.25} = 10.63$ while utility at $B = (7,30)$ is $u(7,30) = 7^{0.75}30^{0.25} = 10.07$. Thus, the real optimum when $\alpha = 0.25$ is bundle A with no work and all leisure.

(d) *As α increases, what happens to the MRS at each bundle?*

Answer: The MRS for $u(\ell, c) = \ell^{1-\alpha}c^{\alpha}$ is

$$MRS = -\frac{\partial u/\partial \ell}{\partial u/\partial c} = -\frac{-(1-\alpha)\ell^{-\alpha}c^{\alpha}}{\alpha \ell^{1-\alpha}c^{\alpha-1}} = -\frac{(1-\alpha)c}{\alpha \ell}. \quad (6.15.iii)$$

Thus, at any bundle (ℓ, c), the MRS becomes larger in absolute value as α decreases and smaller in absolute value as α increases. Put differently, the slope of an indifference curve at any bundle becomes steeper as α gets smaller and shallower as α gets larger.

(e) *Repeat B(c) for* $\alpha = 0.3846$ *and for* $\alpha = 0.4615$. *What can you now say about this worker's choice for any* $0 < \alpha < 0.3846$*? What can you say about this worker's leisure choice if* $0.3846 < \alpha < 0.4615$*?*

Answer: When $\alpha = 0.3846$, $\ell = (1-0.3846)40/5 = 4.92$ at the lower budget line and $\ell = (1-0.3846)65/5 = 8$ on the higher budget line. The solution on the lower budget line lies inside the AFDC budget and is therefore not optimal. The solution of 8 hours of leisure on the higher budget, on the other hand, is within the AFDC budget — it is bundle A. Thus, when $\alpha = 0.3846$, the highest possible indifference curve on the AFDC budget is just tangent to the extended budget line $c = 65-5\ell$ at A. Since lower α's mean steeper indifference curves at every point, we can conclude from that that A will be optimal for all α's that lie between 0 and 0.3846. When $\alpha = 0.4615$, $\ell = (1-0.4615)40/5 = 4.31$ at the lower budget line and $\ell = (1-0.4615)65/5 = 7$ on the higher budget line. The solution on the lower budget is again inside the AFDC budget — so it cannot be optimal. The solution of 7 leisure hours on the higher budget, on the other hand, corresponds to B on the AFDC budget. Thus, when $\alpha = 0.4615$, the highest indifference curve on the AFDC budget is just tangent to the extended budget line $c = 65-5\ell$ at B. Since the slope of indifference curves becomes steeper as α falls, this implies that, for α between 0.3846 and 0.4615, the optimal leisure choice will lie in between A and B on the AFDC budget at $\ell = (1-\alpha)65/5 = 13(1-\alpha)$.

(f) *Repeat B(c) for* $\alpha = 0.9214$ *and calculate the utility associated with the resulting choice. Compare this to the utility of consuming at the kink point (7,30) and illustrate what you have found on a graph. What can you conclude about this worker's choice if* $0.4615 < \alpha < 0.9214$*?*

Answer: When $\alpha = 0.9214$, $\ell = (1-0.9214)40/5 = 0.629$ giving consumption of $w(8-\ell) = 5(8-0.629) = 36.856$. (On the higher budget line, $\ell = (1-0.9214)40/5 = 1.02$ which lies outside the AFDC budget). The bundle on the lower $c = 40-5\ell$ line, (0.629,36.856), gives utility $u(0.629, 36.856) = 0.629^{(1-0.9214)}36.856^{0.9214} = 26.76$. At B, the consumer would get utility $u(7,30) = 7^{(1-0.9214)}30^{0.9214} = 26.76$. Thus, the optimal bundle H on the budget line $c = 40-5\ell$ lies on the same indifference curve as B — as depicted in panel (b) of Exercise Graph 6.15(2). For $\alpha < 0.9214$, the indifference curve at H would be steeper and would therefore cut the AFDC budget while passing below B — and thus B is optimal for α just below 0.9214. Thus B is the optimal bundle for $0.4615 < \alpha < 0.9214$.

(g) *How much leisure will the worker take if* $0.9214 < \alpha < 1$*?*

Answer: Given that indifference curves become shallower at every bundle as α increases, we know that the indifference curve at H will be shallower for $\alpha > 0.9214$ than the one depicted in panel (b) of Exercise Graph 6.15(2). This implies that the optimal bundle for $\alpha > 0.9214$ lies to the left of H at $\ell = (1-\alpha)40/5 = 8(1-\alpha)$.

(h) *Describe in words what this tells you about what it would take for a worker to overcome the work disincentives under the AFDC program.*

Answer: The exponent α tells us how much weight a person places in his tastes on consumption rather than leisure. When α is high, consumption is valued much more than leisure — so even a small increase in consumption can justify giving up a lot of leisure. Thus, for very high α, it is possible that someone with the AFDC budget constraint will in fact work close to full time despite the work disincentives. But that person's tastes would have to be pretty extreme — he would have to place virtually no value on leisure time. For anyone that places some non-trivial value on leisure time — which implies α isn't close to 1 or, to be more precise, $\alpha < 0.9214$ — the payoff from working close to full time is simply not high enough to sacrifice that much leisure. Thus, for most values of α, the person will choose to work less than 1 hour per day.

Conclusion: Potentially Helpful Reminders

1. Keep in mind the distinction between how the MRS changes along an indifference curve (which tells us about substitutability) and how the MRS changes across the indifference map (which leads to ideas like homotheticity and quasilinearity).

2. The idea of substitutability will become critical in Chapter 7 when we introduce substitution effects (which will depend only on the shape of an indifference curve). The ideas of homotheticity and quasiliearity become important as we introduce income effects (in Chapter 7) — which will be measured across an indifference map (rather than along an indifference curve).

3. Extremes like perfect substitutes and perfect complements are useful to keep in mind because they make it easy to remember which way an indifference map looks if the goods are relatively more substitutable as opposed to relatively more complementary and vice versa.

4. Special cases like homothetic and quasilinear tastes will become useful borderline cases in Chapter 7 — with homothetic tastes being the borderline case between luxury goods and necessities, and with quasilinear tastes being the borderline case between normal and inferior goods. (These terms are defined in Chapter 7.)